Telecommunications Systems Exercises 5: Solutions

May 2025

5.1 Entropy

Consider a ternary symbol set $s \in \{a, b, c\}$. Two out of the three symbols have equal probability of occurrence:

$$P_S(a) = p, \quad P_S(b) = p$$

- **Probability of the third symbol $P_S(c)$:**

  Since the total probability must sum to 1:

  $$P_S(a) + P_S(b) + P_S(c) = 1 \Rightarrow 2p + P_S(c) = 1 \Rightarrow P_S(c) = 1 - 2p$$

  So,

  $$\boxed{P_S(c) = 1 - 2p}$$

- **Entropy $H(s)$ of the source:**

  The entropy of a discrete source is given by:

  $$H(s) = -\sum_i P_S(i) \log_2 P_S(i)$$

  Substituting the values:

  $$H(s) = -p \log_2 p - p \log_2 p - (1 - 2p) \log_2(1 - 2p)$$
  $$H(s) = -2p \log_2 p - (1 - 2p) \log_2(1 - 2p)$$

  $$\boxed{H(s) = -2p \log_2 p - (1 - 2p) \log_2(1 - 2p)}$$

  This expression is valid for $0 < p < 0.5$, since probabilities must be non-negative and $1 - 2p \geq 0$.

- **Value of p that maximizes the entropy:**

  Entropy is maximized when all outcomes are equally likely. For a ternary distribution:

  $$P_S(a) = P_S(b) = P_S(c) = \frac{1}{3}$$

  So the value of p that maximizes the entropy is:

  $$\boxed{p = \frac{1}{3}}$$

5.2 Entropy and Conditional Entropy

Suppose that X is a random variable with entropy $H(X) = 8$ bits. Let $Y = f(X)$ be a deterministic function that maps each value of X to a unique value of Y, i.e., f is injective.

- **What is $H(Y)$, the entropy of Y?**

  Since Y is a deterministic and injective function of X, the randomness in Y is entirely due to the randomness in X. Also, every unique value of X maps to a unique value of Y, so they have the same distribution (up to renaming).

  $$\boxed{H(Y) = H(X) = 8 \text{ bits}}$$

- **What is $H(Y|X)$, the conditional entropy of Y given X?**

  Since Y is a deterministic function of X, knowing X completely determines Y. Thus, there is no uncertainty about Y once X is known:

  $$\boxed{H(Y|X) = 0}$$

- **What is $H(X|Y)$, the conditional entropy of X given Y?**

  Since f is injective (1-to-1), knowing Y uniquely determines X. Therefore, once Y is known, there is no uncertainty about X:

  $$\boxed{H(X|Y) = 0}$$

- **What is $H(X, Y)$, the joint entropy of X and Y?**

  Using the identity:

  $$H(X, Y) = H(X) + H(Y|X)$$

  Substituting known values:

  $$H(X, Y) = 8 + 0 = \boxed{8 \text{ bits}}$$

5.3 BSC Mutual Information and Supported Rate

1. Consider a Binary Symmetric Channel (BSC) with crossover (error) probability p.
The maximum supported rate over a channel corresponds to its capacity. For a BSC, the channel capacity C (in bits per channel use) is given by:

$$C = 1 - H(p)$$

where $H(p)$ is the binary entropy function:

$$H(p) = -p \log_2 p - (1 - p) \log_2(1 - p)$$

Therefore, the maximum supported rate (i.e., the mutual information between input and output when the input is uniformly distributed) is:

$$\boxed{R_{\text{max}} = C = 1 + p \log_2 p + (1 - p) \log_2(1 - p)}$$

This is valid for $0 < p < 0.5$. When $p = 0$, the channel is noiseless and $C = 1$. When $p = 0.5$, the channel is completely noisy (output is independent of input), and $C = 0$.
2.Consider a Binary Erasure Channel (BEC) with erasure probability p_ϵ. That is, each transmitted bit is either received correctly or erased (with probability p_ϵ).
The capacity of a Binary Erasure Channel (BEC) is the maximum reliable transmission rate over the channel. Since the receiver knows which bits are erased (but not their values), only the unerased bits contribute to the information rate. The channel capacity C in bits per channel use is:

$$C = 1 - p_\epsilon$$

Therefore, the maximum supported rate is:

$$\boxed{R_{\text{max}} = 1 - p_\epsilon}$$

This expression means that to reliably communicate over the BEC, the code rate R must satisfy:

$$R \leq 1 - p_\epsilon$$

As $p_\epsilon \to 0$, the channel becomes noiseless and $C \to 1$. As $p_\epsilon \to 1$, the channel becomes completely unreliable and $C \to 0$.

5.4 Analyzing Channel Capacity and Spectral Efficiency

Given:

- Transmit power: $P = 30\,\text{dBm} = 10^{\frac{30-30}{10}} = 1\,\text{W}$
- Noise spectral density: $N_0 = 10^{-9}\,\text{W/Hz}$

Capacity formula:

$$C = B \cdot \log_2 \left(1 + \frac{P}{N_0 \cdot B}\right)$$

1. **Compute channel capacity:**

   - For $B = 1\,\text{MHz} = 10^6\,\text{Hz}$

   $$\frac{P}{N_0 \cdot B} = \frac{1}{10^{-9} \cdot 10^6} = 1000$$

   $$C = 10^6 \cdot \log_2(1 + 1000) \approx 10^6 \cdot \log_2(1001) \approx 10^6 \cdot 9.97 = 9.97\,\text{Mbps}$$

   - For $B = 5\,\text{MHz} = 5 \times 10^6\,\text{Hz}$

   $$\frac{P}{N_0 \cdot B} = \frac{1}{10^{-9} \cdot 5 \cdot 10^6} = 200$$

   $$C = 5 \cdot 10^6 \cdot \log_2(1 + 200) \approx 5 \cdot 10^6 \cdot 7.65 = 38.25\,\text{Mbps}$$

   - For $B = 20\,\text{MHz} = 20 \cdot 10^6\,\text{Hz}$

   $$\frac{P}{N_0 \cdot B} = \frac{1}{10^{-9} \cdot 20 \cdot 10^6} = 50$$

   $$C = 20 \cdot 10^6 \cdot \log_2(1 + 50) \approx 20 \cdot 10^6 \cdot 5.67 = 113.4\,\text{Mbps}$$

2. **Spectral Efficiency** $\frac{C}{B}$ in bits/s/Hz:

   - $B = 1\,\text{MHz} \Rightarrow \frac{9.97 \times 10^6}{10^6} = 9.97\,\text{bps/Hz}$
   - $B = 5\,\text{MHz} \Rightarrow \frac{38.25 \times 10^6}{5 \cdot 10^6} = 7.65\,\text{bps/Hz}$
   - $B = 20\,\text{MHz} \Rightarrow \frac{113.4 \times 10^6}{20 \cdot 10^6} = 5.67\,\text{bps/Hz}$

3. **Trend Explanation:**

   - As bandwidth increases, total capacity C increases.
   - However, the spectral efficiency $\frac{C}{B}$ decreases with bandwidth.
   - This occurs because the SNR $= \frac{P}{N_0 B}$ decreases with increasing B, reducing the efficiency of information transfer per Hz.

4. **Saturation of Capacity:**

   - As B becomes large, $\frac{P}{N_0 B} \ll 1$, and

   $$\log_2 \left(1 + \frac{P}{N_0 B}\right) \approx \frac{P}{N_0 B} \cdot \log_2 e \Rightarrow C \approx \frac{P}{N_0} \cdot \log_2 e$$

   - This means that capacity tends to a constant value as $B \to \infty$, regardless of further increases in bandwidth.

- In this example:

$$\frac{P}{N_0} = \frac{1}{10^{-9}} = 10^9, \quad \Rightarrow C_{\max} \approx 10^9 \cdot \log_2 e \approx 1.44\,\text{Gbps}$$

- This is the saturation point. Beyond this, increasing B does not yield proportional capacity gains.

5.5 Power Increase and Channel Coding Implications

1. (a) Compute capacity at two SNR levels:

$$\text{For SNR} = 1: \quad C = \log_2(1+1) = \log_2(2) = 1\,\text{bps}$$

$$\text{For SNR} = 100: \quad C = \log_2(1+100) \approx \log_2(101) \approx 6.6582\,\text{bps}$$

   (b) After increasing power by a factor of 4 (+6 dB), SNR becomes:

$$\text{Low SNR: SNR} = 4 \quad \Rightarrow \quad C = \log_2(1+4) = \log_2(5) \approx 2.32\,\text{bps}$$

$$\text{High SNR: SNR} = 400 \quad \Rightarrow \quad C = \log_2(1+400) \approx \log_2(401) \approx 8.64\,\text{bps}$$

   (c) **Interpretation:**
   - At low SNR, capacity increases *significantly* (more than doubles) with a modest power increase.
   - At high SNR, the capacity increase is *marginal* — higher power becomes inefficient.
   - **Conclusion:** In high SNR regimes, designers should focus on bandwidth or coding gains rather than increasing power.

2. (a) Uncoded modulation schemes suffer from high bit error rates. Shannon capacity assumes perfect error correction. The gap (around 9 dB) represents the inefficiency of uncoded systems.

   (b) Channel coding reduces the required $\frac{E_b}{N_0}$ for a given error rate. It allows operation close to Shannon capacity. It enables reliable transmission even at lower power or narrower bandwidth.

   (c) **Minimum $\frac{E_b}{N_0}$:**

$$\left(\frac{E_b}{N_0}\right)_{\min} = \lim_{R \to 0} \frac{2^R - 1}{R} = \ln(2) \approx 0.6931 \Rightarrow -1.6\,\text{dB}$$

   - As $R \to 0$, the required $\frac{E_b}{N_0}$ approaches its theoretical minimum of -1.6 dB.
   - This defines the lower bound for error-free transmission regardless of coding or modulation.

5.6 Automatic Repeat Request

(1) Total number of bits:

$$\text{File size} = 200 \times 8{,}000 = 1{,}600{,}000 \text{ bits}$$

(2) Block error probability (single block):

$$P_{\text{block}} = 1 - (1 - P_e)^L \approx 1 - e^{-P_e \cdot L}$$

$$P_{\text{block}} \approx 1 - e^{-10^{-6} \cdot 1{,}600{,}000} = 1 - e^{-1.6} \approx 1 - 0.2019 = 0.7981$$

So, there is approximately a 79.8% chance that the block contains at least one error.

(3) Splitting the file into 100 blocks:
Each block has

$$L = \frac{1{,}600{,}000}{100} = 16{,}000 \text{ bits}$$

$$P_{\text{block}} = 1 - (1 - 10^{-6})^{16{,}000} \approx 1 - e^{-0.016} \approx 1 - 0.9841 = 0.0159$$

So, each smaller block has about a 1.59% chance of being corrupted.

(4) Trade-off Discussion:

- **Fewer large blocks:**
  - Lower overhead per block
  - Higher chance of block error, which increases retransmissions
- **More smaller blocks:**
  - Higher reliability per block (lower error probability)
  - Higher relative overhead (more checksums and feedback messages)
  - Lower initial latency

In practice, block size is optimized to balance throughput, reliability, and system latency.

5.7 Detection and Correction Using Block Codes in BSC

1. Code Rate:

$$R = \frac{k}{n} = \frac{4}{7}$$

2. Number of detectable errors:

$$t' = d_{\min} - 1 = 3 - 1 = 2$$

3. Number of correctable errors:

$$t = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1$$