EE-432
Systeme de Telecommunication

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Sampling and Quantization

Week 4: Table of Contents

- **Sampled Signals in Time and Frequency Domain**
- **Discrete Fourier Transform**
- **Up-/Down-Sampling**
- **Applications**
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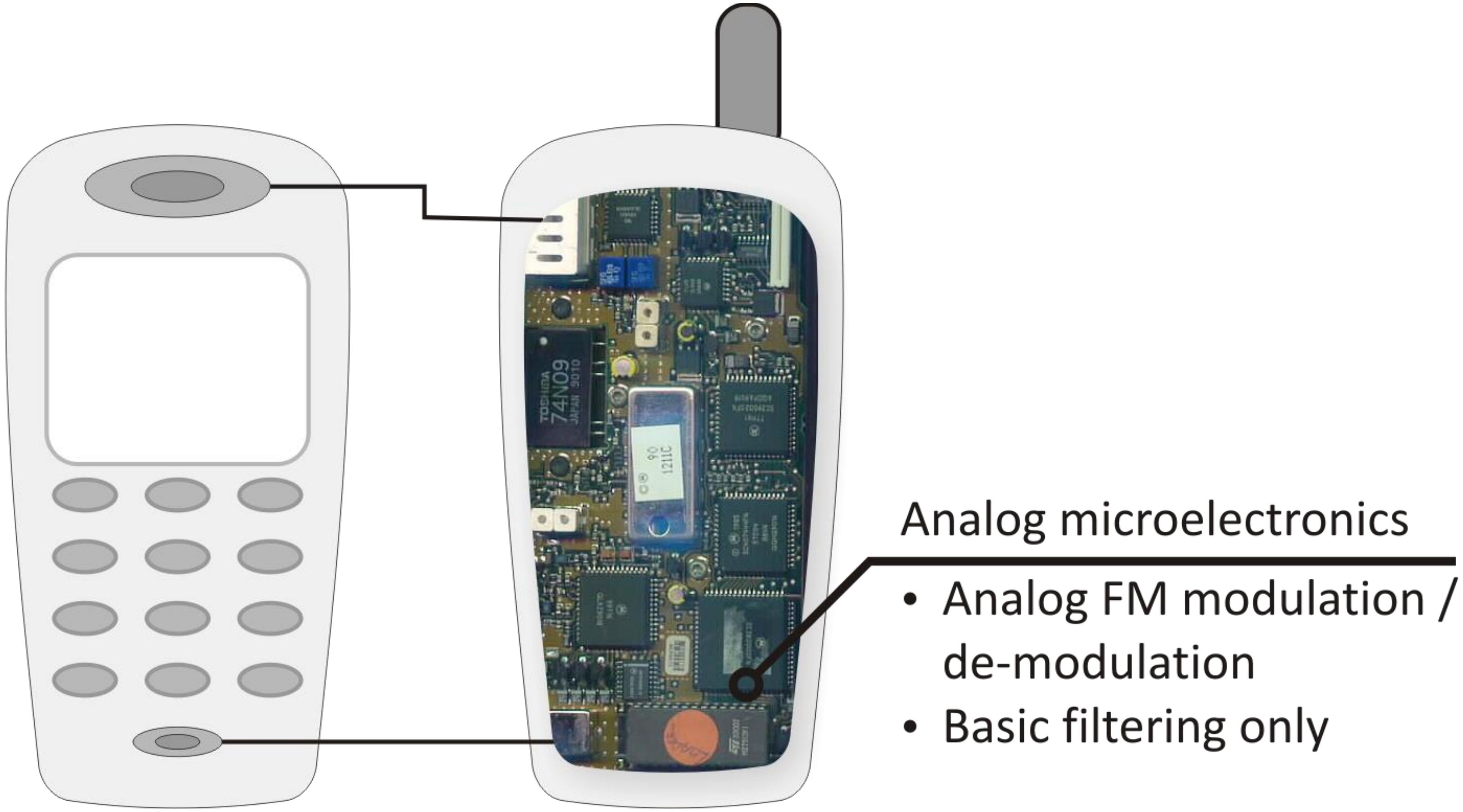
  - Digital AM Audio Transceiver

From Analog to Digital Communications (1)

- **Analog systems: direct link from the analog source signal to the analog RF signal and vice versa through analog circuits**

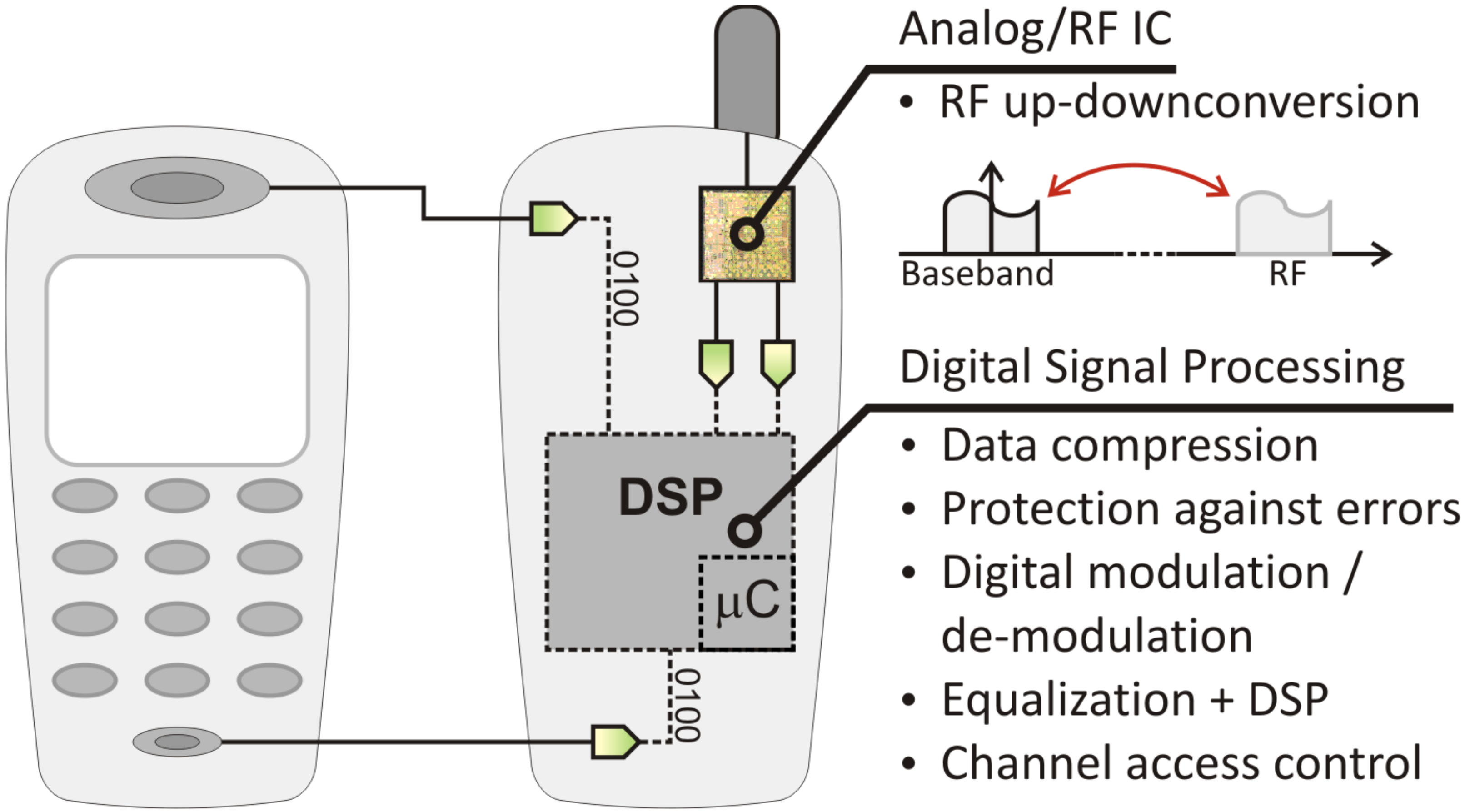
- **Many drawbacks:**
  - Analog circuits are complex
  - Analog circuits are noisy
  - Analog circuits can only realize a very limited set of signal processing functions



From Analog to Digital Communications (2)

- **Digital systems: a digitized source signal generates a digital baseband signal that is then converted to an analog RF signal and vice versa**

- **Advantages:**
  - Digital Signal Processing (DSP) provides unlimited flexibility for processing the source signal and the baseband signal
  - **DSP** can be kept fully noise free and is 100% reproducible

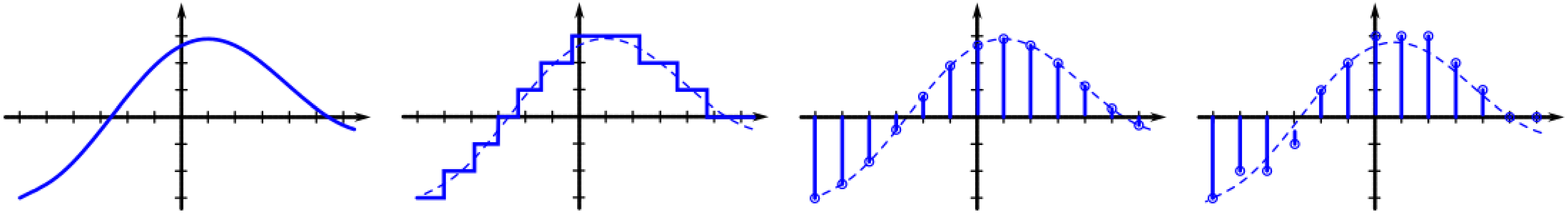


Requirements for Digital Signal Processing

- **Analog (physical) signals: continuous in time and amplitude.**
  - They can not be processed directly by a computer

- **Digital Signal Processing requires digital signals.**
  - A digital signal is often meant as a sufficiently close representation of an analog signal

- **Digital signals: discrete in time and in amplitude.**



Analog
(cont. Time and Amplitude) | (discrete in amplitude) | (discrete in time) | Digital
(discrete in time and amplitude)

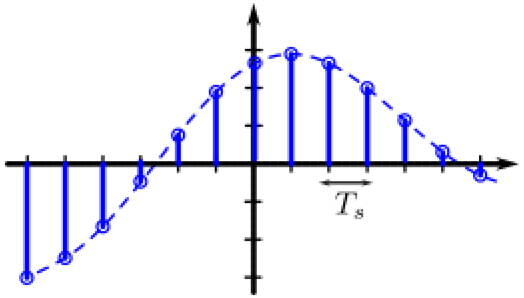
Discrete Time / Sampled Signals

- **A sampled signal describes the value of an originally continuous time signal only at times $\mathrm{t} = \mathrm{n} \cdot T_s$, where** **we call T_s the "sampling period"**, **and where $n = -\infty, \dots, +\infty$ is an integer.**

- **Two options to think of the sampled version of the signal $g(t)$**

Sampled Signal

A continuous time, but "sampled" waveform $\ddot{g}(t)$ with $t \in \mathbb{R}$



Discrete Time Signal

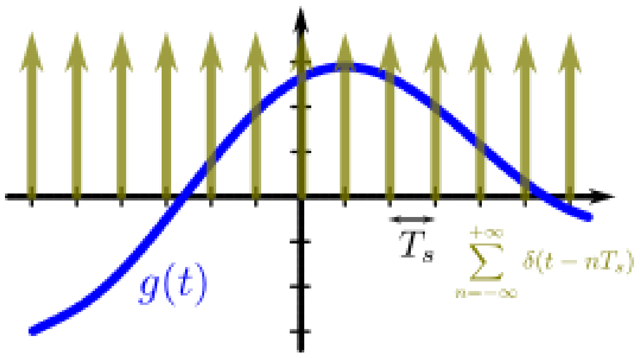
A list of values $g[n]$ with $n = -\infty, \dots, +\infty$ and sampling period T_s

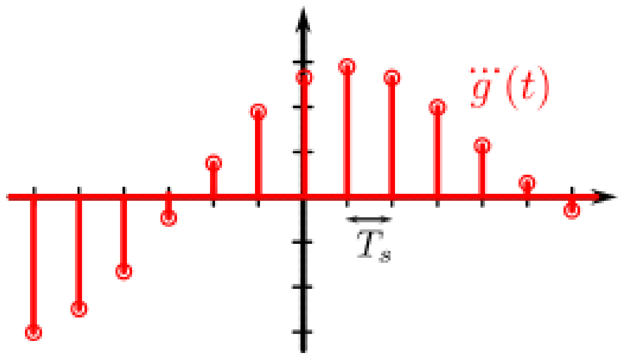
n	...	-3	-2	-1	0	1	2	3	4	...
$g[n]$	...	0.4	0.5	1	1.2	1.25	1.19	1	0.6	...

Sampled Signal

- **Obtain a sampled, but still continuous time representation of a signal by multiplying the signal $g(t)$ with a pulse-train**
  - Sampling period is T_s , i.e., sampling frequency $f_s = \frac{1}{T_s}$
  - A suitable pulse train is $\delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n \cdot T_s)$

$$\ddot{g}(t) = g(t) \cdot \delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} g(t) \cdot \delta(t - n \cdot T_s) = \sum_{n=-\infty}^{+\infty} g(n \cdot T_s) \cdot \delta(t - n \cdot T_s)$$

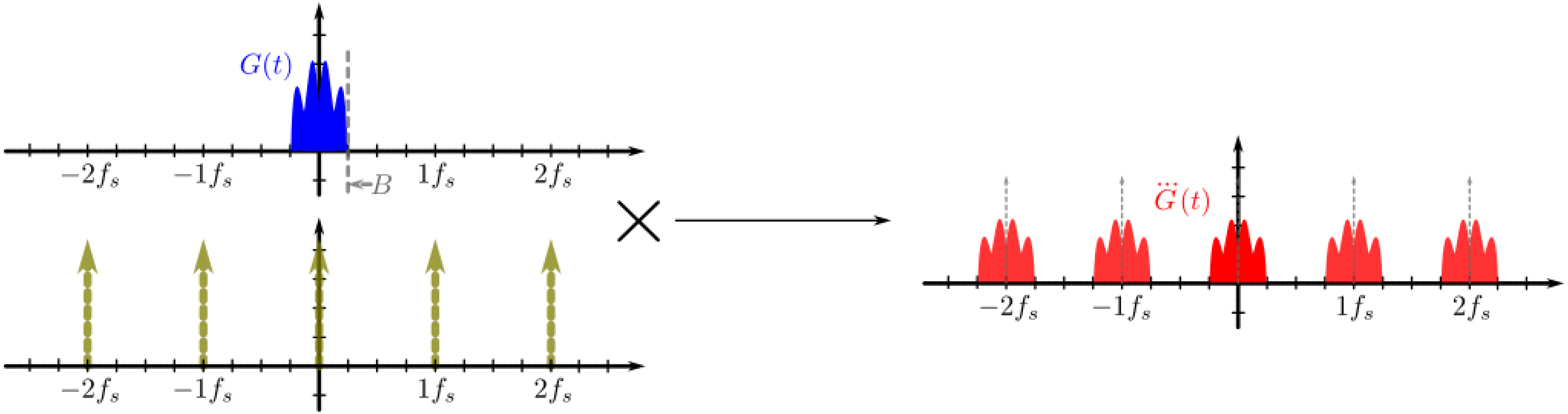




  - Keeps only information about $g(t)$ at the sample points

Spectrum of a Sampled Signal

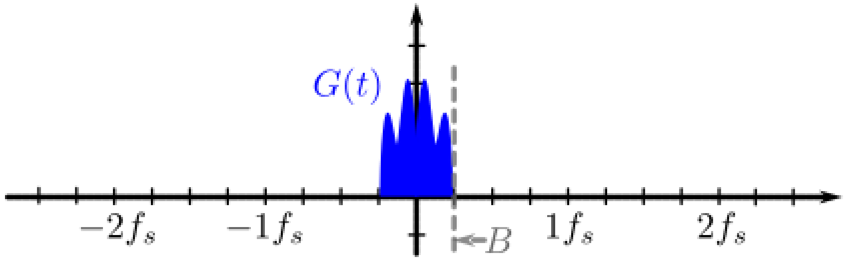
- **Derive the spectrum of the sampled signal (with bandwidth $B < \frac{1}{2 \cdot T_s}$, $B < \frac{1}{2} f_s$)**
  - Multiplication in time domain with pulse-train ⇔ convolution in FD of the spectra
  - Recall spectrum of a pulse-train with spacing T_s : $\mathcal{F}\{\delta_{T_s}(t)\} = \frac{1}{T_s}\delta_{f_s}(t)$ with $f_s = \frac{1}{T_s}$

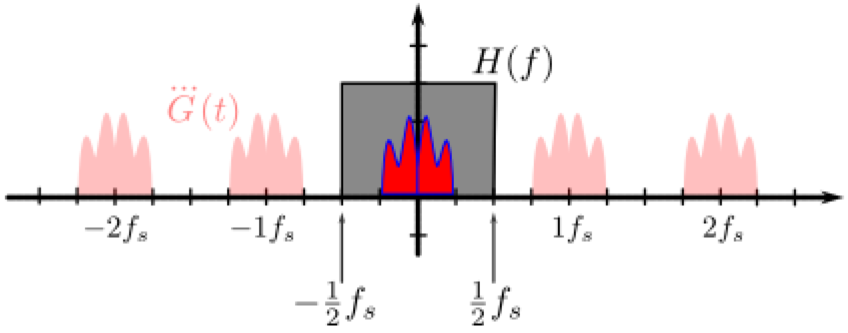


- **Observation:** **sampling creates periodic "images" of the spectrum** **with period $f_s = \frac{1}{T_s}$, leading to a periodic spectrum**

Perfect Reconstruction of a Sampled Signal (FD)

- **The reconstruction of a sampled signal is easy to find in the FD by comparing at the spectra of the sampled and the original signal**





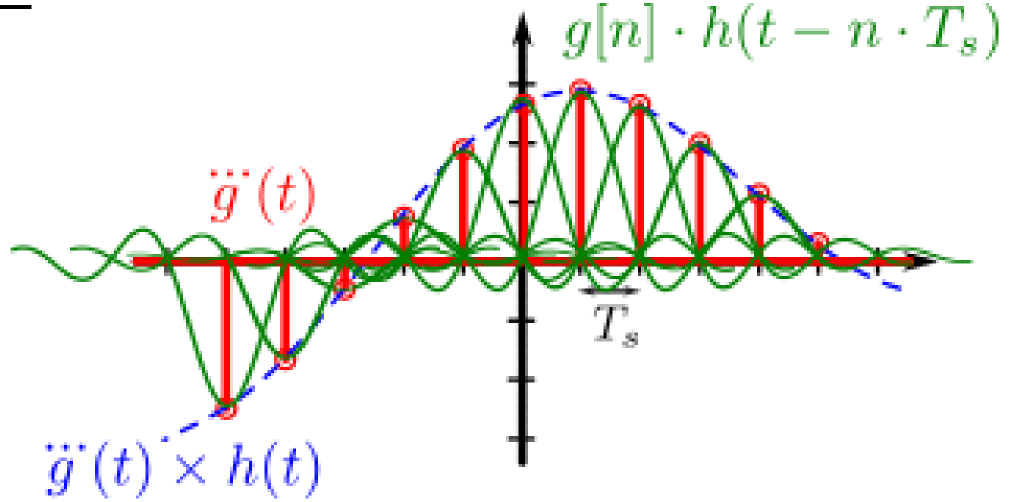
  - Need to remove the replica of the spectrum around $n \cdot f_0$ for $n = -\infty, -1, +1, \dots, +\infty$
  - Only the spectrum around DC for $f = \left[-\frac{1}{2}f_s, +\frac{1}{2}f_s\right]$ remains
- **FD: A brick-wall low-pass filter $H(f) = \Pi_{f_s}(f)$ with cut-off frequency $\frac{1}{2}f_s$ perfectly recovers $G(f)$**
- **TD: see next slide**

Perfect Reconstruction (DAC) of a Sampled Signal

- **The reconstruction is performed in the TD by simply applying the brick-wall low-pass filter derived in the FD.**
  - Brick-wall low-pass filter in FD, corresponding to a SINC filter $h(f)$ in TD
  - The zero-crossing duration of the TD filter impulse response results from the BW of the brick-wall width f_s in the FD as $\frac{1}{f_s} = T_s$

$$h(t) = \frac{\sin \pi \cdot T_s \cdot t}{\pi \cdot T_s \cdot t}$$

$$g(t) = h(t) \times \ddot{g}(t) = \sum_{n=-\infty}^{+\infty} g[n] \cdot h(t - n \cdot T_s)$$

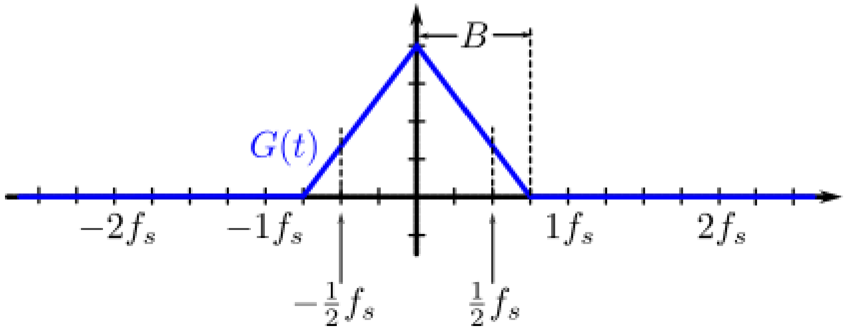


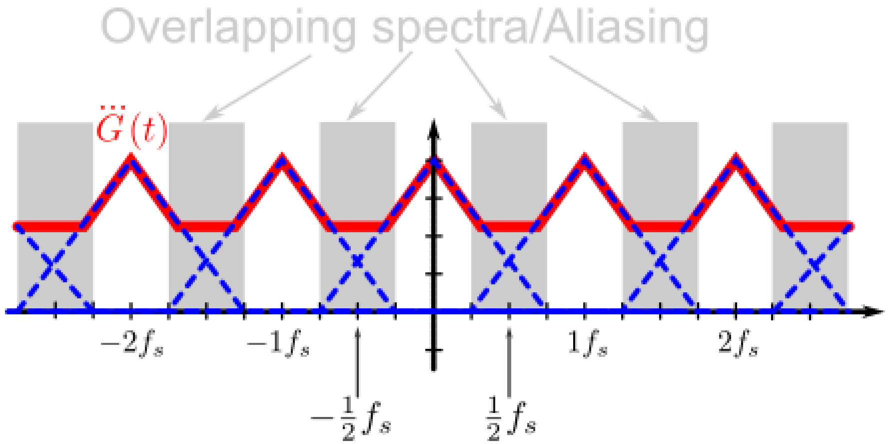
- Note that $h(t) = \begin{cases} 1 & t = 0 \\ 0 & t = n \cdot T_s, n \neq 0 \end{cases}$

Nyquist-Shannon Theorem (1)

- **We have considered signals with bandwidth $B < \frac{1}{2} f_s$ (slide 10), but why?**

- **Consider sampling of a signal with bandwidth $B > \frac{1}{2} f_s$:**



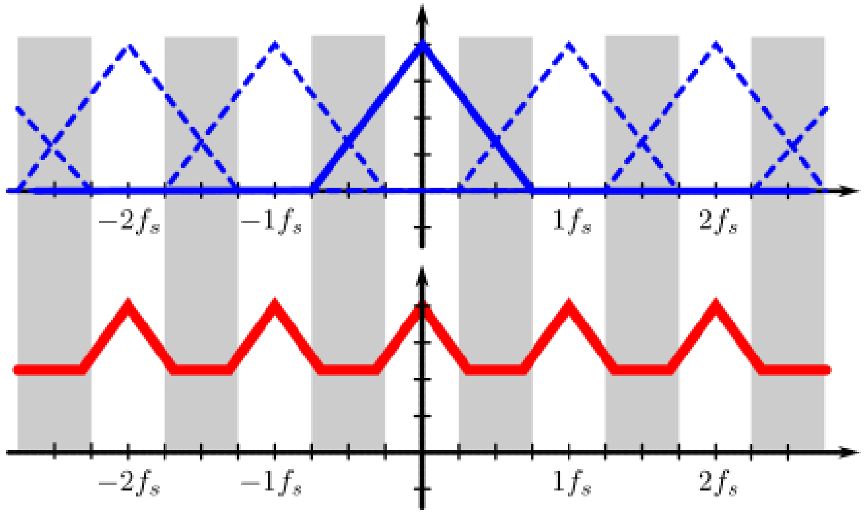


- **Observation: the original spectrum and all periodic "images" overlap**
  - Width of the overlapping spectrum region: $2\left(B - \frac{1}{2} f_s\right) = 2B - f_s$

This overlap of images of the spectrum is called "ALIASING"

Nyquist-Shannon Theorem (2)

- **When a signal is sampled with $f_s < 2 \cdot B$, aliasing "destroys" the original spectrum in the overlapping bands around $\mathrm{n} \cdot \frac{f_s}{2}$.**



- **These parts of the spectrum affected by aliasing can not be restored**
  - Faithful reconstruction of the original signal is no longer possible!

Sampling theorem*:

Perfect reconstruction of signal from its samples taken at sampling frequency f_s, requires that the signal must have a bandwidth $\mathrm{B} < \frac{1}{2} f_s$

$$G\left(|f| > \frac{1}{2} f_s\right) = 0$$

* This is actually only the case if no further information (e.g., sparsity) is available

Discrete Time Representation

- **The representation of a sampled signal as a continuous time signal $\ddot{g}(t)$ with Dirac-pulses at $\boldsymbol{n \cdot T_s}$ is often convenient for illustration**
- **But, we typically prefer to simply express a sampled signal through the value at the discrete sampling points $g[n]$**
  - The integer index n now replaces the time t
  - We need to separately keep track of the sampling period T_s to relate back to $t = n \cdot T_s$

- **We can go back and forth between the two representations as**

$$\ddot{g}(t) = \sum_{n=-\infty}^{+\infty} g[n] \cdot \delta(t - n \cdot T_s)$$

$$g[n] = \ddot{g}(n \cdot T_s)$$

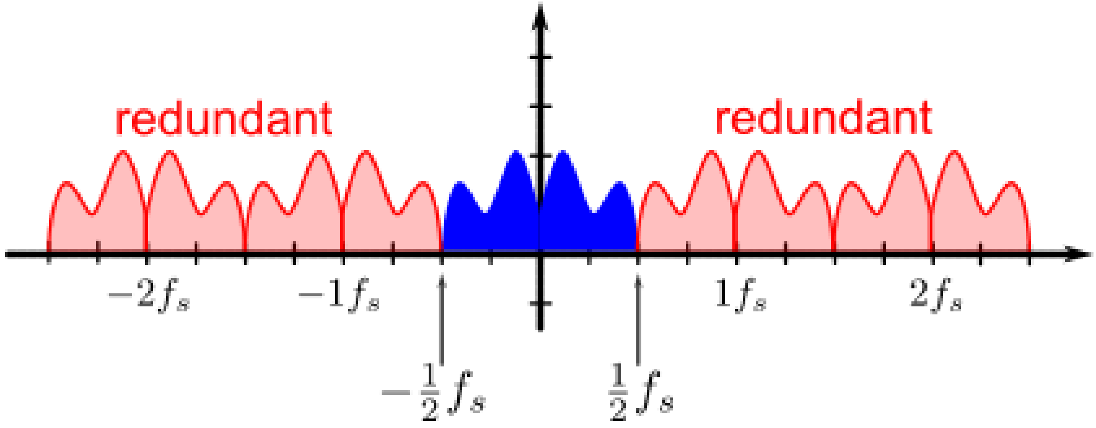
Discrete Fourier Transform (1)

- **We are often interested in numerically computing the spectrum of a sampled signal, typically based on a limited number of samples**

- **As the signal is sampled (frequency f_s), we know that the spectrum is periodic with period f_s ➔ we only need to know one period of the spectrum**

$$G(f + n \cdot f_0) = G(f)$$

  - While we could find any section of the spectrum we typically consider $G(f)$ for $-\frac{1}{2}f_s < f < \frac{1}{2}f_s$



Any signal sampled with frequency f_s can only define the spectrum $G(f)$ for $-\frac{1}{2}f_s < f < \frac{1}{2}f_s$
All other frequencies are then automatically defined

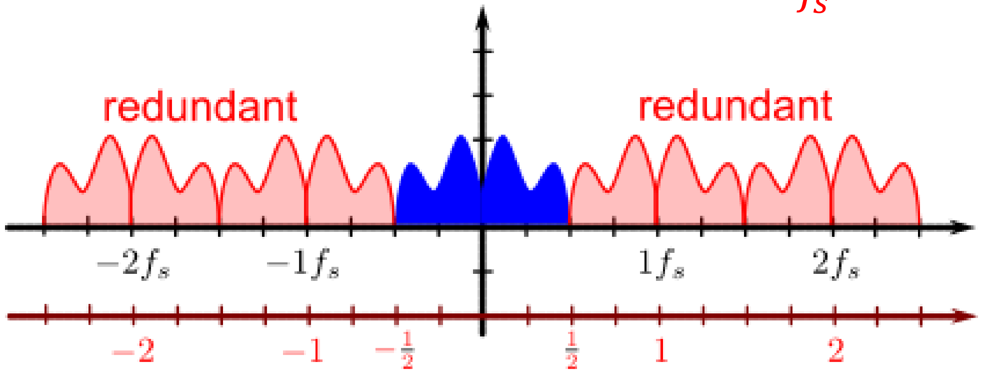
Discrete Fourier Transform (2)

- **Start from the sampled signal representation (assuming we have infinite number of samples)**

$$\ddot{G}(f) = \int_{-\infty}^{+\infty} \ddot{g}(t) \cdot e^{-j\cdot 2\cdot\pi\cdot f\cdot t}\, dt = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} g[n] \cdot \delta(t - n \cdot T_s) \cdot e^{-j\cdot 2\cdot\pi\cdot f\cdot t}\, dt =$$

$$= \sum_{n=-\infty}^{+\infty} g[n] \cdot e^{-j\cdot 2\cdot\pi\cdot f\cdot n\cdot T_s} = \sum_{n=-\infty}^{+\infty} g[n] \cdot e^{-j\cdot 2\cdot\pi\cdot n\cdot \frac{f}{f_s}}$$

  - Note: the FT of the sampled signal is a function of the ratio $\frac{f}{f_s}$. We therefore often consider $G\left(\frac{f}{f_s}\right)$



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Discrete Fourier Transform (3)

- **The spectrum $\ddot{G}(f)$ is computed from an infinite number of a samples, BUT in practice we only have a finite number of samples, so**

$$\ddot{G}(f) \approx \sum_{n=0}^{N-1} g[n] \cdot e^{-j\cdot 2\cdot\pi\cdot f\cdot n\cdot T_s}$$

  - Note that $\ddot{G}(f)$ is still continuous in f
  - HOWEVER, since we only have N samples, there can only be N independent values for $\ddot{G}'(f)$!
  - All other values of $\ddot{G}(f)$ can be reconstructed from those values.
- **The DFT extracts only N independent (orthogonal) frequency components**
  - Since the spectrum is periodic with period f_s, we extract $f = k \cdot \frac{f_s}{N}$ for $k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$

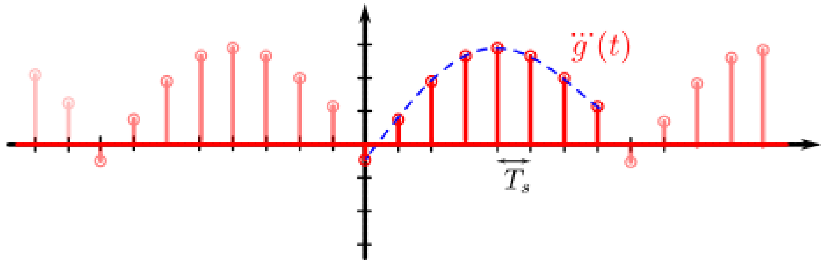
$$\ddot{G}\left(k \cdot \frac{f_s}{N}\right) = \ddot{G}_k = \sum_{n=0}^{N-1} g[n] \cdot e^{-j\cdot 2\cdot\pi\cdot k\cdot\frac{f_s}{N}\cdot n\cdot\frac{1}{f_s}} = \sum_{n=0}^{N-1} g[n] \cdot e^{-j\cdot\frac{2\cdot\pi}{N}\cdot k\cdot n}$$

  - Note that since the DFT spectrum is sampled, reconstruction yields a periodic TD signal

Discrete Fourier Transform (Summary)

- **The Discrete Fourier Transform (DFT) produces a discrete sampled, spectrum with spectral components at $f = k \cdot \frac{f_s}{N}$ for $k = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$**

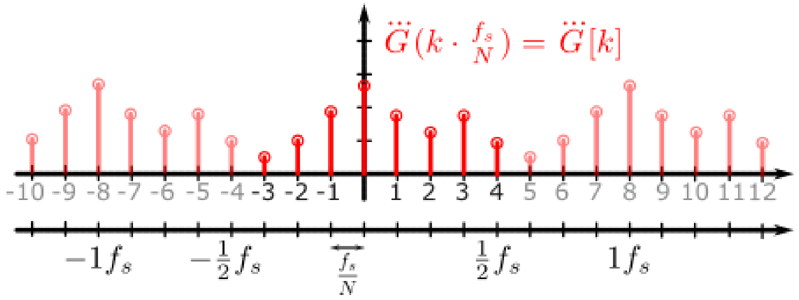
$$\dddot{G}_k = \sum_{n=0}^{N-1} g[n] \cdot e^{-j \cdot \frac{2 \cdot \pi}{N} \cdot k \cdot n}$$



  - For $k < -\frac{N}{2}$ and $k > \frac{N}{2}$ the spectrum is periodic

- **The discrete time signal can be reconstructed as**

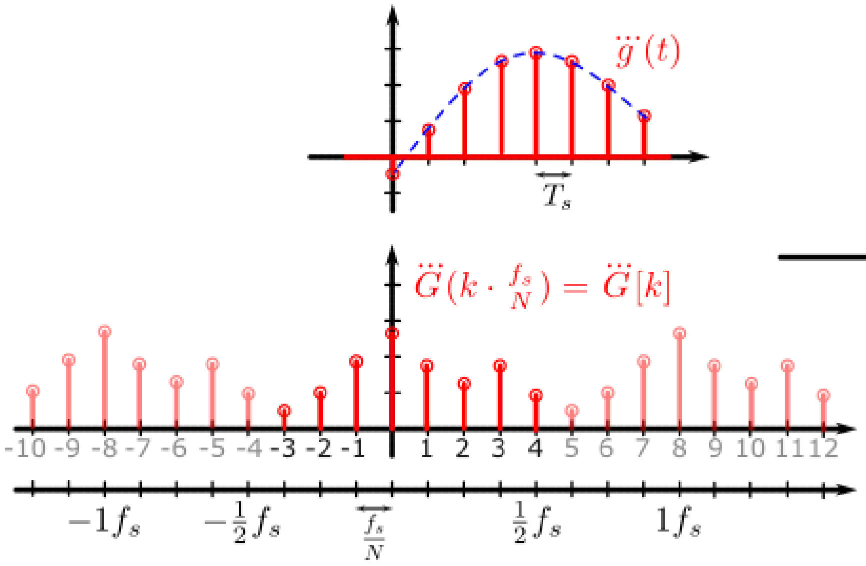
$$g[n] = \frac{1}{N} \sum_{n=0}^{N-1} \dddot{G}_k \cdot e^{j \cdot \frac{2 \cdot \pi}{N} \cdot k \cdot n}$$

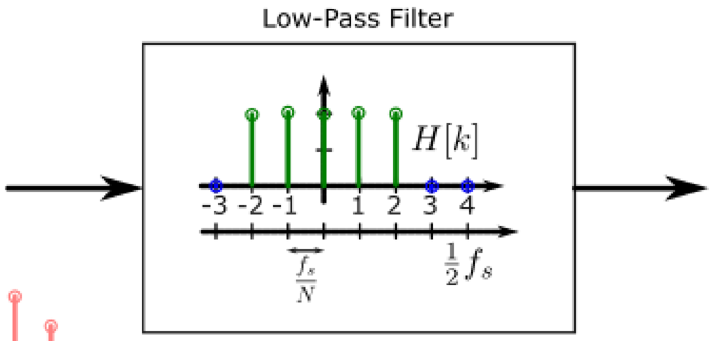


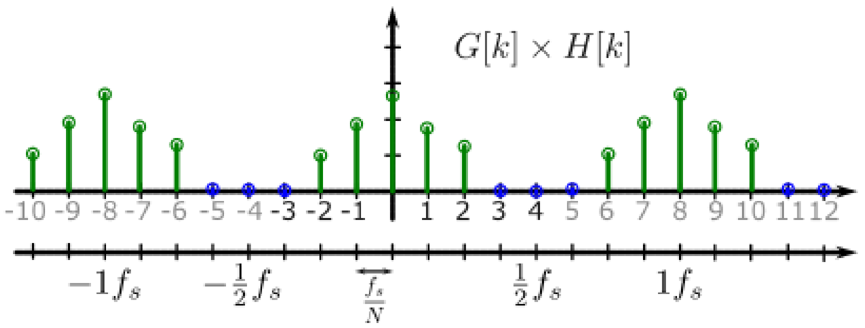
  - For $n < -\frac{N}{2}$ and $n > \frac{N}{2}$ the spectrum is periodic

Up-Sampling (Motivation)

- **Remember: the spectrum of a sampled (discrete time) signal outside $\left]-\frac{f_s}{2}, \frac{f_s}{2}\right]$ is just a periodic repetition of the spectrum in $\left]-\frac{f_s}{2}, \frac{f_s}{2}\right]$.**
  - Any manipulation of the sampled signal leaves the periodicity of the spectrum in tact
  - To manipulate the spectrum outside $\left]-\frac{f_s}{2}, \frac{f_s}{2}\right]$ we first have to increase the sampling rate







Filter affects baseband signal frequencies and all periodic images in the same way

Up-Sampling

- **UP-SAMPLING increases the sampling rate of a sampled signal.**
  - Provides access to the “images” of the original sampled signal to manipulate them
  - Allows to combine/process the signal with other signals (including filters) at higher sampling rate
- **Up-sampling from $\boldsymbol{f_s}$ to $\mathbf{P} \cdot \mathbf{f_s}$ in two steps**
  1. Increase the number of samples, while leaving the cont. time sampled signal waveform in tact
  2. Remove the now accessible image to keep only the original part of the non-periodic spectrum

1. Zero Insertion

$$g'[k] = \begin{cases} g\left[\frac{k}{P}\right] & \frac{k}{P} = \text{integer} \\ 0 & else \end{cases}$$

$$\dddot{G}'_k = \dddot{G}_k$$

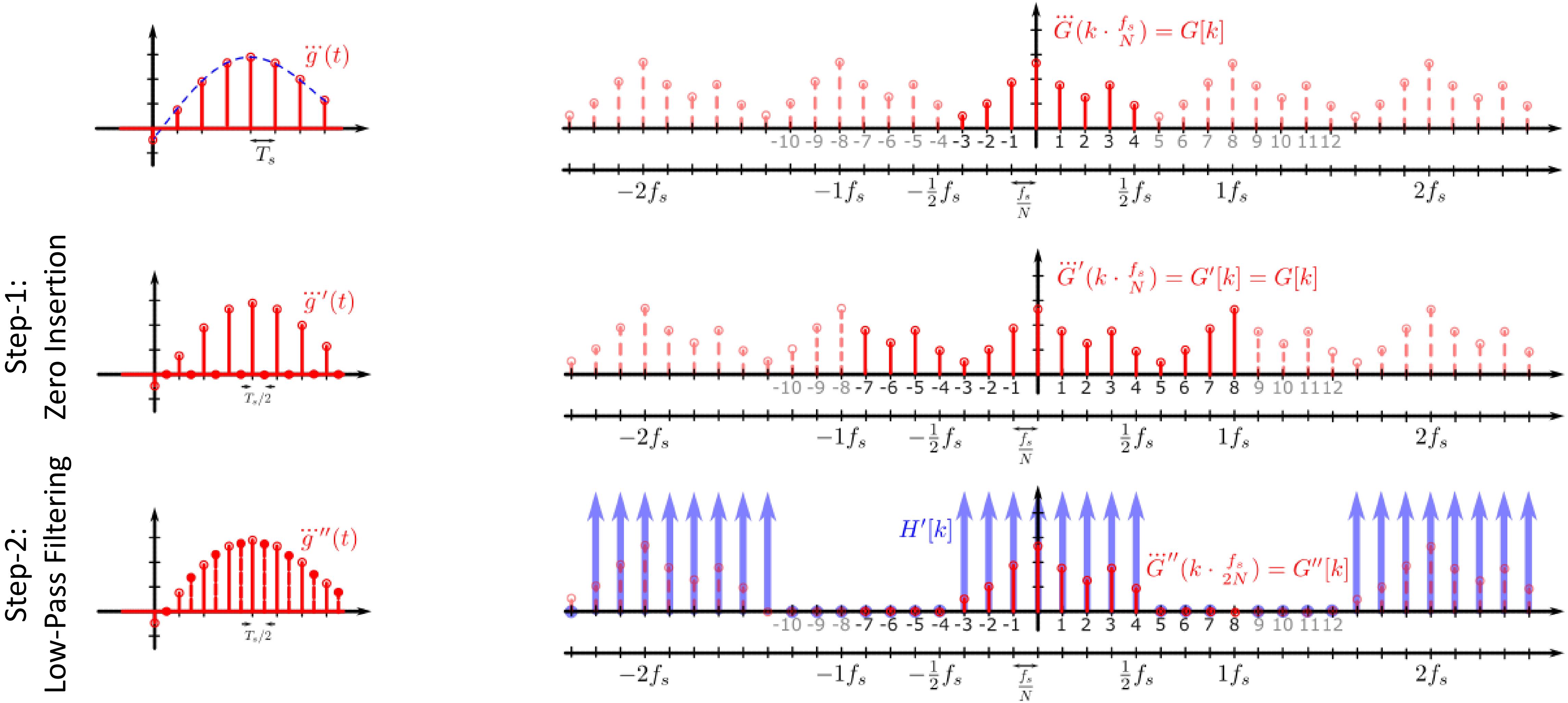
2. Low-Pass Filtering

$$g''[k] = \mathrm{LPF}_{\frac{\mathrm{N}}{2}}\{g'[k]\}$$

$$\dddot{G}''_k = \mathrm{LP}F_{\frac{N}{2}}\{\dddot{G}'_k\}$$

Up-Sampling

- **Example: up-sampling by 2x from f_s to $2f_s$**



Down-Sampling

- **When a signal of interest only occupies a small part of the spectrum covered by the Nyquist Bandwidth $\left]-\frac{f_s}{2}, \frac{f_s}{2}\right]$ we can reduce the sampling rate**
  - Before reducing the number of samples, we need to ensure that the Nyquist-Shannon criterion is met also for the lower sampling rate
- **Down-Sampling from $\boldsymbol{f_s}$ to $\boldsymbol{f'_s = \frac{f_s}{P}}$ in two steps:**

1. Remove all spectral components above $\frac{f'_s}{2}$ (outside $\left]-\frac{f'_s}{2}, \frac{f'_s}{2}\right]$)
2. Keep only every $\boldsymbol{P}$*th* sample

2. Low-Pass Filtering

$$g'[k] = \mathrm{LPF}_{\frac{\mathrm{N}}{2\mathrm{P}}}\{g[k]\}$$

$$\dddot{G}'_k = \mathrm{LP}F_{\frac{\mathrm{N}}{2\mathrm{P}}}\{\dddot{G}_k\}$$

1. Decimation

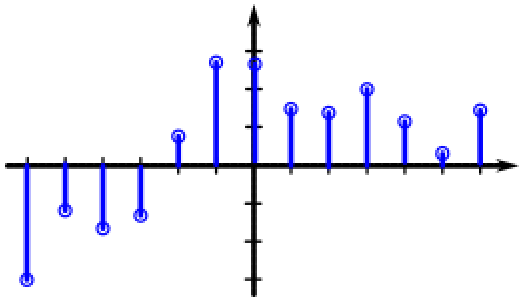
$$g''[k] = g'[P \cdot k]$$

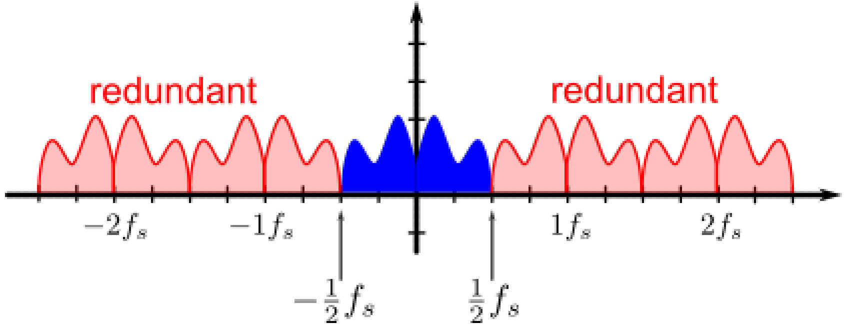
$$\dddot{G}''_k = \dddot{G}'_k \times \delta_{\frac{N}{P}}[k]$$

Recap from Week-4

- **Modern communication systems are based on sampled signals to allow for advanced Digital Signal Processing**
  - DSP used for both data signals and communication waveforms
- **We can think of sampled signals in two ways:**
  - A list of discrete time samples (indexed with an integer, keeping the sampling period in mind)
  - A continuous time sequence of weighted dirac pulses

- **The spectrum of a signal sampled with f_s is periodic with period f_s**
  - We refer to the periodic repetitions of the baseband spectrum around DC as “images”





Recap from Week-4

- **To be able to reconstruct a signal perfectly, it must be sampled with a frequency that is at least twice the signal bandwidth: $\boldsymbol{f_s > 2 \cdot BW}$**
  - Sampling below this frequency leads to aliasing (overlap of “images”)
- **Perfect reconstruction of sampled signals: remove images with a low-pass**
  - Convolution with a brick-wall filter (with signal bandwidth $\boldsymbol{BW}$)

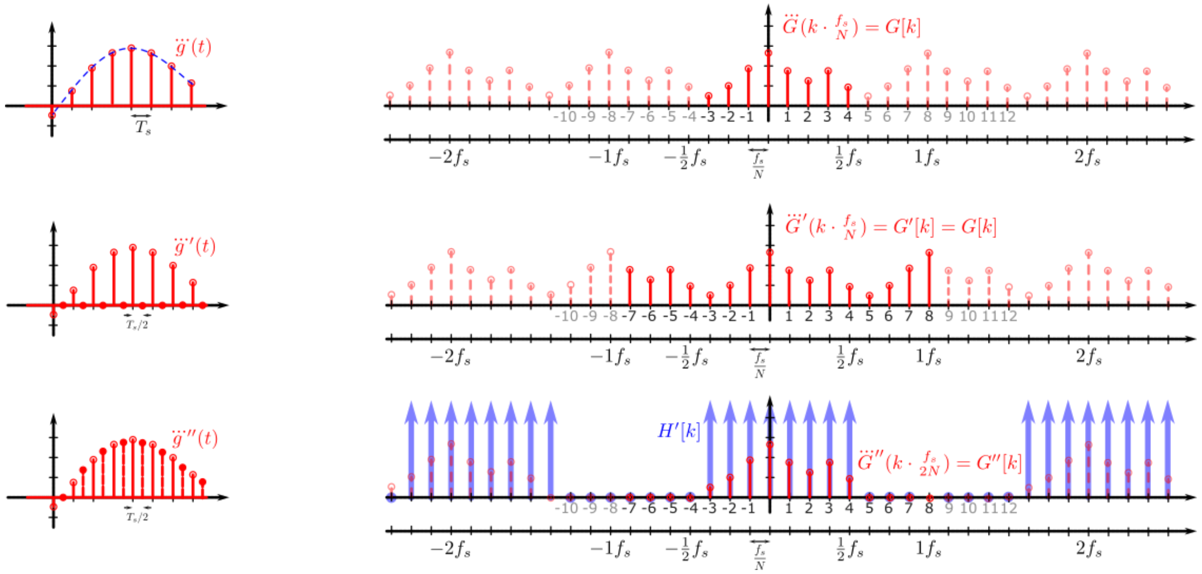
- **The (inverse) discrete fourier transfor computes a discrete spectrum of a from a finite number of $\boldsymbol{N}$ samples**

$$\ddot{G}_k = \sum_{n=0}^{N-1} g[n] \cdot e^{-j \cdot \frac{2 \cdot \pi}{N} \cdot k \cdot n} \qquad g[n] = \frac{1}{N} \sum_{n=0}^{N-1} \ddot{G}_k \cdot e^{j \cdot \frac{2 \cdot \pi}{N} \cdot k \cdot n}$$

  - For $\boldsymbol{N}$ samples with sampling rate $\boldsymbol{f_s}$ (spaced $\mathrm{T}_s = \frac{1}{f_s}$) we obtain a spectrum sampled at a resolution of $\frac{f_s}{N}$ => increasing the number of DFT samples increases spectrum resolution f_s

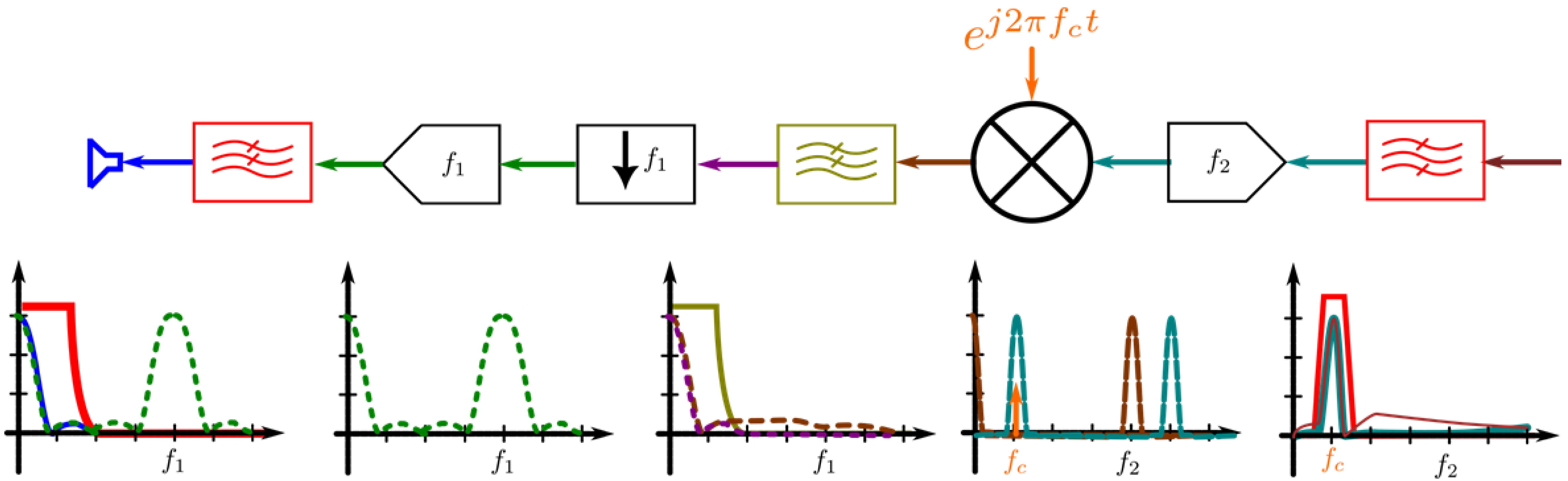
Recap from Week-4

- **With a sampled signal, we can only "manipulate" the spectrum between $-\frac{f_s}{2}$ and $+\frac{f_s}{2}$**
- **To "access" the images, we need to first up-sample the signal**
  - Insert zeros according to up-sampling factor
  - Filter out the now "accessible" images with a low-pass filter



Applications: Software Radio (Digital) AM Receiver for Audio

- **Example: A digital radio performs all modulation in the digital domain, using a very high sampling frequency to directly output an RF signal after DA conversion**
  - Multiple levels of filtering and sample rate conversion are common



Applications: Software Radio (Digital) AM Transmitter for Audio

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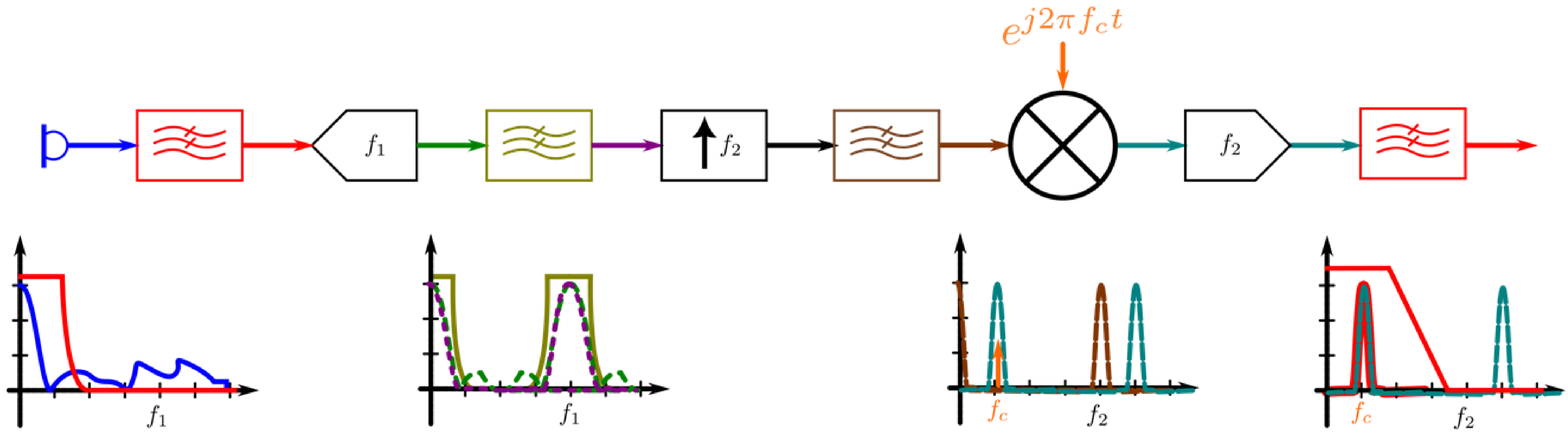
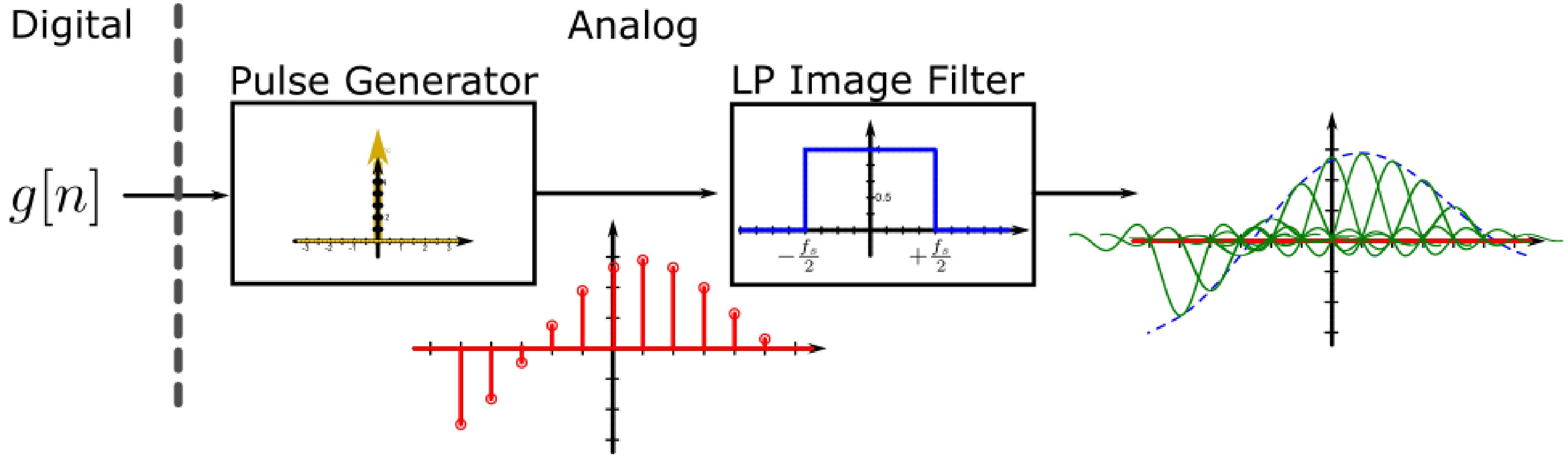


Applications: Practical D-to-A Conversion / Reconstruction

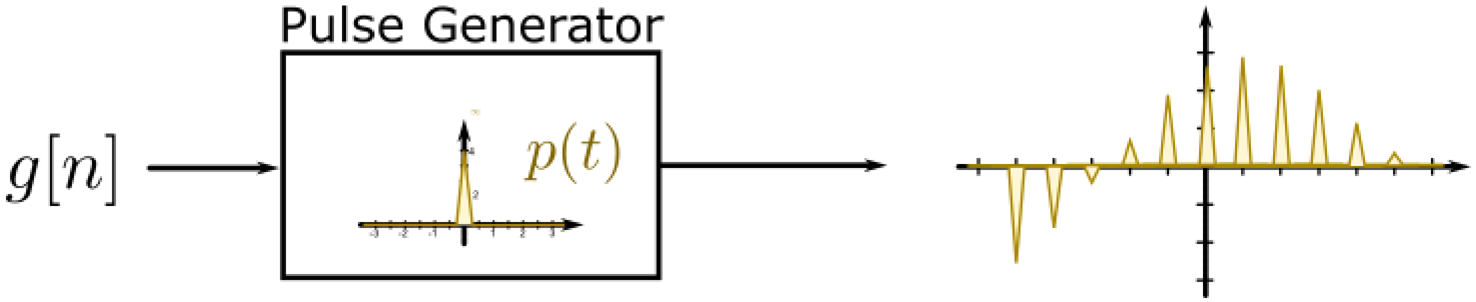
- **Digital-to-Analog Conversion reconstructs a continuous time signal analog from a sampled digital signal**
  - Reminder: ideal reconstruction is analog LP-filtering of an ideal pulse-train (sampled signal)



- **There are two main issues with this ideal setup**
  - An ideal Dirac-Pulse generator does not exist
  - An ideal LP filter to perfectly remove images while leaving the signal spectrum in tact also does not exist (sharp filters are difficult to realize as analog circuits)

Applications: Practical D-to-A Conversion / Reconstruction

- **To solve the pulse-generator problem, consider a non-ideal pulse generator**

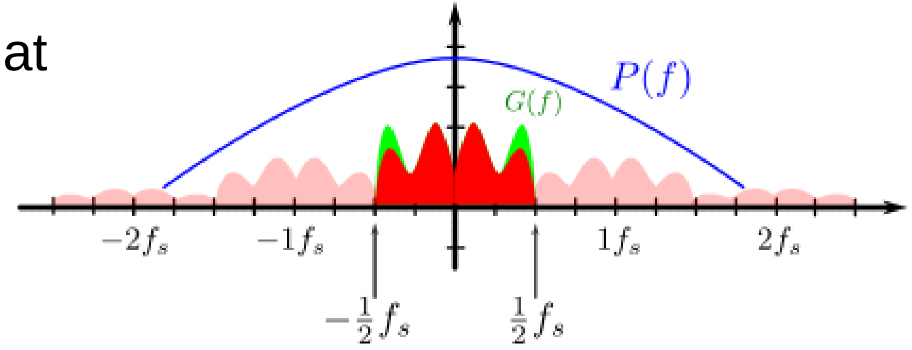


  - The resulting analog signal is now a train of pulses $p(t)$ and even though it is generated directly, the pulse generator can be mathematically described as a filter that receives an ideal pulse train

$$\tilde{g}(t) = p(t) \times \sum_{-\infty}^{+\infty} g(nT_s) \cdot \delta\left(t - nT_s\right)$$

  - The resulting spectrum is a replica of the original spectrum at multiples of f_s, BUT filtered with the spectrum of the pulse

$$\tilde{G}(f) = P(t) \cdot \frac{1}{T_s} \sum_{-\infty}^{+\infty} G\left(f - nf_s\right)$$

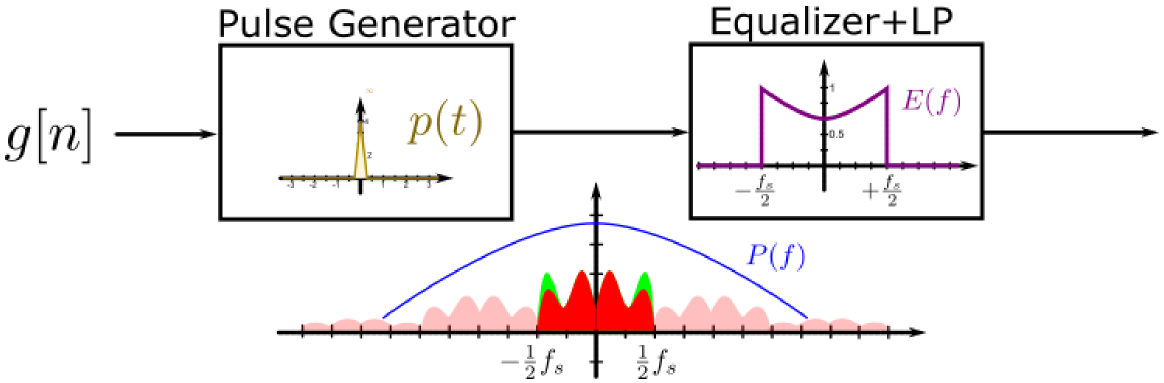


Applications: Practical D-to-A Conversion / Reconstruction

- **To perfectly recover the original signal we need to do two things:**
  1. Remove the images (trivial: using a LP filter)
  2. Correct the distortion of the non-ideal pulse (based on its frequency response) within the band of interest

- **Both objectives can be achieved with a single filter ("equalizer" + LP)**

$$E(f) = \begin{cases} \frac{T_s}{P(f)} \cdot & |f| < B \\ 0 & |f| > B \end{cases}$$

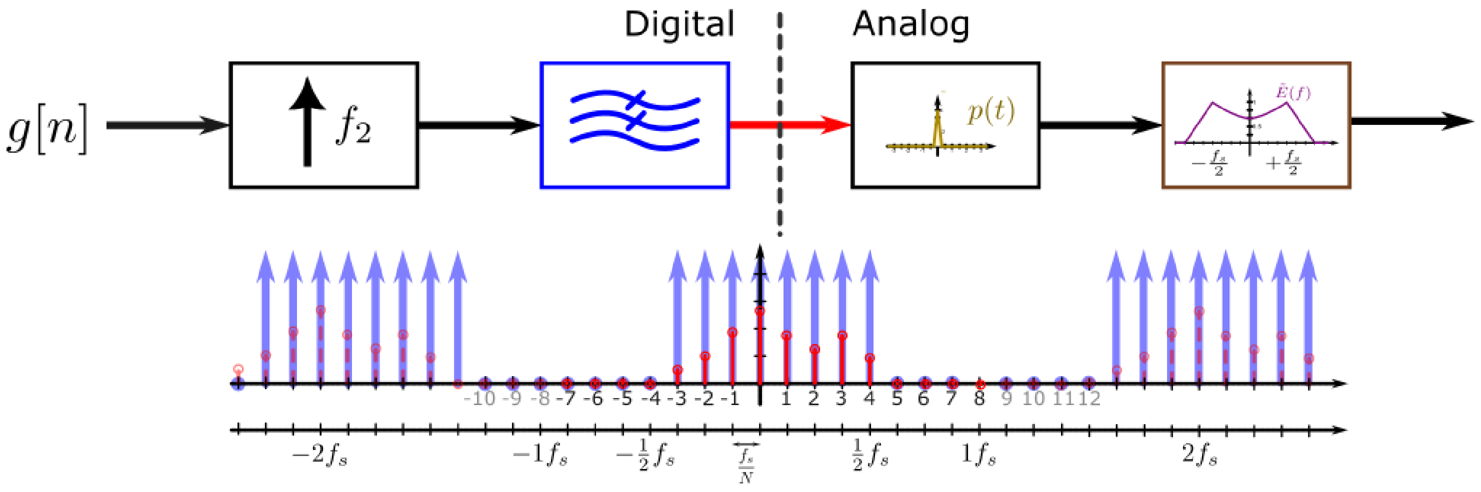


- Note: since even non-ideal pulses are still short, much shorter than T_s, $P(f)$ decays slowly. The shorter the pulses, the smaller the impact

Applications: Practical D-to-A Conversion / Reconstruction

- **To solve the non-ideal LP problem, we realize that implementing an almost ideal LP filter is much easier in the digital than in the analog domain.**

- **Idea: upsample the signal in the digital domain to “expose” the closest images and remove them by a digital LP filter**



  - Significantly relaxes the requirements on the analog filter
  - Note: the digital filter can also contribute to the compensation of the non-ideal pulse (not shown)