EE-432
Systeme de Telecommunication

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Fundamental Signals and Signal Representations

Table of Contents

Table of Contents

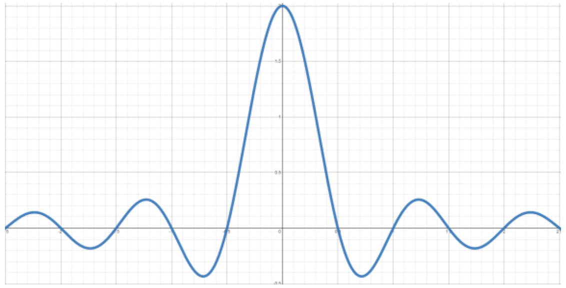
Fundamental Definitions

In digital communications we deal with two main concepts that a physical interpretation, but also translate well into precise mathematical counterparts

- **Signals: a sequence of values that evolves as a function of one or independent parameters (often time)**
  - 1-dimensional (1D) signals are a function of one variable (often, but not always) time
  - Signals can also be n-dimensional (e.g., images)
  - We represent physical quantities with signals, BUT many signals have no physical counter part
  - Different ways to represent a signal:

$$y(t) = \frac{\sin 2 * \pi * t}{\pi * t}$$

Equation



Graph

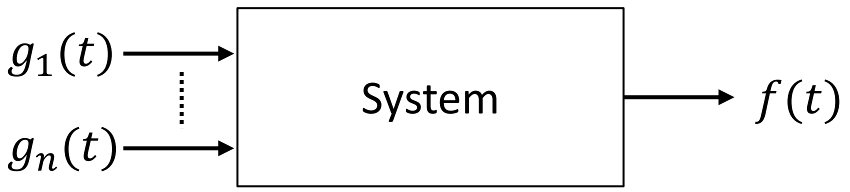
Time	Analog Signal Value (V)
0	4.7
1	12.3
2	-6.8
3	-28.3
4	20.3

List of numerical values (sampled signals)

Fundamental Definitions

In digital communications we deal with two main concepts that a physical interpretation, but also translate well into precise mathematical counterparts

- **Systems: receive and operate on one or multiple input signals to generate one (or multiple) new signals according to well-defined rules**



$$f(t) = \frac{d}{dt} g(t)$$

$$f(t) = g_1(t) \times g_2(t)$$

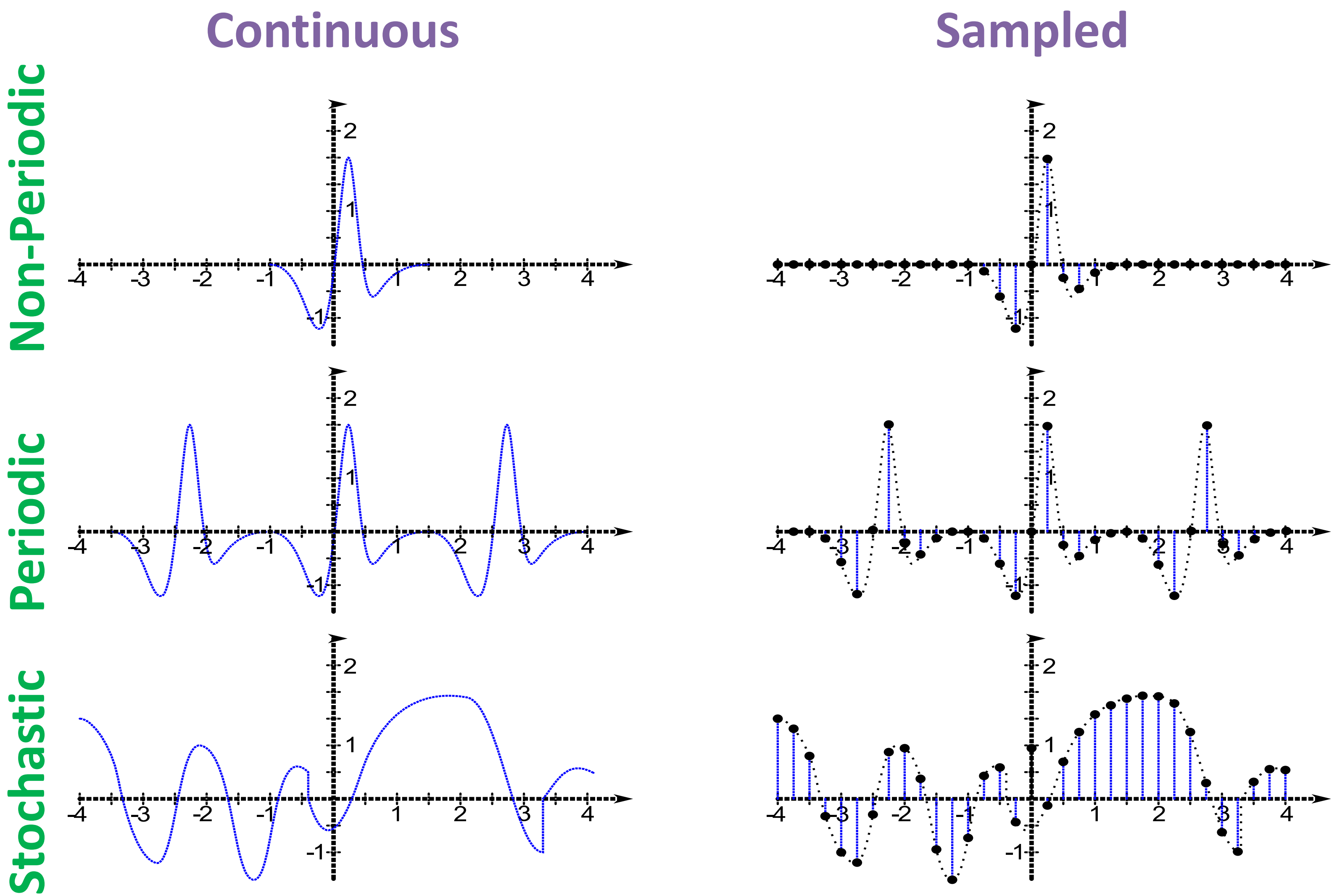
...

  - Realized either from physical components (carefully constructed (e.g., RF circuit) or naturally given (e.g., the wireless channel)) or as abstract calculations (math)
  - Challenges:
    - For physical systems: find a proper mathematical description to include it in an abstract model
    - For systems to be constructed: given a set of input signals, engineer the “system” to achieve an output with desired properties

Signal Classification

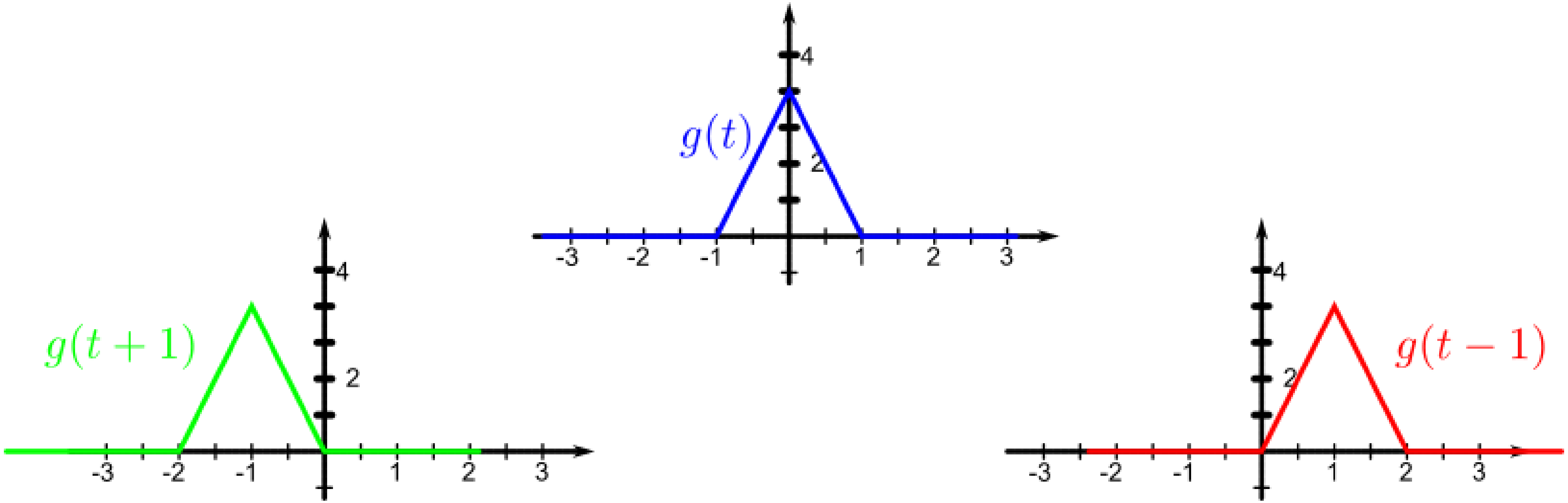
- **Representation**
  - Continuous
  - Sampled

- **Behaviour over time**
  - Non-periodic
  - Periodic
  - Random (stochastic)



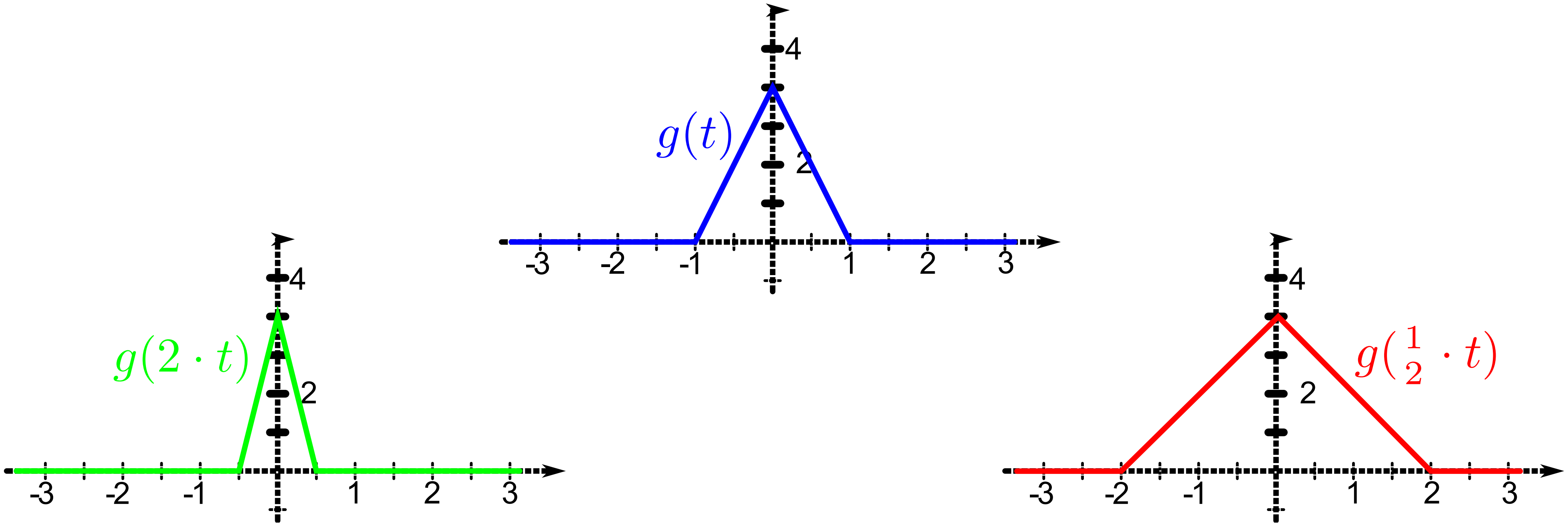
Useful Basic Operations on Signals: Shifting

- **Consider a signal $g(t)$ and a time (frequency) offset/shift $\Delta T > 0$**
  - Delaying the signal: to shift the signal to the “right” replace $t \leftarrow t - \Delta T$
  - Advancing the signal: to shift the signal to the “left”, replace $t \leftarrow t + \Delta T$



Useful Basic Operations on Signals: Stretching

- **Consider a signal $g(t)$ and a time/frequency stretching factor α**
  - Stretching the signal: to stretch the signal replace $t \leftarrow \alpha t \text{ with } \boldsymbol{\alpha} < \mathbf{1}$
  - Compress the signal: to compress a signal, replace $t \leftarrow \alpha t \text{ with } \boldsymbol{\alpha} > \mathbf{1}$



Instantaneous Signal Power

- **We define the instantaneous "power" of a signal as**

$$P_g(t) = |g(t)|^2$$

  - As signals often represent levels of physical quantities such as "voltages", "currents", "field strengths", ... , $P_g(t)$ is indicative for "physical" instantaneous power
  - CAVEAT: as constants are often omitted, for simplicity and generality, when defining signals, actual "physical" power in SI units requires careful consideration of appropriate constants

- **Instantaneous signal power varies over time**

Signal Energy

- **We define the “energy” of a signal as**

$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 \, dt$$

  - As signals often represent levels of physical quantities such as “voltages”, “currents”, “field strengths”, ... , E_g is indicative for “physical” energy
  - CAVEAT: as constants are often omitted, for simplicity and generality, when defining signals, actual “physical” energy in SI units requires careful consideration of appropriate constants

- **Signal energy is only meaningful if $E_g < \infty$. This is the case for**

  - Signals **with finite duration**: $|g(t)|^2 = 0$ for $t < T_-$ and $|g(t)|^2 = 0$ for $t > T_+$

  - Signals with **rapidly decaying tails**: $\lim_{T \to \infty} \left[\int_{-\infty}^{-T} |g(t)|^2 \, dt + \int_{T}^{+\infty} |g(t)|^2 \, dt \right] = 0$

Average Signal Power

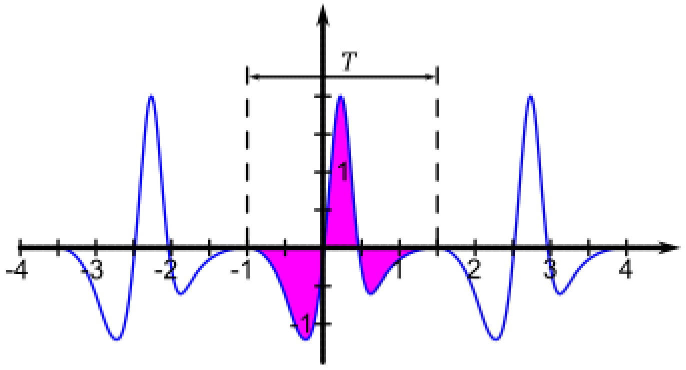
- **For signals with "infinite" energy, considering "power" (mean square value) makes more sense**
  - Normalizing by signal duration, while still considering the entire (potentially infinitely long) signal

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |g(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} P_g(t) \, dt$$

  - Sometimes the "root mean square" (RMS) value is also considered as $\bar{g}_{RMS} = \sqrt{P_g}$

- **Periodic signals g' (inf. Energy) with period T**
  - Integration over any full period with arbitrary offset Δ

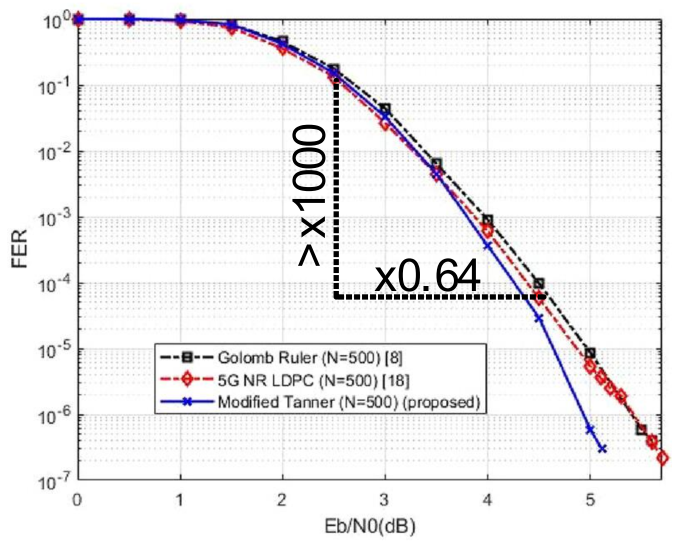
$$P_g = \frac{1}{T} \int_{0+\Delta}^{+T+\Delta} |g'(t)|^2 \, dt$$



Power/Energy Units in Communications

- **Specific properties when analysing communication signals/systems**
  - Communication signal power varies over an extremely wide range
  - Amplification and attenuation are multiplicative effects
  - Many metrics of interest (e.g., error rates) are exponential functions of the signal power and vary also over a wide range
- **Example**

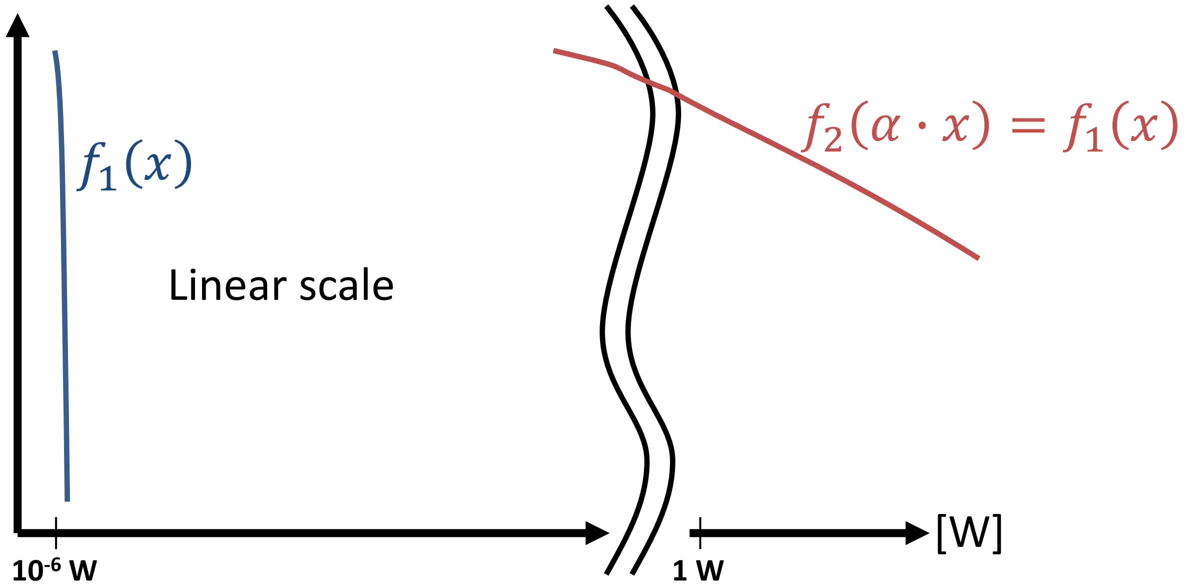
Received Signal Strength vs Distance



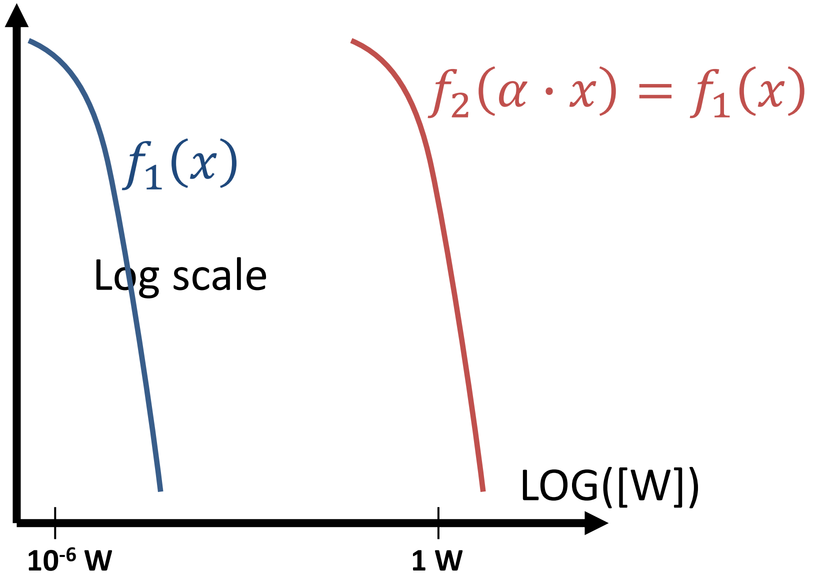
Error Rate vs received Signal Strength

Power/Energy Units in Communications

- **Expressing power on a linear scale is inconvenient and leads to highly compressed, unreadable graphs**



Curve shapes that are invariant to multiplications (scaling) of their input power appear stretched



Log-scale: stretching preserves curve shape

Power/Energy Units in Communications

- **We measure power always in decibels [dB]**

$$P[dB] = 10 \cdot \log_{10} P[lin]$$

  - dB is a relative metric and when signals are often unit-free, we follow the above expression

- **If we are interested in the actual "physical" power in Watts we relate the power in dB to a known reference**

  - Power in dBW is $10 \cdot \log_{10} P[W]$ (less commonly used)
  - **Power in dBm** is $10 \cdot \log_{10} P[mW] = 30 + 10 \cdot \log_{10} P[W]$

P[dBm]	P[dBW]	P[mW]
-20 dBm	-50 dBW	0.01 mW
-10 dBm	-40 dBW	0.1 mW
0 dBm	-30 dBW	1 mW
10 dBm	-20 dBW	10 mW
20 dBm	-10 dBW	100 mW
30 dBm	0 dBW	1 W

Orthogonal Signals

- **When considering two signals, it is interesting to see if these can be "separated" perfectly even when they are superimposed (e.g., in time)**

- **DEFINITION: two signals $g(t)$ and $x(t)$ are "orthogonal" if**

$$\langle g(t), x(t)\rangle = \int_{-\infty}^{+\infty} g(t) \cdot x^*(t)\, dt = 0$$

Why is this useful?

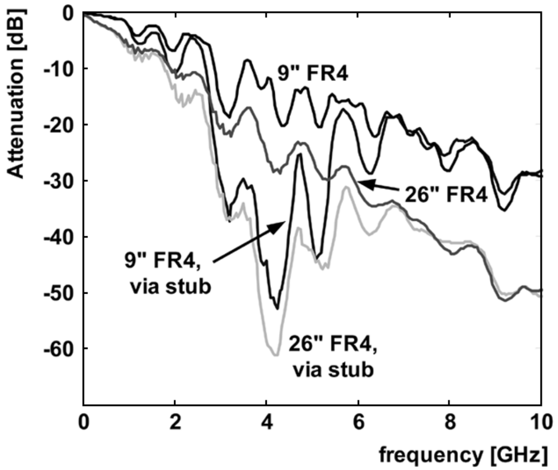
- Consider $y(t) = \alpha g(t) + \beta x(t)$ with $\langle g(t), x(t)\rangle = 0$, then

$$\int_{-\infty}^{+\infty} \big(\alpha g(t) + \beta x(t)\big) \cdot x^*(t)\, dt = \alpha \underbrace{\int_{-\infty}^{+\infty} g(t) x^*(t)\, dt}_{0} + \beta \int_{-\infty}^{+\infty} |x(t)|^2\, dt \sim \beta$$

(Complex) Sinusoids as Orthogonal Signals

- **Real and complex sinusoids are a particularly useful class of signals as they**
  - Are pure oscillations that each contain only a single frequency
  - Relate directly to specific frequencies and to the behaviour of a system or channel at specific frequencies:

Computer Backplane with connections to multiple PCI cards



Attenuation of different frequencies on the connection from the CPU to the different PCI card connectors

(Complex) Sinusoids as Orthogonal Function Basis

- **For real-valued signals, we can consider** $\sin 2\pi t$ **and** $\cos 2\pi t$
  - The functions $\sin 2\pi f t$ and $\cos 2\pi f t$ themselves are orthogonal to each other

- **We use "complex sinusoids" to express deal with both complex-valued and real-valued signals**

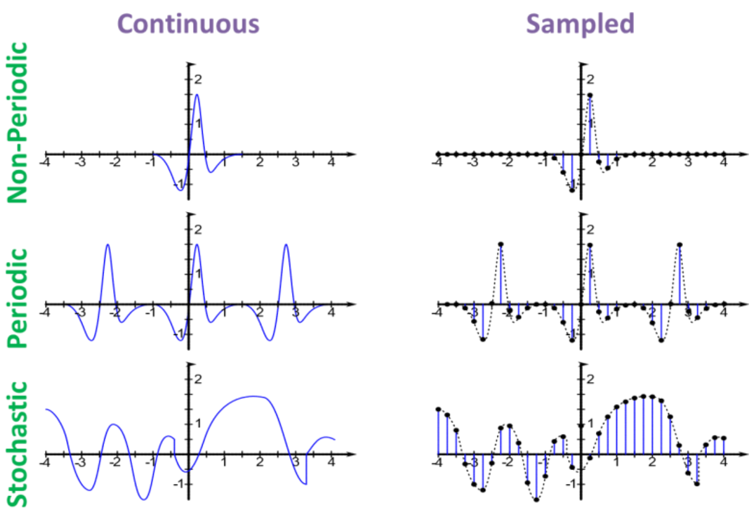
$$\varphi_f(t) = e^{j \cdot 2 \cdot \pi \cdot f \cdot t}$$

  - Each pair of complex sinusoids with $f_1 \neq f_2$ is orthogonal: $\langle \varphi_{f_1}(t), \varphi_{f_2}(t) \rangle = 0$
  - The set of all complex sinusoids forms an orthogonal function basis

- **Any complex finite-energy function can be decomposed into a sum of complex sinusoids**

Recap from Week-1

- **Important signal classes and representations**



- **We measure signal power always in dB**
- **Orthogonality of signals: two signals $g(t)$ and $x(t)$ are "orthogonal" if**

$$\langle g(t), x(t) \rangle = \int_{-\infty}^{+\infty} g(t) \cdot x^*(t)\, dt = 0$$

  - Orthogonality helps us to separate out signal components from a sum of signals
  - Famous orthogonal signals sin() and cos() -> orthogonal components of complex numbers
  - Real/complex sinusoids with different frequrncies

Fourier Transformation

- **The Fourier Transform (FT) extracts a continuous frequency spectrum from a finite energy (therefore also non-periodic) signal:**

$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{-j\cdot 2\cdot \pi \cdot f \cdot t}\, dt$$

  - $|G(f)|$ and $\frac{1}{\mathrm{j}} \log G(f) = \arg G(f)$ are magnitude and phase of frequency component f in $g(t)$

- **The Inverse Fourier Transform (IFT) reconstructs the time domain function from the frequency spectrum:**

$$g(t) = \int_{-\infty}^{+\infty} G(f) e^{j\cdot 2\cdot \pi \cdot f \cdot t}\, df$$

- **Negative frequencies: required to form real-valued signals (see later)**

FT and IFT Duality

- **FT and IFT are almost completely symmetric** (only difference in the sign of the exponent of the complex sinusoid in the integral)

- **This leads to a beautiful duality:**

$$\mathcal{F}\{g(t)\} = G(f) \Longrightarrow \mathcal{F}\{G(t)\} = g(-f)$$

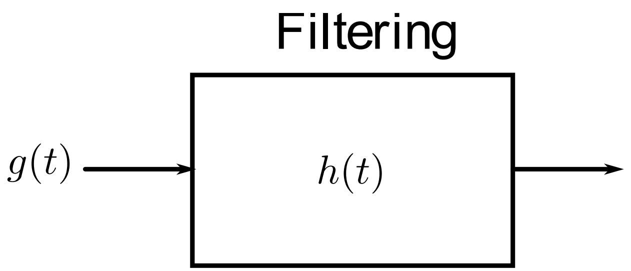
$$\mathcal{F}^{-1}\{G(f)\} = g(t) \Longrightarrow \mathcal{F}^{-1}\{g(f)\} = G(-t)$$

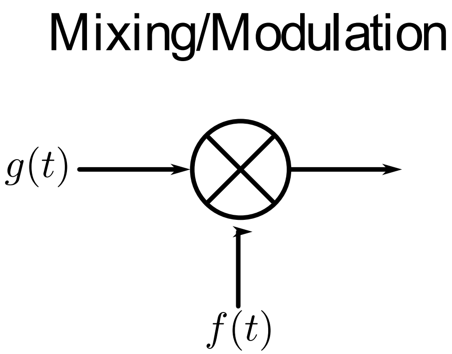
  - The FT/IFT of one function also reveals easily the FT/IFT of another function
  - Time domain (TD) and frequency domain (FD) are fundamentally related

- All **linear operations** on a signal **can be carried out in TD and FD**

Convolution/Mixing in TD and FD (FT/IFT Property)

- **Two TD operations on signals that are frequently used:**





- **Convolution in time domain:**

$$g_1(t) \times g_2(t) = \int_{-\infty}^{+\infty} g_1(u) \times g_2(t-u)dt$$

$$\mathcal{F}\{g_1(t) \times g_2(t)\} = G_1(t) \cdot G(t)$$

$$g_1(t) \times g_2(t) = \mathcal{F}^{-1}\{G_1(t) \cdot G(t)\}$$

- **Multiplication in time domain:**

$$g_1(t) \cdot g_2(t)$$

$$\mathcal{F}\{g_1(t) \cdot g_2(t)\} = G_1(t) \times G(t)$$

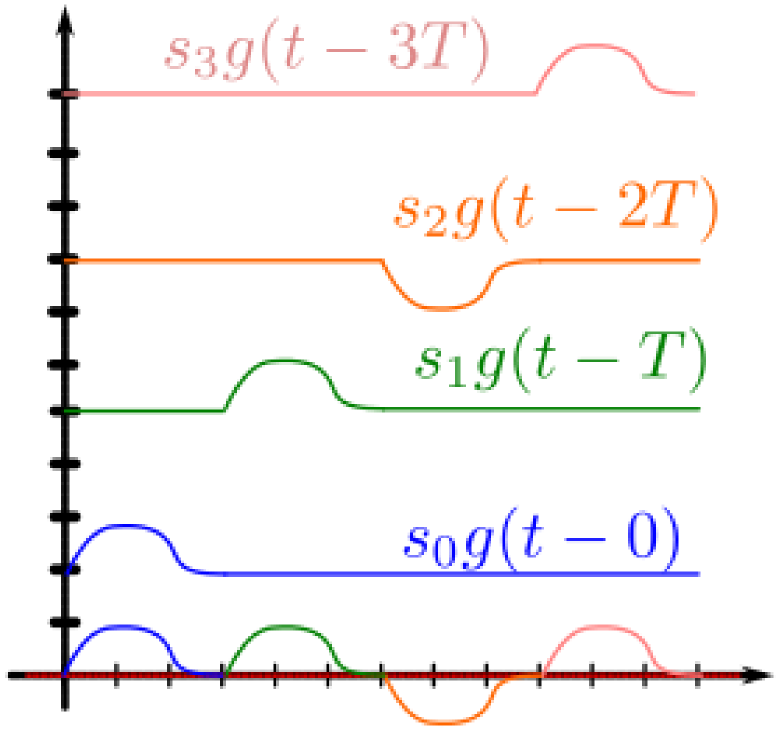
$$g_1(t) \cdot g_2(t) = \mathcal{F}^{-1}\{G_1(t) \times G(t)\}$$

FT/IFT of Time/Frequency Shifted Signals

- **We often encounter signals that are either delayed or shifted in frequency**

- **Example from digital communications:** construction of a pulse-train (sending multiple symbols) from a single (prototype) pulse

$$\boldsymbol{y}(\boldsymbol{t}) = \sum_{\boldsymbol{n}=-\infty}^{\boldsymbol{n}=+\infty} \boldsymbol{s}_{\boldsymbol{n}} \boldsymbol{g}(\boldsymbol{t} - \boldsymbol{n} \cdot \boldsymbol{T})$$



$$y(t) = s_0 g(t-0) + s_1 g(t-T) + \\ + s_2 g(t-2T) + s_3 g(t-3)$$

Given $G(F)$, how can we compute $Y(f)$?

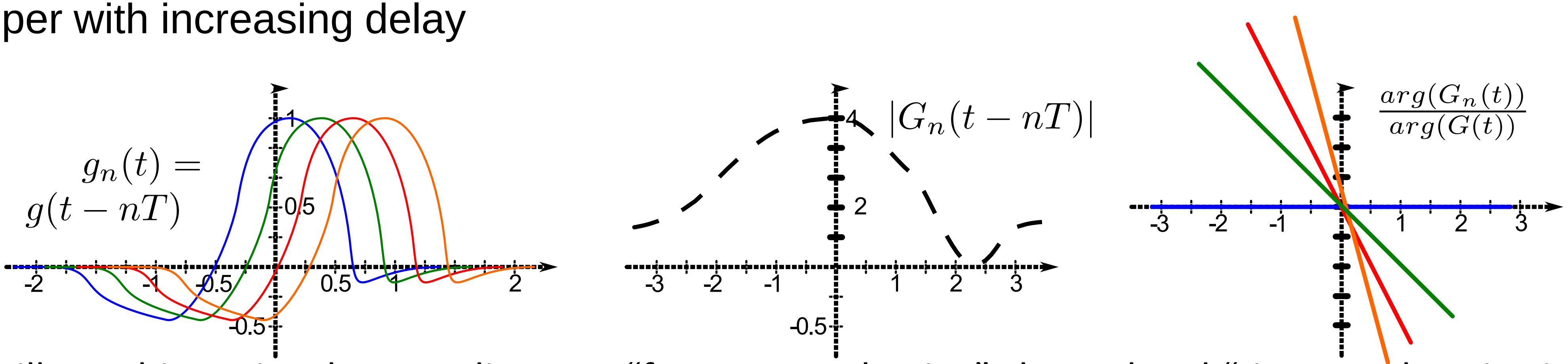
FT/IFT of Time/Frequency Shifted Signals

- **Consider the FT of $\boldsymbol{g(t-T)}$**

$$\mathcal{F}\{g(t-T)\} = \int_{-\infty}^{+\infty} g(t-T)e^{-j\cdot 2\cdot\pi\cdot f\cdot t}\,dt = \int_{-\infty}^{+\infty} g(u)e^{-j\cdot 2\cdot\pi\cdot f\cdot (u+T)}\,du$$

$$= \int_{-\infty}^{+\infty} g(u)e^{-j\cdot 2\cdot\pi\cdot f\cdot u}\cdot e^{-j\cdot 2\cdot\pi\cdot f\cdot T}\,du = e^{-j\cdot 2\cdot\pi\cdot f\cdot T}\cdot G(f)$$

  - Delaying a signal leaves the magnitude of its FT unaltered
  - The argument (phase) of its FT experiences a frequency dependent linear shift that becomes steeper with increasing delay



*We will see this again when we discuss a “frequency selective” channel and “time synchronization”

FT/IFT of Time/Frequency Shifted Signals

- **Consider the IFT of G$(\boldsymbol{t}-\boldsymbol{F})$**

$$\mathcal{F}^{-1}\{G(t-F)\} = \int_{-\infty}^{+\infty} G(t-F)e^{j\cdot 2\cdot\pi\cdot f\cdot t}\,df = \int_{-\infty}^{+\infty} G(u)e^{j\cdot 2\cdot\pi\cdot t\cdot(u+F)}\,du$$

$$= \int_{-\infty}^{+\infty} G(u)e^{j\cdot 2\cdot\pi\cdot \boldsymbol{t}\cdot u}\cdot e^{-j\cdot 2\cdot\pi\cdot f\cdot \boldsymbol{F}}\,du = e^{j\cdot 2\cdot\pi\cdot \boldsymbol{t}\cdot \boldsymbol{F}}\cdot g(t)$$

  - Moving a signal in frequency leaves the magnitude of its IFT unaltered
  - The argument (phase) of its IFT experiences a frequency dependent linear shift that becomes steeper with increasing frequency shift

- **Summary**

$$\mathcal{F}^{-1}\{G(t-F)\} = e^{j\cdot 2\cdot\pi\cdot \boldsymbol{t}\cdot \boldsymbol{F}}\cdot g(t) \qquad \mathcal{F}\left\{e^{j\cdot 2\cdot\pi\cdot \boldsymbol{t}\cdot \boldsymbol{F}}\cdot g(t)\right\} = G(t-F)$$

$$\mathcal{F}\{g(t-T)\} = e^{-j\cdot 2\cdot\pi\cdot f\cdot T}\cdot G(f) \qquad \mathcal{F}^{-1}\left\{e^{-j\cdot 2\cdot\pi\cdot f\cdot T}\cdot G(f)\right\} = g(t-T)$$

*We will see this again when we discuss a “passband modulation” and “multi-carrier modulation”

Fourier Transformation Shift/Duality Example

Example-1a:

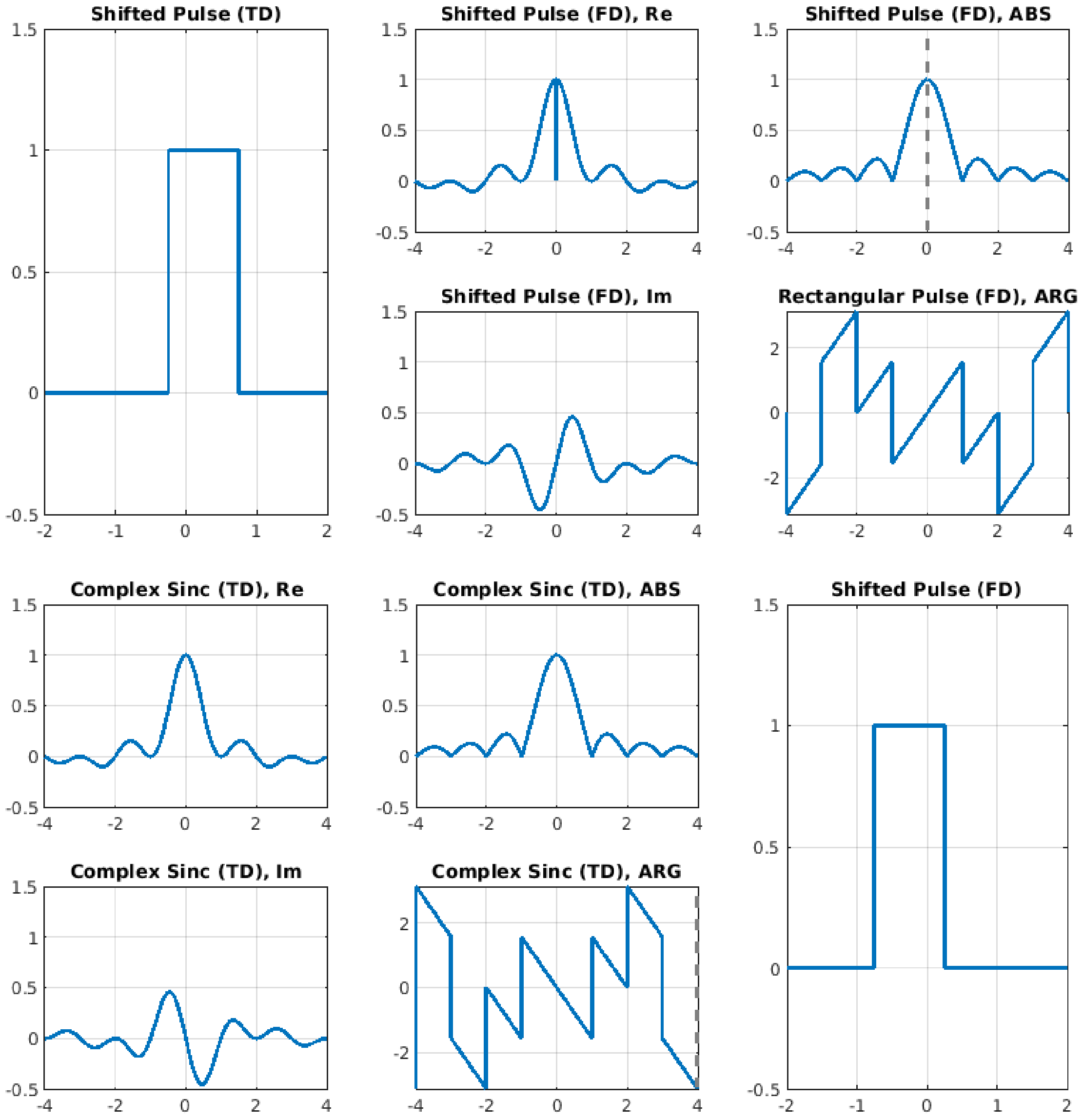
- $g(t) = \begin{cases} 1 & -0.25 < t < 0.75 \\ 0 & else \end{cases}$

- $G(f) = e^{j\cdot\frac{1}{4}\cdot\pi\cdot f} \cdot \frac{\sin \pi\cdot f}{\pi\cdot f}$

Example-1b:

- $g(t) = e^{j\cdot\frac{1}{4}\cdot\pi\cdot t} \cdot \frac{\sin \pi\cdot t}{\pi\cdot t}$

- $G(f) = \begin{cases} 1 & -0.75 < f < 0.25 \\ 0 & else \end{cases}$



Computing and Plotting FT/IFT with MATLAB

- **Manual computation of FT/IFT pairs through integrals is often tedious**
- **Two main approaches to simplify our life:**
  - Derivation by using known FT/IFT pairs and fundamental properties (see next slides)
  - Symbolic math tools (e.g., MATLAB)

Using MATLAB symbolic math toolbox

1. **Set up scaling parameters for FT/IFT (MATLAB defaults to a scaled version) using `sympref('FourierParameters',[1,2*pi]);`**
2. **Define symbols for time and frequency (e.g., `t`, and `v`) using `syms`**
3. **Define a function**
4. **Obtain FT/IFT with `fourier()` and `ifourier()`**
5. **Simplify the resulting expression using `simplify()`**

Computing and Plotting FT/IFT with MATLAB

Example

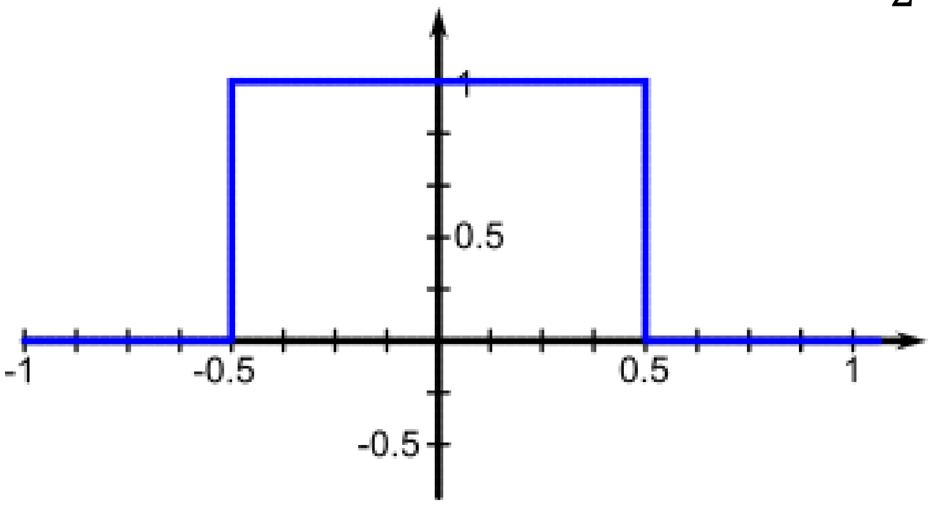
```
sympref('FourierParameters',[1,2*pi]);     % Define the FT/IFT scaling
syms t v;                                  % Declare symbols for time (t)
                                             and frequency (v)
f=rectangularPulse(-0.5,0.5,t);            % Define TD function f(t)

F=simplify(fourier(f,t,v));                % Compute FT and simplify

figure(1); clf;                            % Plot preparation
fplot(f,'LineWidth',2);                    % plot f(t)
axis([-2,+2,-0.5,+1.5]); grid;

figure(2); clf;                            % Plot preparation
fplot(abs(F),'LineWidth',2);                     % plot |F(t)|
axis([-4,+4,-0.5,+1.5]); grid;
```

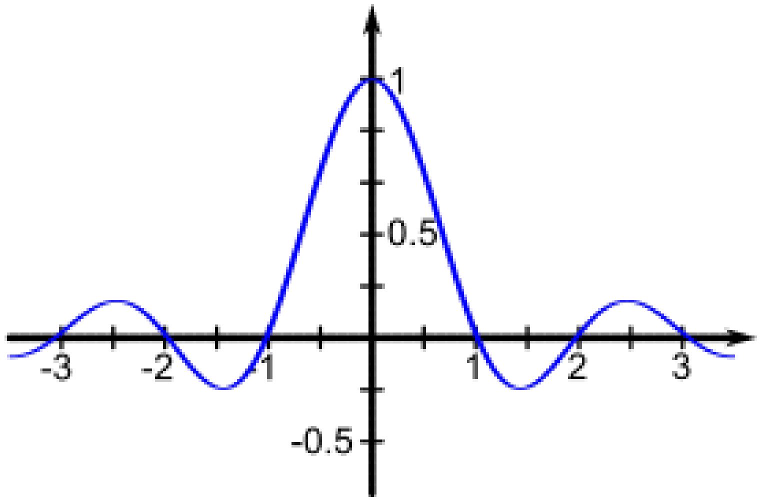
Useful Functions and Their Fourier Transforms

- **The brick-wall (unit rectangle) function**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & else \end{cases}$$

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{+\infty} \Pi(t) e^{-j\cdot 2\cdot\pi\cdot t\cdot f} dt = \int_{\frac{1}{2}}^{\frac{1}{2}} e^{-j\cdot 2\cdot\pi\cdot t\cdot f} dt = \frac{\sin(\pi \cdot f)}{\pi \cdot f} = \text{sinc}(\pi \cdot f)$$

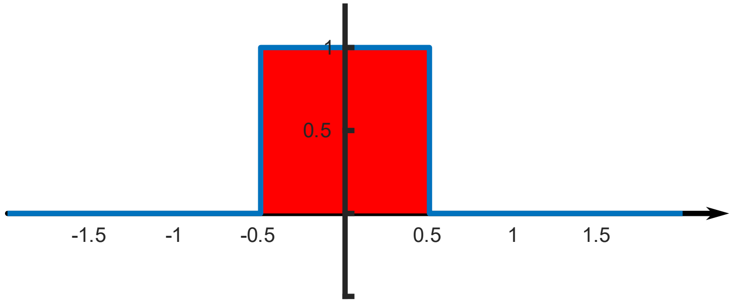


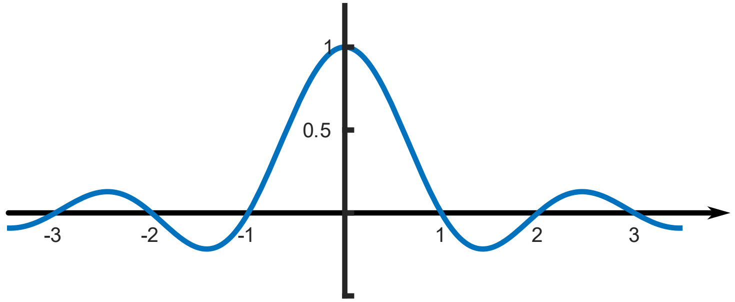


Energy in Time- and Frequency-Domain (Parseval)

- **We are often interested in the total energy of a signal.**
- **Recap: in time domain we have**

$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 \, dt$$

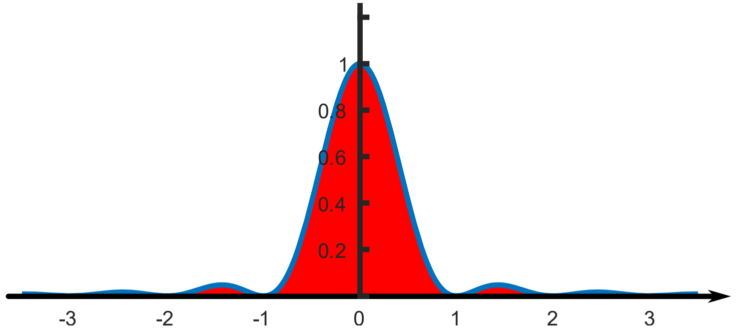




- **Parseval's theorem states that:**
  - The fourier transform preserves the energy of the signal
  - The Integral of the square of a function is equal to the integral of the square of its transform

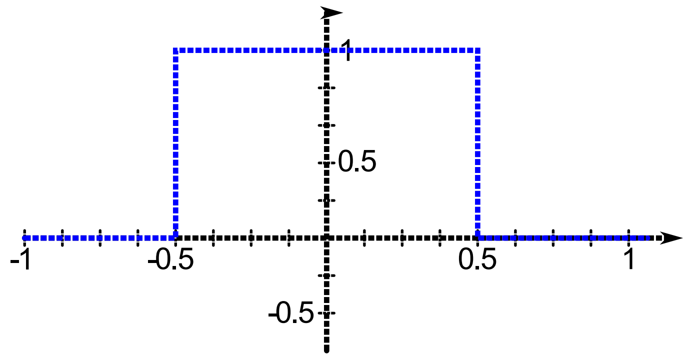
$$E_g = \int_{-\infty}^{+\infty} |G(f)|^2 \, df$$

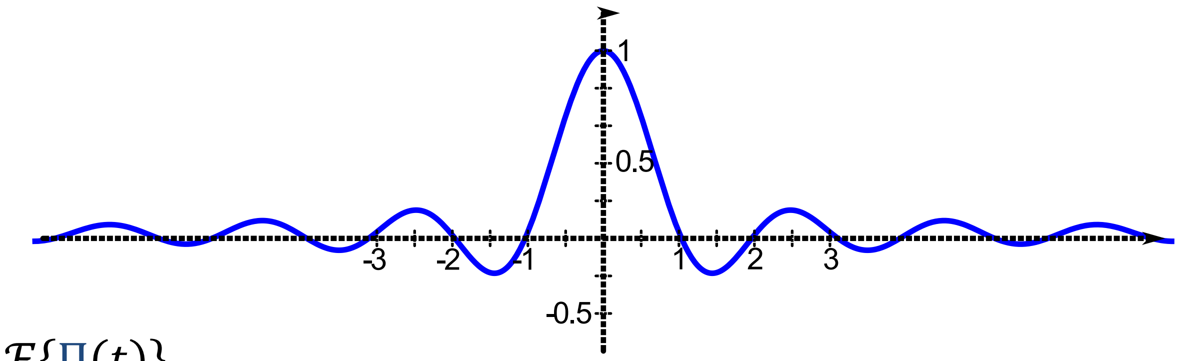
with $|G(f)|^2$: Energy Spectrum



Finite/Infinite Duration vs Infinite/Finite Spectrum

- **From the FT of the Brick-Wall function, we observe that a time limited function can occupy an infinitely wide spectrum**





Time-limited pulse $\xrightarrow{\mathcal{F}\{\Pi(t)\}}$ Infinitely wide spectrum

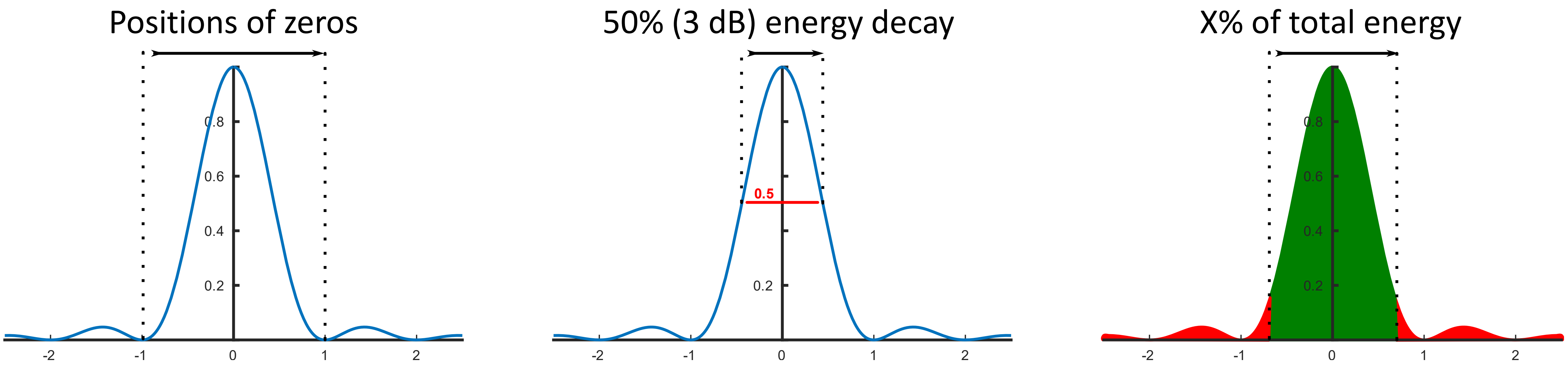
Frequency limited spectrum $\xrightarrow{\mathcal{F}^{-1}\{\Pi(f)\}}$ Infinitely wide pulse

Even more general:

- **Every finite width pulse has a spectrum with unbounded frequencies.**
- **Every finite spectrum results in an infinitely long pulse, BUT with (rapidly) decaying tails** (to preserve finite enery)

Defining Signal Bandwidth/Duration

- **Even for a signal with infinite width (TD or FD), we are interested in defining a relevant bandwidth**
- **Many different ways to define bandwidth(FD) / duration (TD):**



- Depending on the application/situation/requirements, choose the best option
- For the general behaviour/trend (see later), most options behave similarly

Example: Bandwidth of a Brick-Wall

- **Consider the brick-wall in the time domain:**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & else \end{cases}$$

- with Fourier Energy Spectrum

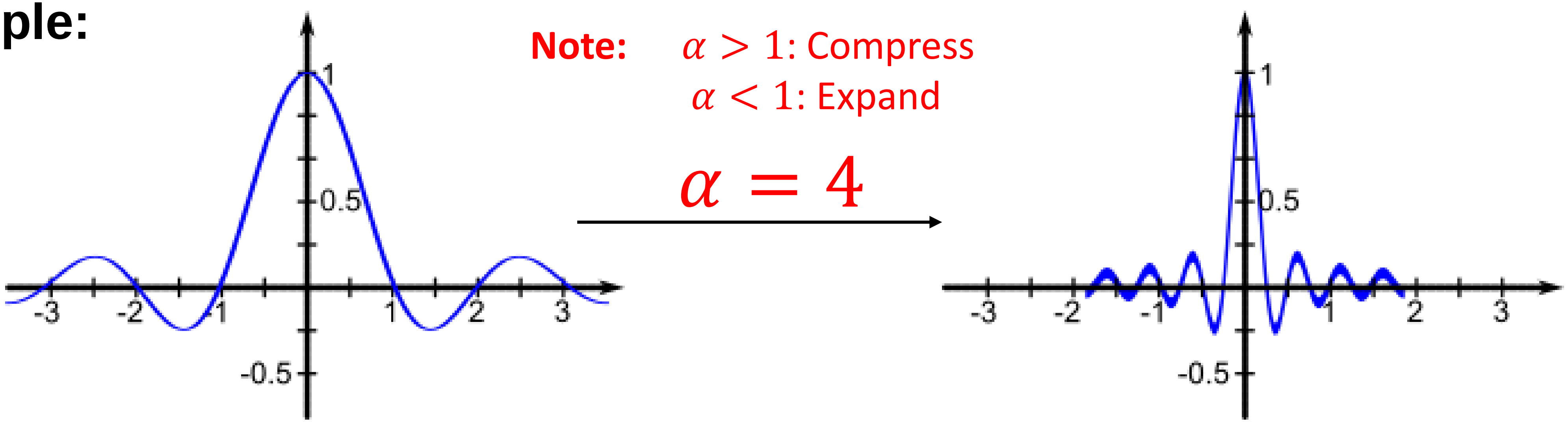
$$|\mathcal{F}\{g(t)\}|^2 = |\mathrm{sinc}(\pi \cdot f)|^2$$

- **Zero-energy bandwidth:**

$$|\mathrm{sinc}(\pi \cdot f_{BW})|^2 = \left|\frac{\sin \pi f_{BW}}{\pi f_{BW}}\right|^2 == 0 \Rightarrow \qquad \pi f_{BW} = \pi \qquad \Rightarrow \qquad f_{BW} = 1$$

Fourier Transform Time Scaling

- **Often, we are interested of the FT of a time-scaled (frequency-scaled) signal**
- **Example:**



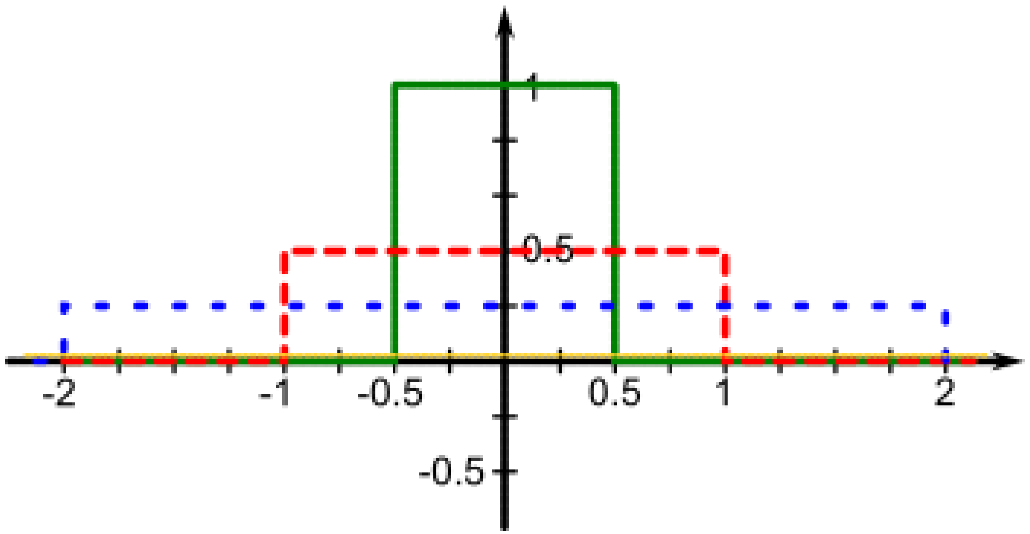
- **Compute the FT/IFT of a time-scaled function with scaling factor $\boldsymbol{\alpha}$**

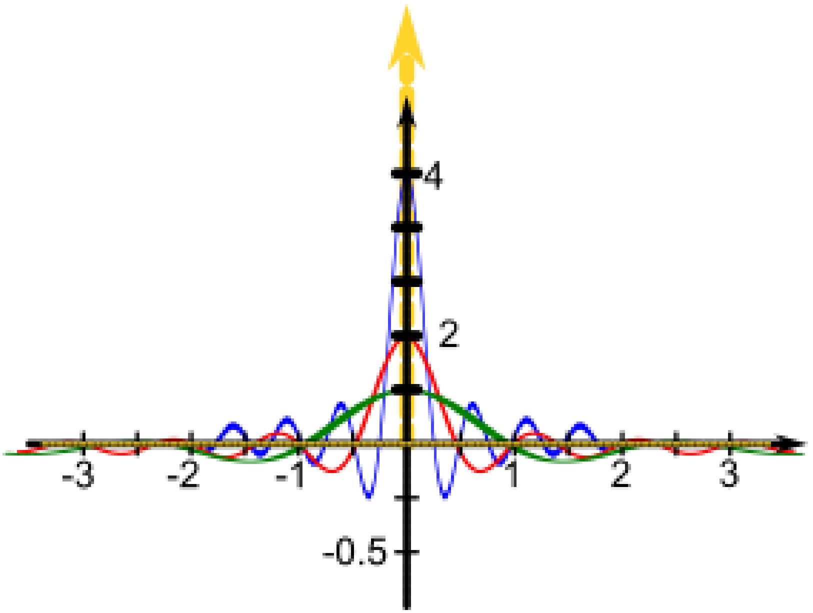
$$\mathcal{F}\{g(\alpha t)\} = \int_{-\infty}^{+\infty} g(\alpha t) e^{-j\cdot 2\cdot\pi\cdot f\cdot t}\, dt = \int_{-\infty}^{+\infty} g(u) e^{-j\cdot 2\cdot \pi\cdot \frac{f\cdot \boldsymbol{u}}{\boldsymbol{\alpha}}} \frac{du}{\alpha} = \frac{1}{|\alpha|} G\left(\frac{f}{\alpha}\right)$$

$$\mathcal{F}^{-\mathbf{1}}\{G(\beta f)\} = \frac{1}{|\beta|} g\left(\frac{t}{\beta}\right)$$

Time Scaling Example -> Dirac Delta

- **Consider a brick-wall spectrum that expands, but preserves the area under the spectrum and its FT:**





$$\mathcal{F}\left\{\frac{1}{B}\Pi\left(\frac{f}{B}\right)\right\} = \operatorname{sinc}\pi Bt = \frac{\sin \pi Bt}{\pi Bt}$$

Time-domain of a widening spectrum converges to an infinitely short pulse

Observation: Bandwidth & Time Duration

- **Consider now the duration T of a sinc-pulse created by brick-wall spectrum with increasing bandwidth**
  - Choose the zero-crossing bandwidth as easy reference

$$\mathcal{F}\left\{\frac{1}{B}\Pi\left(\frac{f}{B}\right)\right\} = \operatorname{sinc}\pi B t$$

$$|\operatorname{sinc}(\pi \cdot B \cdot T)|^2 = \left|\frac{\sin \pi\cdot B\cdot T}{\pi\cdot B\cdot T}\right|^2 == 0 \qquad \Rightarrow \qquad T = \frac{1}{B}$$

Signal Bandwidth and Pulse Duration are inversely proportional

$$T = C \cdot \frac{1}{B}$$

(holds in general with different constants C)

Triangle Function

- **The "triangle function" is defined in the TD as**

$$\boldsymbol{\Delta(t) = \begin{cases} 1 - 2|x| & |x| < 1 \\ 0 & else \end{cases}}$$

- **To find the triangle FT, we observe that it can be written as the convolution of two scaled Brick-Wall functions**

$$\boldsymbol{\Delta(t) = 2 \cdot \Pi(2t) \times \Pi(2t)}$$

  - Scaling to preserve the width with a boundary of ½ and to preserve the hight after scaling

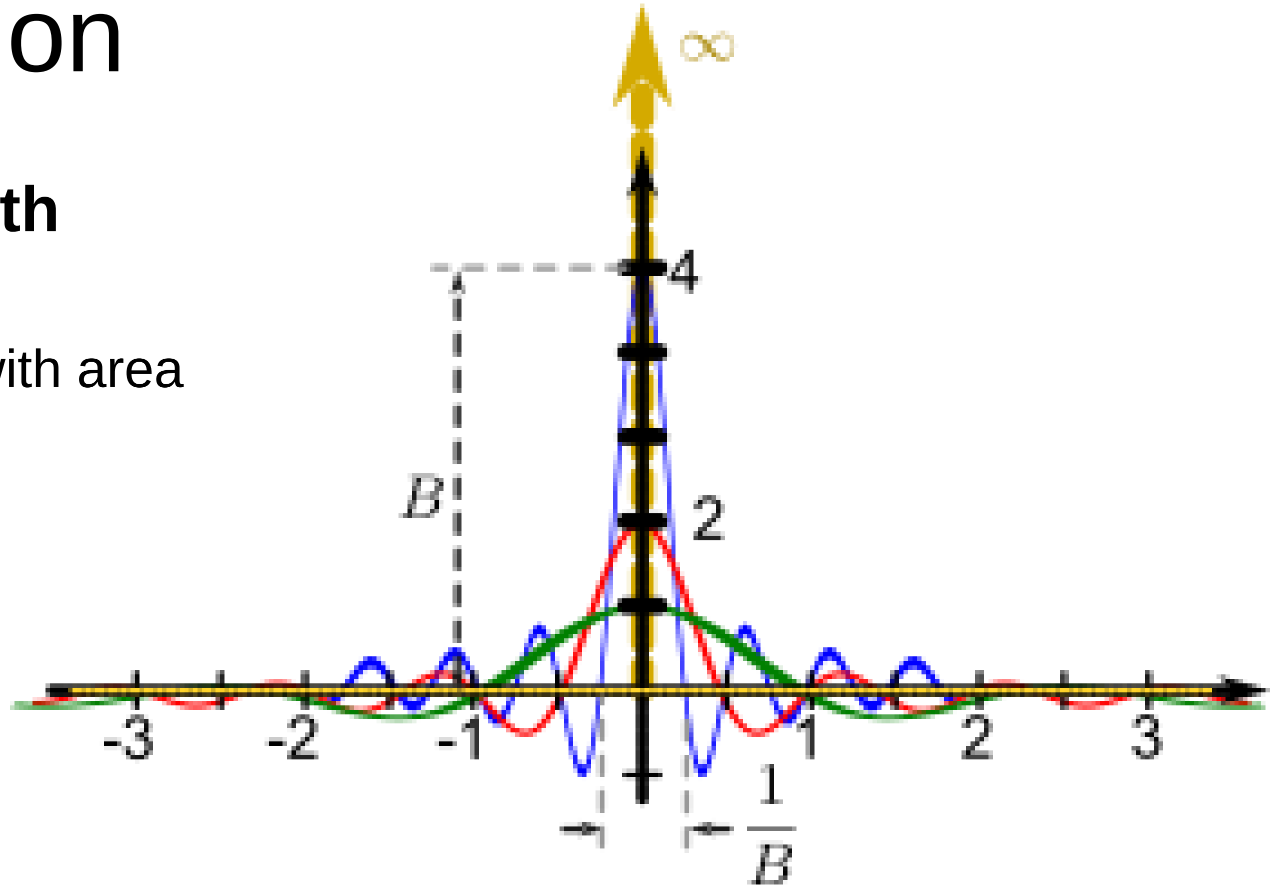
$$\boldsymbol{\mathcal{F}\{\Delta(t)\} = 2 \cdot \mathcal{F}\left\{\Pi(2t) \times \mathcal{F}\{\Pi(2t)\}\right\} =}$$

$$\boldsymbol{\mathcal{F}\{\Pi(2t)\}^2 = 2 \cdot \left(\frac{1}{2}\mathrm{sinc}\frac{\pi}{2}f\right)^2 = \frac{1}{2}\mathrm{sinc}^2\frac{\pi}{2}f}$$

- **Zero-Crossing Bandwidth:** $\frac{\pi}{2} f_{BW} = \pi \rightarrow f_{BW} = 2$

The Dirac Delta Function

- **Consider a sinc-function (TD) with increasing bandwidth**
  - Converges to an infinitely short pulse with area below the pulse remaining 1

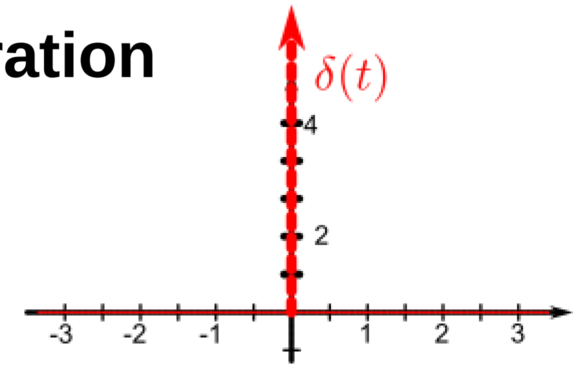


Define the Dirac delta function as

$$\boldsymbol{\delta(t)} = \lim_{B\to\infty} \mathcal{F}\left\{\frac{1}{B}\Pi\left(\frac{f}{B}\right)\right\} = \lim_{B\to\infty} \operatorname{sinc} \pi B t$$

Dirac Delta Function Properties

- **Unit area under the function, despite its infinitely short duration**



$$\int_{-\infty}^{-\infty} \delta(t)\, dt = 1$$

- **Convolution $\times$ of a function $\boldsymbol{f(t)}$ with a shifted Dirac delta $\delta(t)$, evaluated at $\boldsymbol{\tau}$ extracts the function value at time $\boldsymbol{\tau}$ (as a constant)**

$$\big(\boldsymbol{f(t)} \times \boldsymbol{\delta(t)}\big)(\boldsymbol{\tau}) = \int_{-\infty}^{+\infty} \boldsymbol{f(t) \cdot \delta(t-\tau)\, dt} = \int_{-\infty-\tau}^{+\infty-\tau} \boldsymbol{f(t+\tau) \cdot \delta(t)\, dt}$$

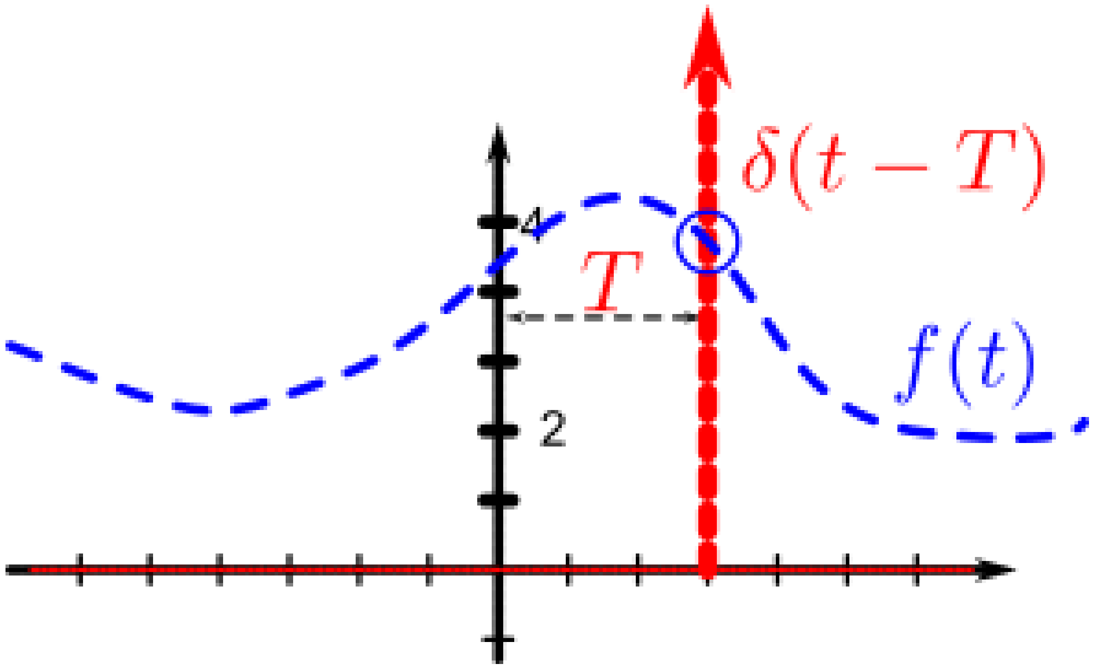
$$= \int_{-\infty}^{+\infty} \boldsymbol{f(\tau) \cdot \delta(t)\, dt} = \boldsymbol{f(\tau)} \cdot \int_{-\infty}^{+\infty} \boldsymbol{\delta(t)\, dt} = \boldsymbol{f(\tau)}$$

Dirac Delta Function Properties

- **Multiplying a function/signal $f(t)$ with a shifted Dirac delta $\delta(t)$**

$$f(t) \cdot \delta(t - T) \; = f(T) \cdot \delta(t - T)$$

- **Result depends only on $f(T)$**
  - removes any dependency from $f(\tau)$ for $\tau \neq T$



Product of a Signal with a shifted Dirac Delta corresponds to ideal Sampling

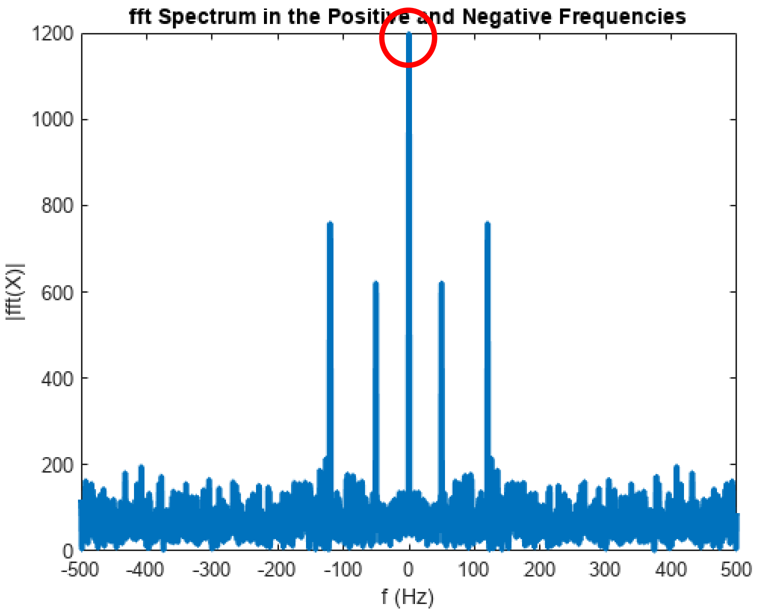
FT/IFT of a Dirac Delta

- **Consider the inverse Fourier Transform of a Dirac Delta in the FD**

$$g(t) = \mathcal{F}^{-1}\{\delta(f)\} = \int_{-\infty}^{+\infty} \delta(f) e^{j\cdot 2\cdot\pi\cdot f\cdot t} dt = 1$$

  - A Dirac delta (at DC) corresponds to a DC offset of the signal

- **NOTE: When looking at the Spectrum of a signal, you often see a "spike" at zero frequency. This means that your signal has a significant DC value**



FT of (Complex and Real) Sinusoids

- **Consider the inverse Fourier Transform of a shifted Dirac Delta in the FD**
  - Use shifted FT property

$$\boldsymbol{g(t)} = \boldsymbol{\mathcal{F}^{-1}\{\delta(f-F)\}} = \boldsymbol{\mathcal{F}^{-1}\{\delta(f)\}} \cdot e^{j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}} = e^{j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}}$$

  - Dirac in FD at frequency F corresponds to a complex sinusoid in TD at frequency F

$$\boldsymbol{g(t)} = e^{j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}} \qquad \Leftrightarrow \qquad \boldsymbol{G(f)} = \boldsymbol{\delta(f-F)}$$

- **FT of real-valued sinusoids obtained by writing $\sin(2\pi tF)$ and $\cos(2\pi tF)$ as sums of complex sinusoids**

$$\boldsymbol{\mathcal{F}}\left\{\frac{\mathbf{1}}{\mathbf{2}}\left(e^{j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}} + e^{-j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}}\right)\right\} = \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{\delta(f-F)} + \boldsymbol{\delta(f+F)}\right)$$

$$\boldsymbol{\mathcal{F}}\left\{-\frac{\boldsymbol{i}}{\mathbf{2}}\left(e^{j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}} - e^{-j\cdot 2\cdot\pi\cdot\boldsymbol{t}\cdot\boldsymbol{F}}\right)\right\} = -\frac{\boldsymbol{i}}{\mathbf{2}}\left(\boldsymbol{\delta(f-F)} - \boldsymbol{\delta(f+F)}\right)$$

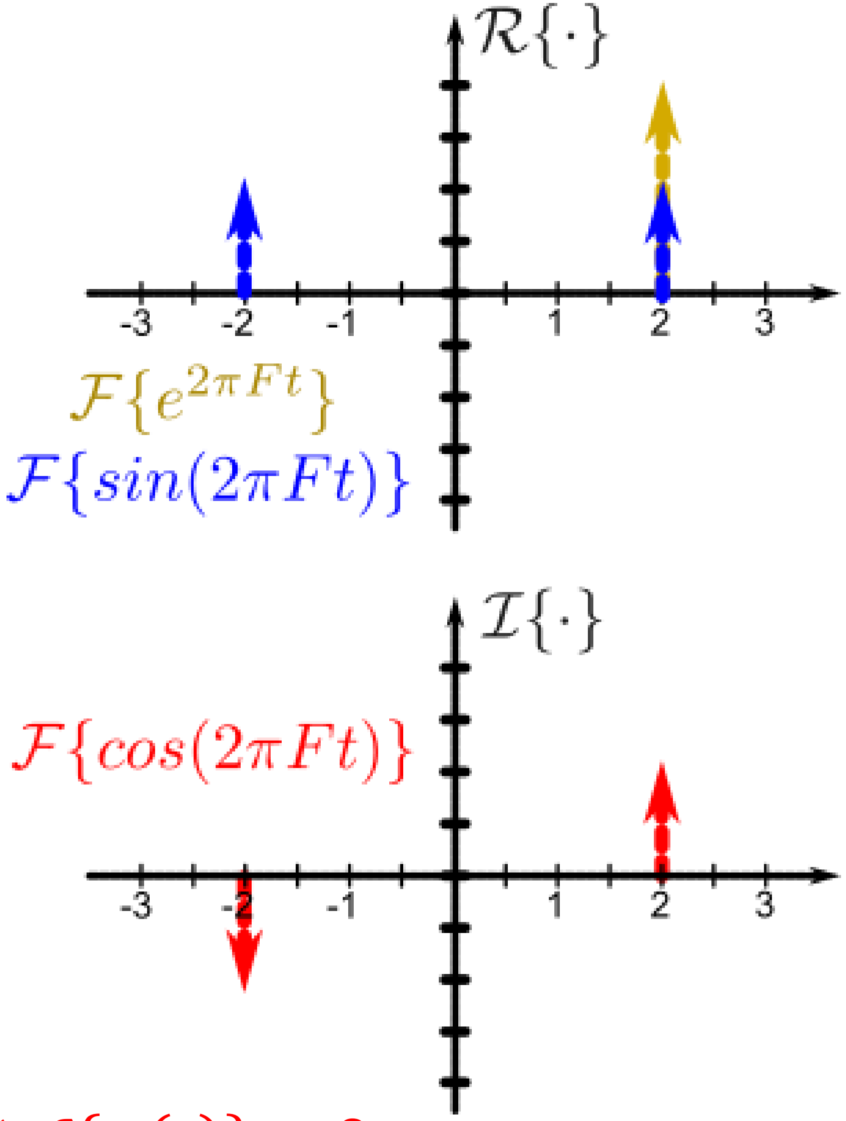
Spectrum of Real-Valued Signals

- **Compare the spectra of complex and real-valued sinusoids**

- **Two interesting observations for real-valued signals**
  - Magnitudes are symmetric $|G(f)| = |G(-f)|$
  - Imaginary part is inverse symmetric $\mathcal{I}\{G(f)\} = -\mathcal{I}\{G(f)\}$

$$\boldsymbol{G(f) = G(f)^*}$$

$\boldsymbol{G(f)}$ **and** $\boldsymbol{G(-f)}$ **are complex-conjugate pairs**



In general, if and only if $\boldsymbol{G(f) = G(-f)^*}$, $\boldsymbol{\mathcal{I}\{g(t)\} = 0}$

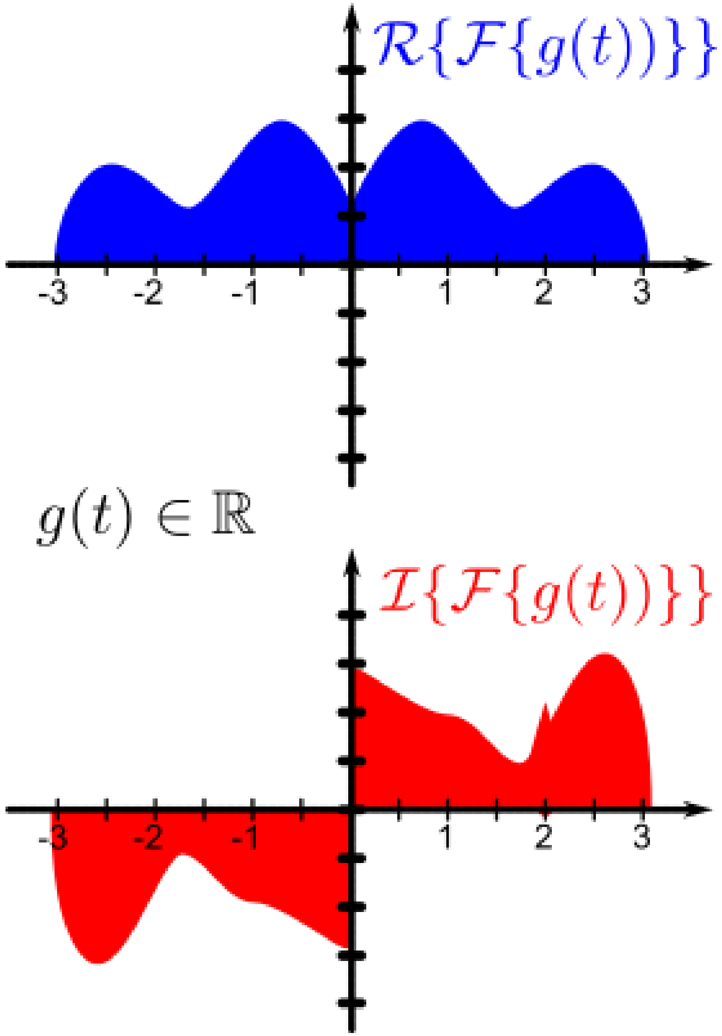
Removing Redundancy

- **The complex two-sided spectrum of a strictly real-valued signal is highly redundant**
  - Positive and negative parts of the spectrum have identical real-part
  - Positive and negative parts of the spectrum have complex-conjugate imaginary part

50% of the spectrum are redundant

- **To remove the "useless" part of the spectrum we can compute:**

$$\mathrm{F}(f) = \frac{1}{2}\left[G(f) + \mathrm{sgn}(f)\, G(f)\right]$$



Fourier Transform Cheat Sheet

Property / Pair	Signal	FT in f	FT in ω
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(f) + bX_2(f)$	$aX_1(\omega) + bX_2(\omega)$
Time delay	$(xt - t_0)$	$(X)fe^{-j2\pi f t_0}$	$(X)\omega e^{-j\omega t_0}$
Frequency Translation	$x(t)e^{j2\pi f_0 t}$	$X(f - f_0)$	$X(\omega - \omega_0)$
Convolution	$x_1(t) * x_2(t)$	$X_1(f) \cdot X_2(f)$	$X_1(\omega) \cdot X_2(\omega)$
Multiplication	$x_1(t) \cdot x_2(t)$	$X_1(f) * X_2(f)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Parseval' s Theorem	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt$	$\int_{-\infty}^{\infty} \lvert X(f) \rvert^2 df$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2 d\omega$
Rectangle	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}(f\tau)$	$\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
sinc( )	$2W \operatorname{sinc}(2Wt)$	$\Pi\left(\frac{f}{2W}\right)$	$\Pi\left(\frac{\omega}{4\pi W}\right)$
Triangle	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}^2(f\tau)$	$\tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
Exponential	$e^{-at}(u),t \quad a>0$	$\frac{1}{a + j2\pi f}$	$\frac{1}{a + j\omega}$
Impulse	$A\delta(t)$	A	A
Constant	A	$A\delta(f)$	$2\pi A\delta(\omega)$
Complex exponential	$e^{j2\pi f_0 t}$	$\delta(f - f_0)$	$2\pi\delta(\omega - \omega_0)$

Assume that $x_1(t)$ and $x_2(t)$ have FTs $X_1(f)$ and $X_2(f)$ respectively.

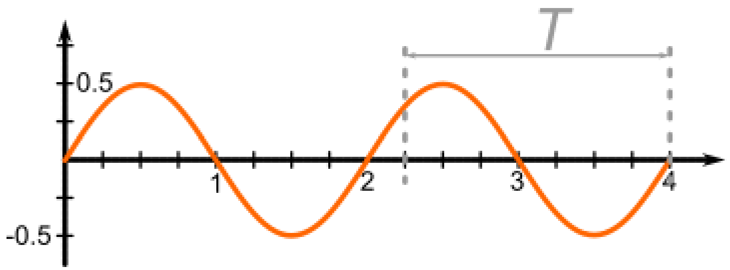
https://www.dummies.com/article/business-careers-money/careers/trades-tech-engineering-careers/signals-and-systems-working-with-transform-theorems-and-pairs-166452/

Periodic Signals

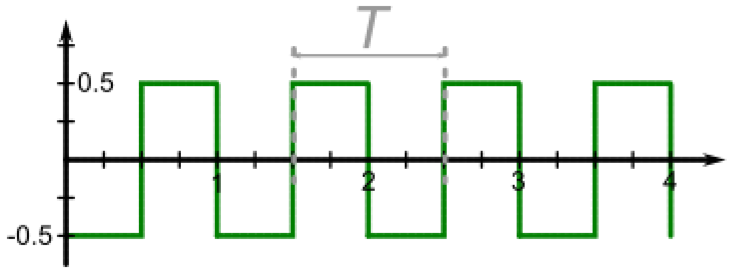
Periodic signals occur in various contexts, mainly as reference signals.

Examples:

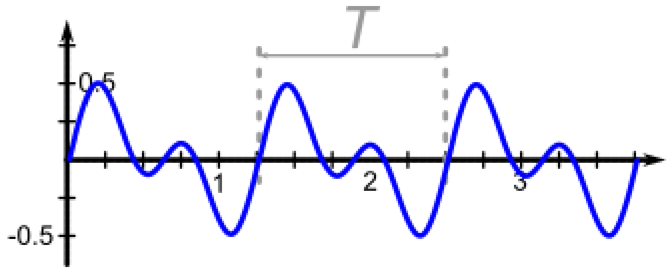
- **Clock signals in digital systems**
- **Carrier signals that define the frequency band of a wireless link**
- **Multi-tone test signals for system analysis**



Carrier Signal



Clock Signal



Two-Tone Signal

Period Signals (more formally and properties)

- **We call a signal $\bar{g}(t)$ periodic with period T_0 if for all integers n**

$$\bar{g}(t) = \bar{g}(t + n \cdot T_0)$$

- **Sometimes, signals are combination of periodic signals, for example sums or products of two (or more) periodic signals.**
  - Combining periodic signals with periods T_0 and T_1, we obtain again another periodic signal
  - Combined signal is only periodic if the ratio between the individual periods is a rational number

$$T_1 = \frac{a}{b} T_2 \rightarrow Combined\ period\ T_3 = \frac{a}{GCD(a,b)} T_2 = \frac{b}{GCD(a,b)} T_1$$

$$a, b\text{: integer, GCD}(a, b)\text{: Greatest Common Divider}$$

  - **CAVE**: For signals with very similar periods $T_0 \approx T_1$, we have $|a - b| \ll a$ and $|a - b| \ll b$. Since $|a - b| \geq 1$ and integer, $T_0 \approx T_1$ also implies that $a, b: very\ large$, which implies very T_3 can be very large even if T_0, T_1 are small

Power Spectral Density for Periodic Signals

- **Reminder: for non-periodic signals, we have derived the**
  - Signal energy: total energy in the signal
  - Energy spectral density: energy per spectral component

- **But, periodic signals have infinite energy:** discussing "energy" makes no sense. **Consider instead for periodic signal:**
  - **Signal power**: average power over the infinite signal duration = energy in one period, normalized with the duration of one period

$$P_{\bar{g}} = \frac{1}{T_0} \int_{\Delta}^{\Delta+T_0} |\bar{g}(t)|^2 \, dt$$

  - **Power spectral density**: power per spectral component = energy per spectral component in one period, normalized with the duration of one period

$P_{\bar{g}} = \int_{-\infty}^{+\infty} |G(f)|^2 \, df$ requires normalization when computing $G(f)$

Fourier Transform of Sinusoids

The FT of some special, useful periodic signals is straightforward

- **Example-1: complex sinusoid**

$$\bar{g}(t) = e^{j\cdot 2\cdot \pi \cdot f_0 \cdot t} \leftrightharpoons G(f) = f_0 \cdot \delta(f - f_0) = \frac{1}{T_0}\delta(f - f_0)$$

$$T_0 = \frac{1}{f_0}$$

Fourier Transform of a Pulse (Dirac) Train

The FT of some special, useful periodic signals is straightforward

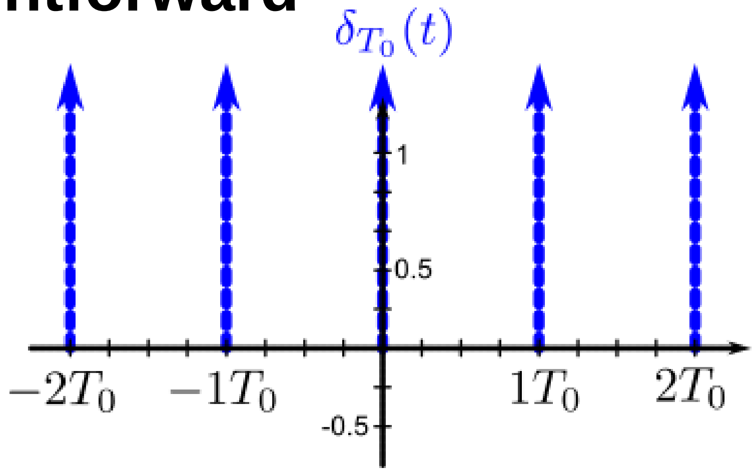
- **Example-2: pulse train ("sampling function")**

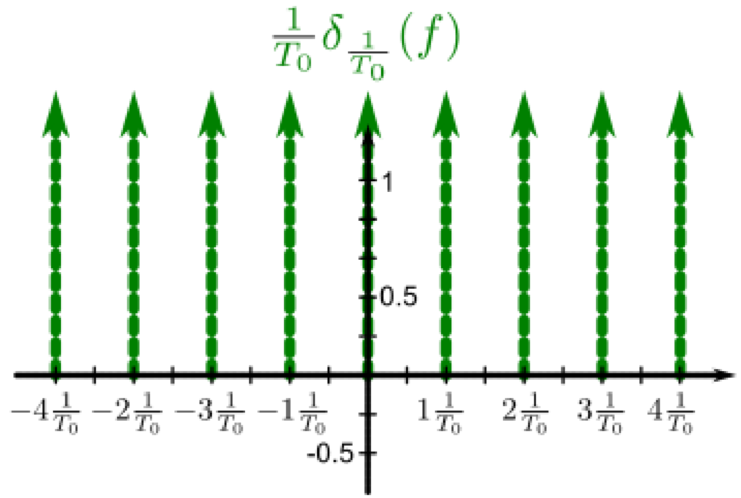
$$\delta_{T_0}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$$

$$\mathcal{F}\{\delta_{T_0}(t)\} = \sum_{n=-\infty}^{+\infty} \frac{1}{T_0} \int_{-\infty}^{+\infty} \delta(t - nT_0) e^{-j\cdot 2\cdot\pi\cdot f\cdot t} dt =$$

$$= \frac{1}{T_0} \underbrace{\sum_{n=-\infty}^{+\infty} e^{-j\cdot 2\cdot\pi\cdot n\cdot T_0\cdot f}}_{\frac{1}{T_0}\sum_{-\infty}^{+\infty}\delta\left(f-\frac{m}{T_0}\right)} = \frac{1}{T_0}\delta_{f_0}(f), \qquad \text{with} \quad f_0 = \frac{1}{T_0}$$

For all $f \neq \frac{m}{T0}$ the infinite sum is zero

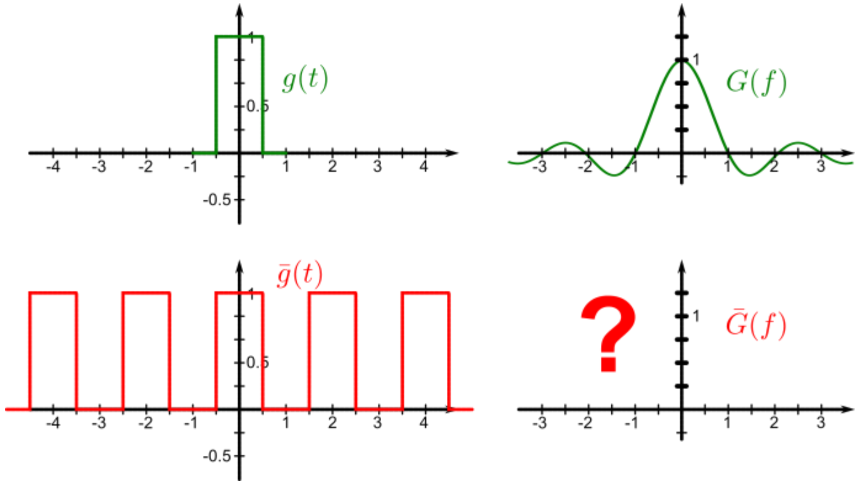




- Normalization with $1/T_0$ maintains equal power in TD and FD

Fourier Transform of Arbitrary Periodic Signals (1)

Infinite duration of periodic signals sometimes makes it hard to calculate the Fourier Transform directly



- **Can we understand the FT of a periodic signal from the FT of a single period?**

- **Write periodic signal as a function of the non-periodic signal**

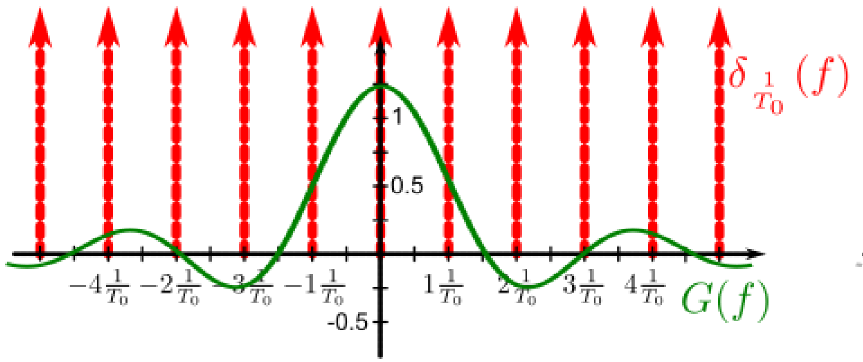
$$\bar{g}(t) = \sum_{n=-\infty}^{+\infty} g(t - nT_0) = g(t) \times \delta_{T_0}(t)$$

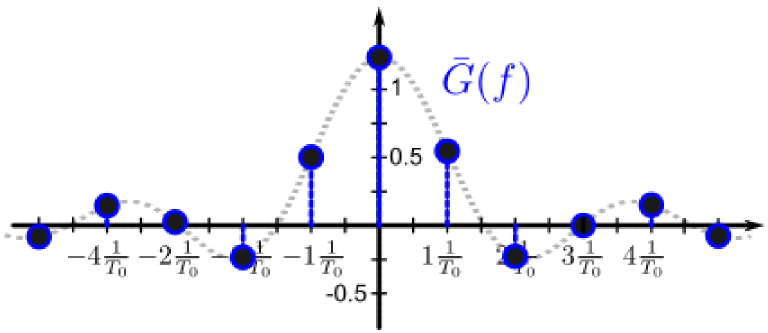
Fourier Transform of Arbitrary Periodic Signals (1)

Infinite duration of periodic signals sometimes makes it hard to calculate the Fourier Transform directly

$$\bar{G}(f) = \mathcal{F}\{g(t) \times \delta_{T_0}(t)\} = \mathcal{F}\{g(t)\} \cdot \mathcal{F}\{\delta_{T_0}(t)\} = G(f) \cdot \frac{1}{T_0}\delta_{\frac{1}{T_0}}(t) =$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} G\left(n\frac{1}{T_0}\right)\delta\left(f - n\frac{1}{T_0}\right)$$





Spectrum of a Periodic Signal is a Dirac (Sampled)

Spectrum of the Spectrum of one Period, with Samples spaced $\frac{1}{T_0}$

The Fourier Series

- **Since the spectrum of a periodic signal is a discrete spectrum, we can write inverse FT as a sum instead of an integral over Dirac pulses**

- **Fourier Series Representation:**

$$\bar{g}(t) = \sum_{n=-\infty}^{+\infty} G_n e^{j \cdot 2 \cdot \pi \cdot n \cdot \frac{1}{T_0} \cdot t}$$

- **The Fourier Coefficients G_n are obtained from the normalized FT of a single Period**

$$G_n = \frac{1}{T_0} \int_0^{T_0} \bar{g}(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot n \cdot \frac{1}{T_0} \cdot t} dt$$

The Fourier Series (Remarks)

- **Some remarks are in order:**
  - The continuous formulation / view as a set of pulses on the continuous frequency axis remains useful to be able to plot the Fourier series coefficients as a function of f
  - We often write the Fourier Series as a function of the **fundamental frequency** $\boldsymbol{f_0 = \frac{1}{T_0}}$

$$\bar{\boldsymbol{g}}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{T_0}} \sum_{n=-\infty}^{+\infty} \boldsymbol{G_n} \boldsymbol{e}^{\boldsymbol{j \cdot 2 \cdot \pi \cdot n \cdot f_0 \cdot t}} \qquad \text{with} \qquad \boldsymbol{f_0 = \frac{1}{T_0}}$$

- **Interpretation of the line spectrum of periodic function**
  - As $\bar{\boldsymbol{g}}(\boldsymbol{t})$ is periodic ($\bar{\boldsymbol{g}}(\boldsymbol{t}) = \bar{\boldsymbol{g}}(\boldsymbol{t} + \boldsymbol{k} \cdot \boldsymbol{T_0})$), it must be composed of basis functions which are also periodic with $\boldsymbol{T_0}$.
  - This is the case for exactly all those sinusoids with frequencies $f = \boldsymbol{n} \cdot \frac{\mathbf{1}}{\boldsymbol{T_0}} = \boldsymbol{n} \cdot \boldsymbol{f_n}$ as

$$\boldsymbol{e}^{\boldsymbol{j \cdot 2 \cdot \pi \cdot n \cdot \frac{(t + k \cdot T_0)}{T_0}}} = \boldsymbol{e}^{\boldsymbol{j \cdot 2 \cdot \pi \cdot n \cdot \frac{t}{T_0}}} \cdot \underbrace{\boldsymbol{e}^{\boldsymbol{j \cdot 2 \cdot \pi \cdot n \cdot k \cdot \frac{T_0}{T_0}}}}_{\mathbf{1}}$$

Power Spectral Density for Periodic Signals

- **Reminder: for non-periodic signals, we have derived the**
  - Signal energy: total energy in the signal
  - Energy spectral density: energy per spectral component

- **But, periodic signals have infinite energy:** discussing "energy" makes no sense. **Consider instead for periodic signal:**
  - **Signal power**: average power over the infinite signal duration = energy in one period, normalized with the duration of one period

$$P_{\bar{g}} = \frac{1}{T_0} \int_{\Delta}^{\Delta+T_0} |\bar{g}(t)|^2 \, dt$$

  - **Power spectral density**: power per spectral component = energy per spectral component in one period, normalized with the duration of one period

$$P_{\bar{g}}(f) = \sum_{n=-\infty}^{+\infty} |G_n|^2 \delta\left(f - n \cdot \frac{1}{T_0}\right)$$

Parseval Theorem with Fourier Series

- **For periodic signals, Parseval's theorem applies as well, but for power instead of energy**
- **The power of a periodic signal can be computed from**
  - The time domain (energy over one period, normalized by the period)
  - The integral of the power spectral density (series of dirac pulses, sampling the FT of one period, normalized by the period)
  - The squared magnitude Fourier Series coefficients

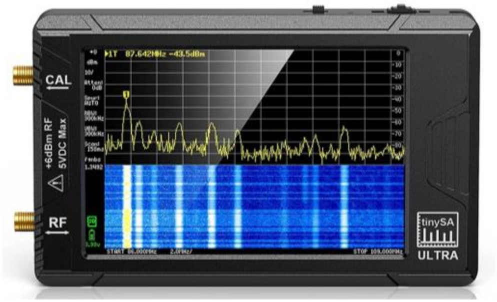
$$\frac{1}{T_0}\int_{\Delta}^{\Delta+T_0} |\bar{g}(t)|^2\, dt = \int_{-\infty}^{+\infty} P_{\bar{g}}(f)df = \sum_{\boldsymbol{n}=-\infty}^{+\infty} |\boldsymbol{G_n}|^2$$

With $P_{\bar{g}}(f)$: Power Spectral Density

$\boldsymbol{G_n}$: Fourier Series Coefficients

Power Spectral Density on a Spectrum Analyzer (1)

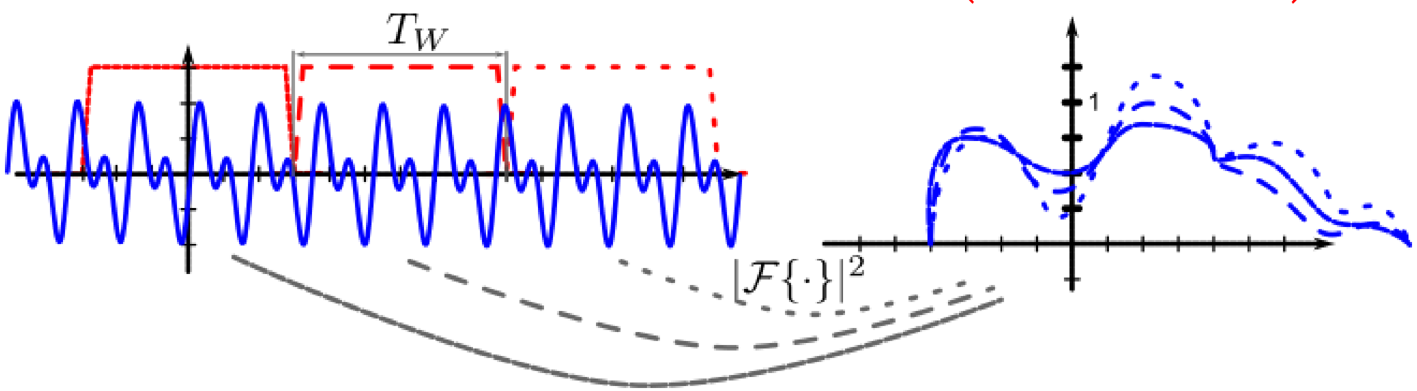
- **In practice, we analyze the PSD on a general-purpose spectrum analyzer**
  - The spectrum analyzer does not “know” the period of the signal, but it can also not analyze the spectrum of an infinitely long signal



- **Solution: consider the spectra of many long windows $T_W \gg T_0$ and average**

$$P_{\bar{g}}(f) = \sum_{n=-\infty}^{+\infty} |G(f)|^2 \delta\left(f - n \cdot \frac{1}{T_0}\right) \approx \frac{1}{M} \sum_{n=1}^{M} |G'_k(f)|^2$$

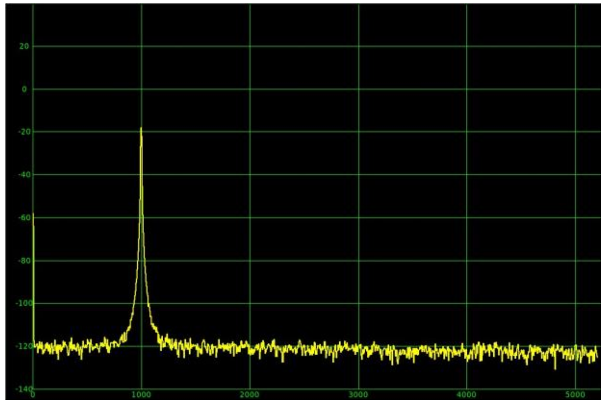
$$G'_k(f) = \mathcal{F}\{g'_k(t)\} \text{ with } g'_k(t) = g(t) \cdot \Pi\left(\frac{1}{T_W} t - k \cdot T_W\right)$$



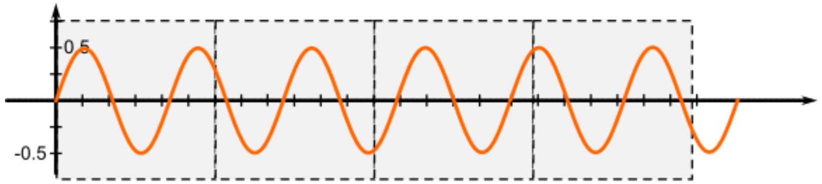
Power Spectral Density on a Spectrum Analyzer (2)

- **This practical implementation produces some unexpected artifacts**

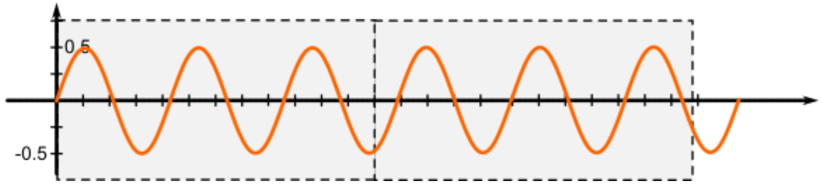
- **Consider the example of a periodic sine wave: we expect a sharp peak in the spectrum at the given frequency**
  - The expected peak (Dirac) shows as a wider peak

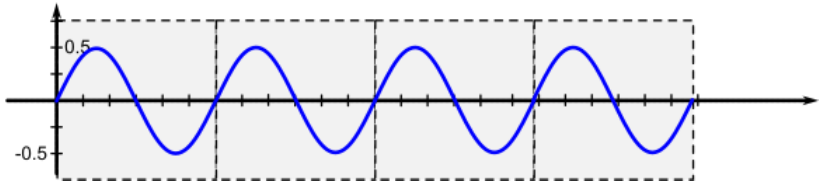


  - Especially in a digital spectrum analyzer, the width of the peak changes with the settings of the analyzer
  - For some frequencies of the sine wave, we even observe a perfect Dirac



  - **What happens here??**





Recap from Week-2

- **The Fourier Transform decomposes a signal into complex sinusoids**

$$G(f) = \int_{-\infty}^{+\infty} g(t)e^{-j\cdot 2\cdot \pi \cdot f \cdot t}\, dt \text{ and } g(f) = \int_{-\infty}^{+\infty} G(t)e^{j\cdot 2\cdot \pi \cdot f \cdot t}\, dt$$

- **Some important properties of the FT/IFT**
  - The FT of a signal and its inverse are closely related

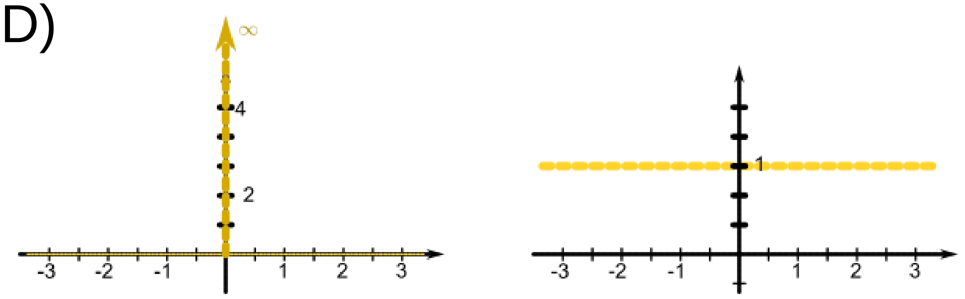
$$\mathcal{F}\{g(t)\} = G(f) \Longrightarrow \mathcal{F}\{G(t)\} = g(-f)$$
$$\mathcal{F}^{-1}\{G(f)\} = g(t) \Longrightarrow \mathcal{F}^{-1}\{g(f)\} = G(-t)$$

  - Convolving two signals in one domain corresponds to multiplying the signals in the other domainand vice versa

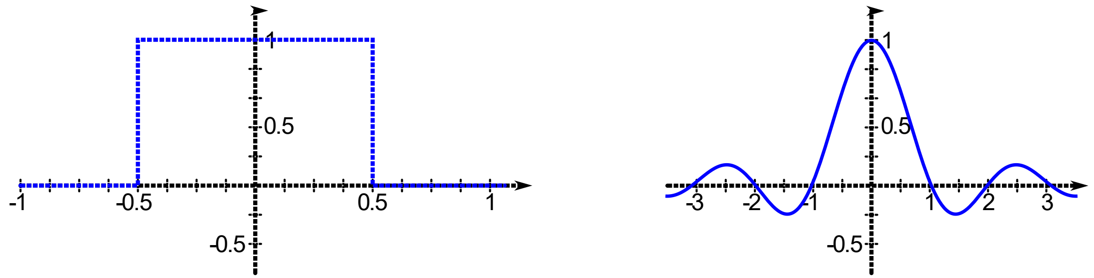
$$\mathcal{F}\{g_1(t) \times g_2(t)\} = G_1(t) \cdot G(t)$$
$$\mathcal{F}^{-1}\{G_1(t) \times G(t)\} = g_1(t) \cdot g_2(t)$$

Recap from Week-2

- **Some important signals are helpful to keep in mind together with their FT**
  - The Dirac delta pulse: an infinitely short, infinitely high pulse with unit-energy and a flat power spectral density (PSD)



  - The Brick-Wall and Sinc signals: a brick-wall signal of duration T has a FT/IFT of the form $\frac{\sin \pi \frac{f}{T}}{\pi \frac{f}{T}}$

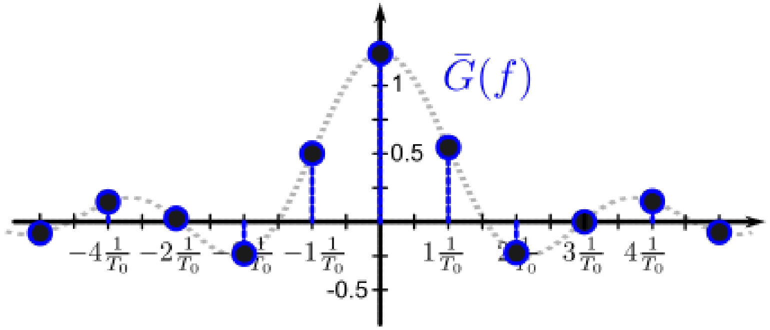


  - In general: duration of a signal T and its bandwidth B are inversely proportional: $T \propto \frac{1}{B}$

Recap from Week-2

- **Periodic signals have infinite energy, but we can describe them by their "power spectral density" (power at each frequency component)**

- **The spectrum of periodic signals is a sequence of Dirac pulses that sample the spectrum of one period**



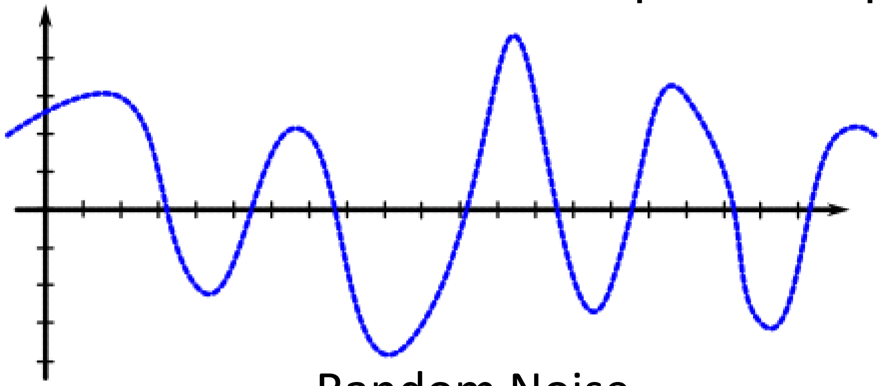
- **For a signal with period T_0 the space between the Dirac pulses is $\mathbf{f_0 = \frac{1}{T_0}}$**
  - We call f_o the fundamental frequency

- **Since the spectrum of a periodic signal is discrete (sampled in f), we can express the signal as an infinite sum of Fourier Coefficeints**

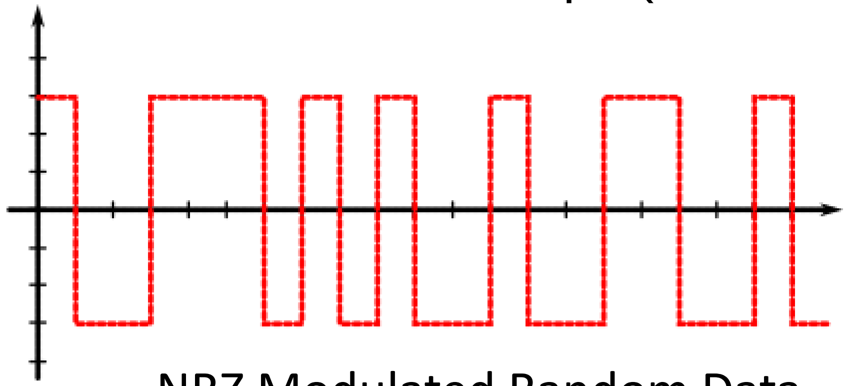
$$\bar{g}(t) = \sum_{n=-\infty}^{+\infty} G_n e^{j \cdot 2 \cdot \pi \cdot n \cdot \frac{1}{T_0} \cdot t}$$ **with** $$G_n = \frac{1}{T_0} \int_0^{T_0} \bar{g}(t) \cdot e^{-j \cdot 2 \cdot \pi \cdot n \cdot \frac{1}{T_0} \cdot t} dt$$

Stochastic Signals

- **Deterministic signals are important, but ultimately rare.**
- **Most signals are actually somewhat random. For example:**
  - Noise: random, but often limited in bandwidth or with a specific frequency characteristic
  - Data to be transmitted: random sequence of pulses of similar nature/shape (see modulation)



Random Noise



NRZ Modulated Random Data

  - **Observation: even random signals are usually not completely random (points that are close in time appear somehow related to each other)**

How can we characterize these signals, especially in terms of their spectral content?

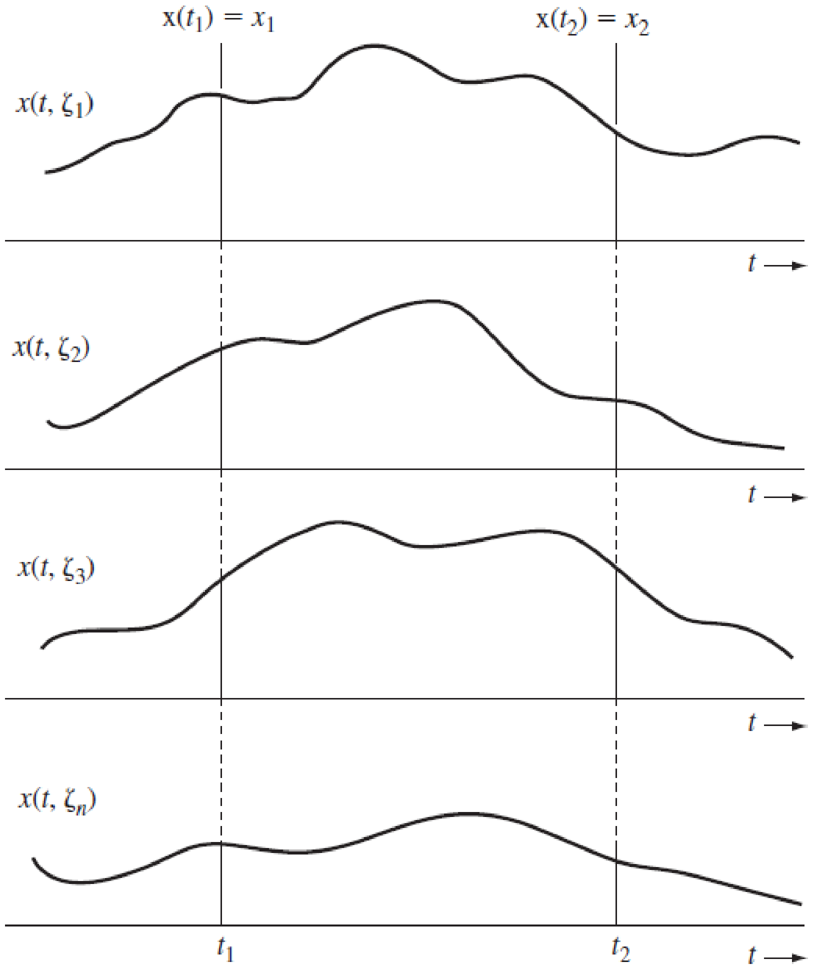
Stochastic Signals

- **Stochastic signals are random processes that generate a sequence of consecutive values $x(t)$**

  - Every observation ζ_k of a stochastic signal $x(t,\zeta_k)$ between t_0 and t_1: $t_0 < t < t_1$ is a realization (sample function) of the same stochastic process

  - Instead of considering only a single value $x(t)$ we consider always realizations of the process $x(t,\zeta_k)$

  - **We refer to the stochastic process** that generates these samples **as X**

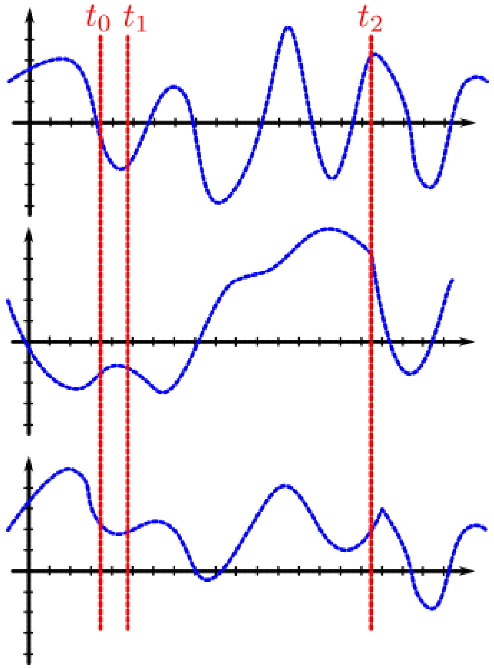


Stochastic Signals (Probabilistic Characterization)

- **A scalar random variable y is characterized by its PDF $P_Y(y)$**

- **Since X always generates (infinitely) many samples, we need a PDF for a random process $P_X(\boldsymbol{x};\boldsymbol{t})$ that jointly characterizes $x(t)$ at all times t**
  - $\boldsymbol{x}$ and $\boldsymbol{t}$ in $P_X(\boldsymbol{x};\boldsymbol{t})$ are intentionally **BOLD** since they represent many values jointly

- **Now consider $x(t)$ at two time instants t_0 and t_1. In the same sample ζ_k, $x(t_0)$, $x(t_1)$, $x(t_2)$ are not necessarily independent (unrelated). For example,**
  - if t_0 and t_1 are close, $x(t_0)$ and $x(t_1)$ are often also close
  - if t_0 and t_2 are far, $x(t_0)$ and $x(t_2)$ are often very different (or at least less likely to be close)



- **Relationship captured by the joint PDF: $P_X(x_0, x_1; t_0, t_1)$**

Stochastic Signals (Probabilistic Characterization)

- **To fully capture this relationship between signal values, we would need to consider the joint PDF of all values in a sample**

$$P_X(x_0, x_1, x_2, x_3, \ldots ; t_0, t_1, t_2, t_3, \ldots)$$

  - In practice this is too complex and not really necessary

- **Three simplifications are common place and sufficient in communications**
  1. Assume that the 1st order statistics of a value is independent of its time in a sample
  $$P_X(x; t) = P_X(x)$$
  2. Consider only joint probability between two time instants (2nd order statistics):
  $$P_X(x_0, x_1; t_0, t_1) \text{ for any } t_0, t_1$$
  3. Assume that only the distance Δt between the two time instances matters
  $$P_X(x_0, x_1; t_0, t_1) \approx P_X(x; \Delta t)$$

Such as process is called Wide-Sense Stationary

Stochastic Signals and Autocorrelation Fct. (ACF)

- **With the 2nd order statistics, we can define the "autocorrelation function"**

$$R_{\boldsymbol{X}}(\Delta t) = E_{\zeta_k}\{x(t) \cdot x^*(t + \Delta t)\}$$

  - Note that the expectation E_{ζ_k} is over the "ensemble", i.e., over many realizations of the process

- **In practice** (ergodic process)**, we can replace the ensemble expectation with an expectation/average over time**

$$R_{\boldsymbol{X}}(\Delta t) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) \cdot x^*(t + \Delta t)\, dt$$

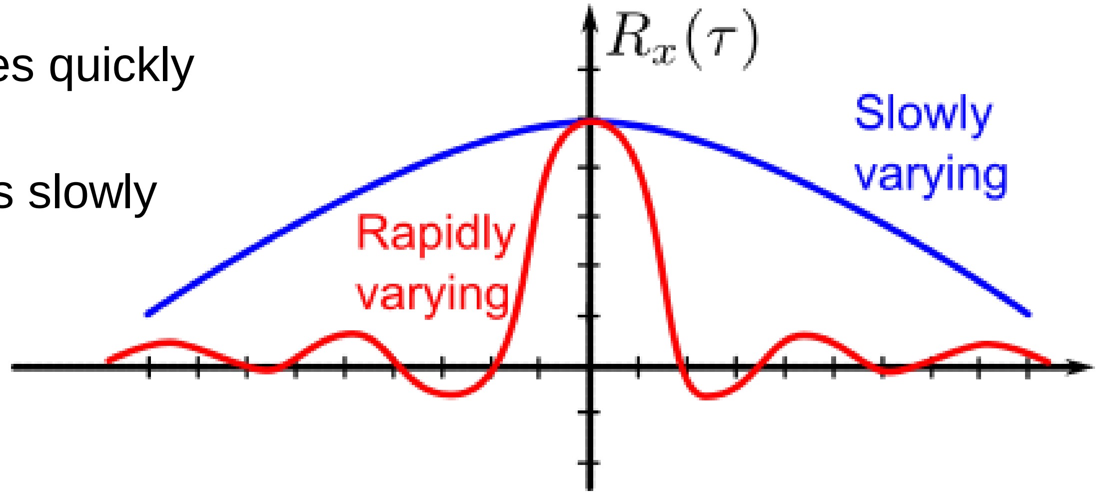
  - We can numerically calculate the ACF from a sample of a random signal
  - With a high number of samples, this ACF characterizes the corresponding random process

Properties of the Autocorrelation Fct. (ACF)

- **The ACF has some very interesting and helpful properties**
  - Power of a Wide-Sense Stationary Stochastic Signals

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)\cdot x^*(t+\Delta t)\,dt = R_{\boldsymbol{X}}(0)$$

  - The ACF has its maximum magnitude at $R_{\boldsymbol{X}}(0) \geq |R_{\boldsymbol{X}}(\tau)|$
  - More handwaving observations: The ACF reflects how "similar" two τ-spaced values of the signal are
    - when $R_{\boldsymbol{X}}(\tau)$ decays rapidly with τ, the signal changes quickly ("appears very much random over time")
    - when it remains high for larger τ, the signal changes slowly ("appears less random over time")



Link between ACF and Power Spectral Density

- **The PSD shows the power in each frequency component**
  - A slowly varying signal has much power in low frequencies and little power in high frequencies
  - A fast varying signal has significant power in higher frequencies

The PSD of a Stochastic Signal is the Fourier Transform of its ACF

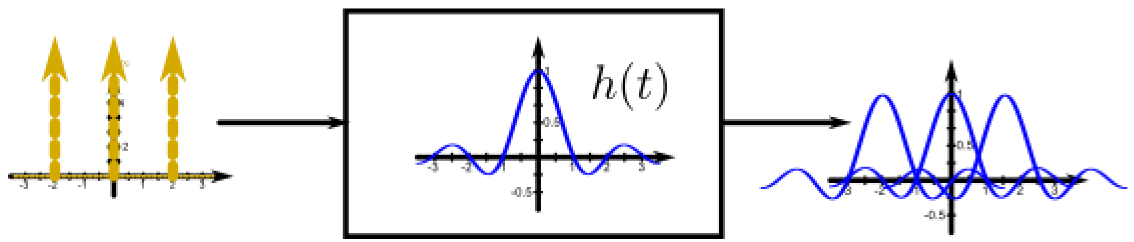
$$P_x(f) = \mathcal{F}\{R_{\boldsymbol{X}}(\tau)\}$$

$$R_{\boldsymbol{X}}(\tau) = \mathcal{F}^{-1}\{P_x(f)\}$$

- We can obtain the power of a signal from the FT of its ACF: $P_x = \int_{-\infty}^{+\infty} P_x(f)df$
- The ACF (as the PSD) are always positive: $P_x(f) \geq 0$
- When the signal is real-valued: $P_x(f) = P_x(-f)$

Signal Generation with Filters (for the Lab)

- **We often want to generate a signal with a specific shape or a specific spectrum and a corresponding signal generator may not be available**
- **Trick: we know that a filter allows us to do two things**
  - Convolve a signal with the impulse response of the filter



  - Multiply the spectrum with the FT of the filter (its frequency characteristics)