EE-432
Systeme de Telecommunication

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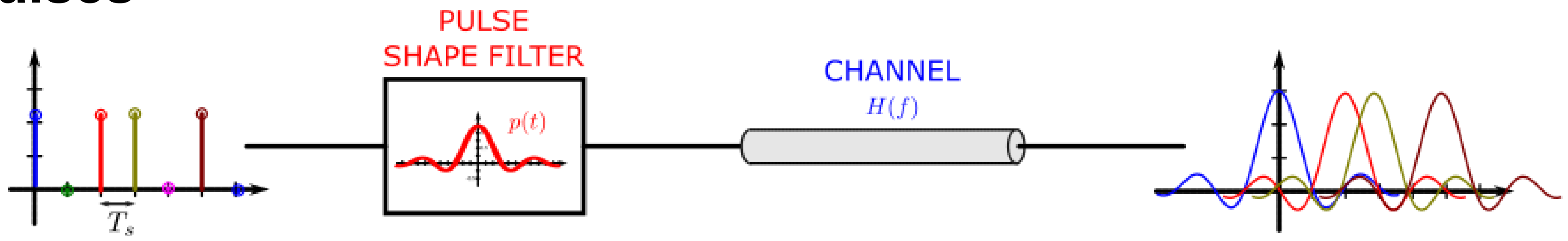
Digital Data Transmission: Error Rates in Digital PAM and Scaling with SNR, Es, Eb

Recap from Week-6

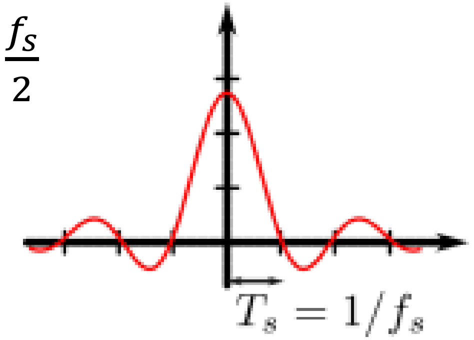
- **Digital Signals are modulated by sending pulses that encode digital bits**

- **Pulse shape filter "translated" the discrete time data signal into continuous time pulses**

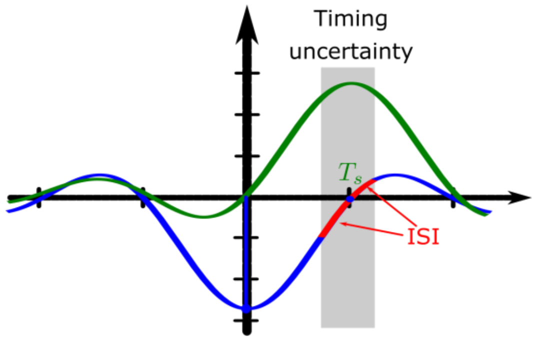


  - The symbol rate and the shape of the corresponding pulse shape filter define the bandwidth of the signal
  - An optimal pulse shape is a SINC filter resulting in a bandwidth of $BW_{ch} = \frac{f_s}{2}$
  - A SINC pulse guarantees ISI free transmission as

$$\frac{\sin(k \cdot \pi)}{k \cdot \pi} = \begin{cases} \mathbf{1} & \boldsymbol{k = 0} \\ \mathbf{0} & \boldsymbol{k \neq 0} \end{cases}$$

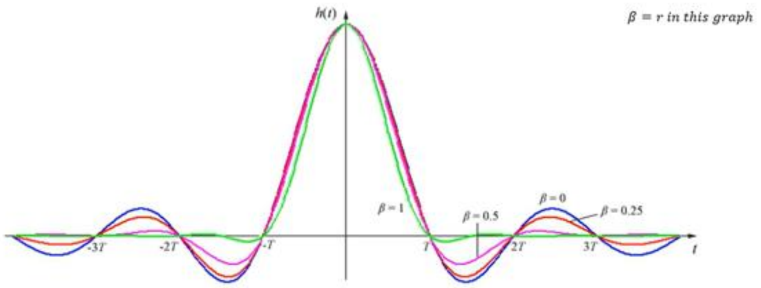


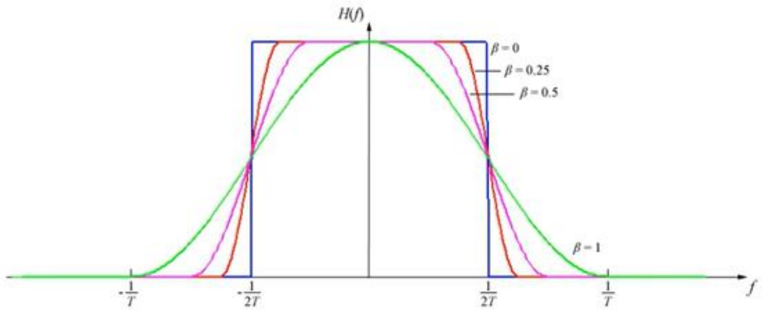
Recap from Week-6



- **SINC pulses are not always ideal. They are very long and decay only slowly**

- **Any pulse that fulfills 2nd Nyquist criterion is sufficient to avoid ISI**
  - Better pulses exist than the SINC pulse: Raised Cosine Pulse





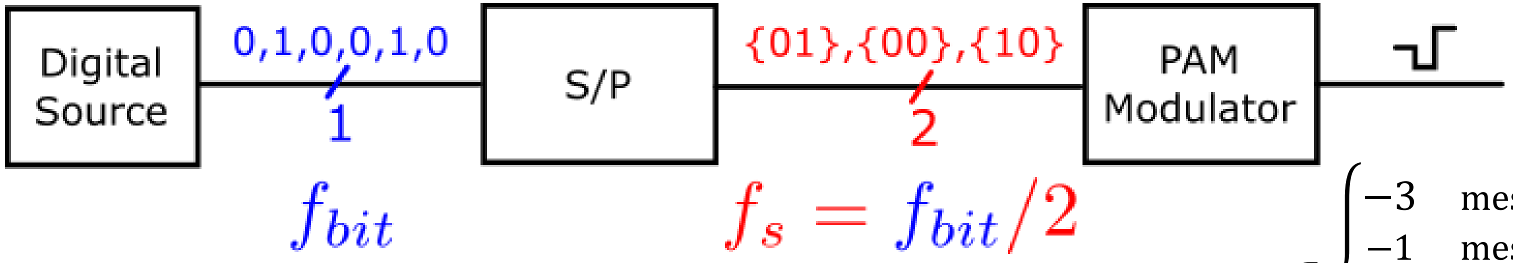
  - The more rapid decay of signal power comes at the expense of additional bandwidth.
  - For a non-SINC pulse, we have

$$\frac{1+\beta}{2} f_s < BW_{ch}$$ with bandwidth expansion factor $\beta \approx 0.6 - 0.8$

Recap from Week-6

- **For a given bandwidth, the data rate with binary PCM is limited by the bandwidth as we send only 1 bit for every “symbol”**

- **To achieve higher data rates, we combine multiple bits into one symbol with multiple levels**
  - Example: 4-PAM

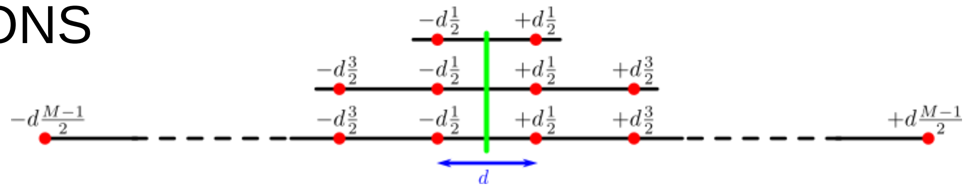


Efficiency: 2 bits / symbol

$$a_k \in \begin{cases} -3 & \text{message bits } 00 \\ -1 & \text{message bits } 01 \\ +1 & \text{message bits } 10 \\ +3 & \text{message bits } 11 \end{cases}$$

  - We call the set of values that encode bits CONSTELLATIONS

$$\mathcal{O} = \{x_0, x_1, x_2, \dots, x_{M-1}\}$$



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Applications

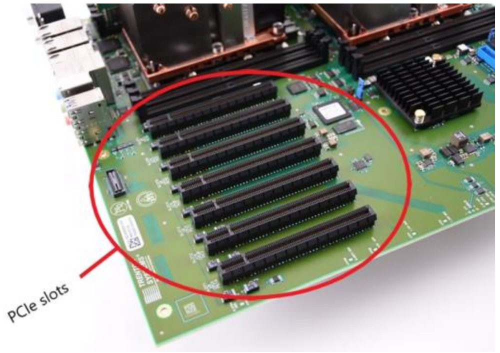
Applications of PCM Systems: Wireline

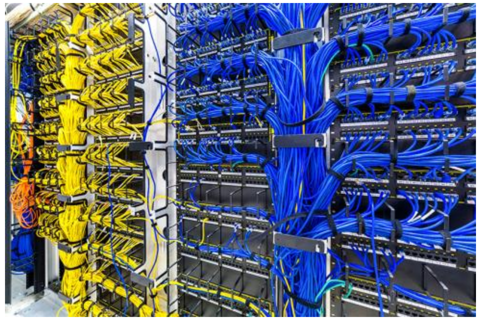
Telephone

USB

HDMI, Display Port, ...

Memory Interfaces

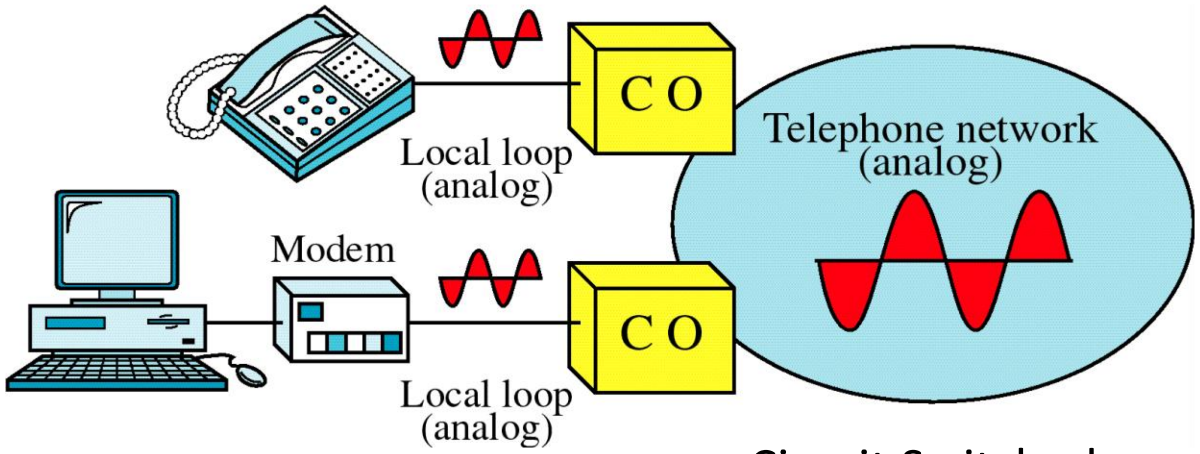




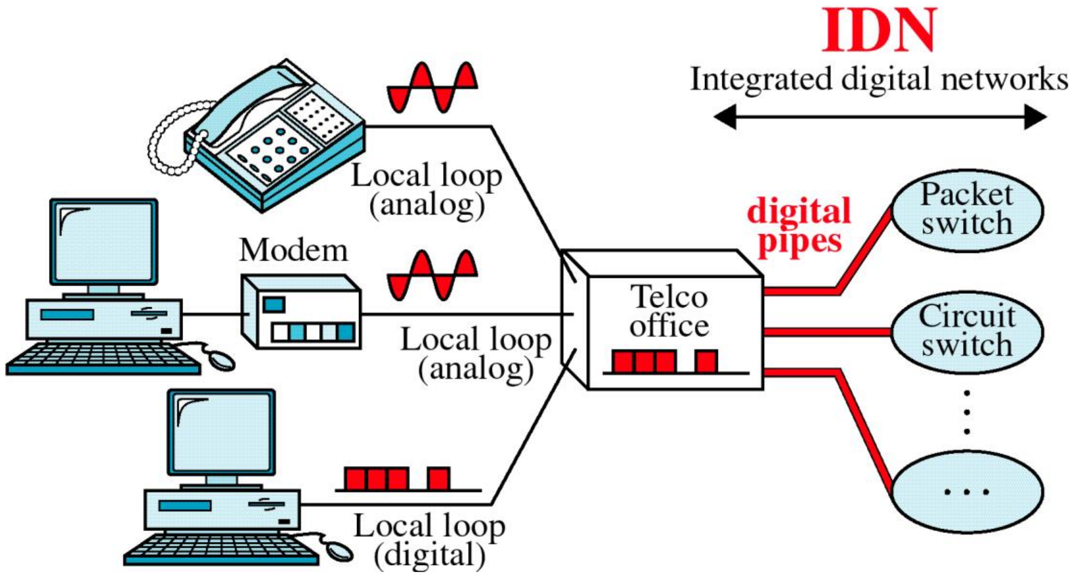
Computer Networks

ISDN Digital Telephony Motivation

- **Motivation in the late 1980s**
  - Increasing digitization of the global telephone network (with only local loop analog)
  - Increasing need for digital data communication over analog telephone network



Circuit Switched Network

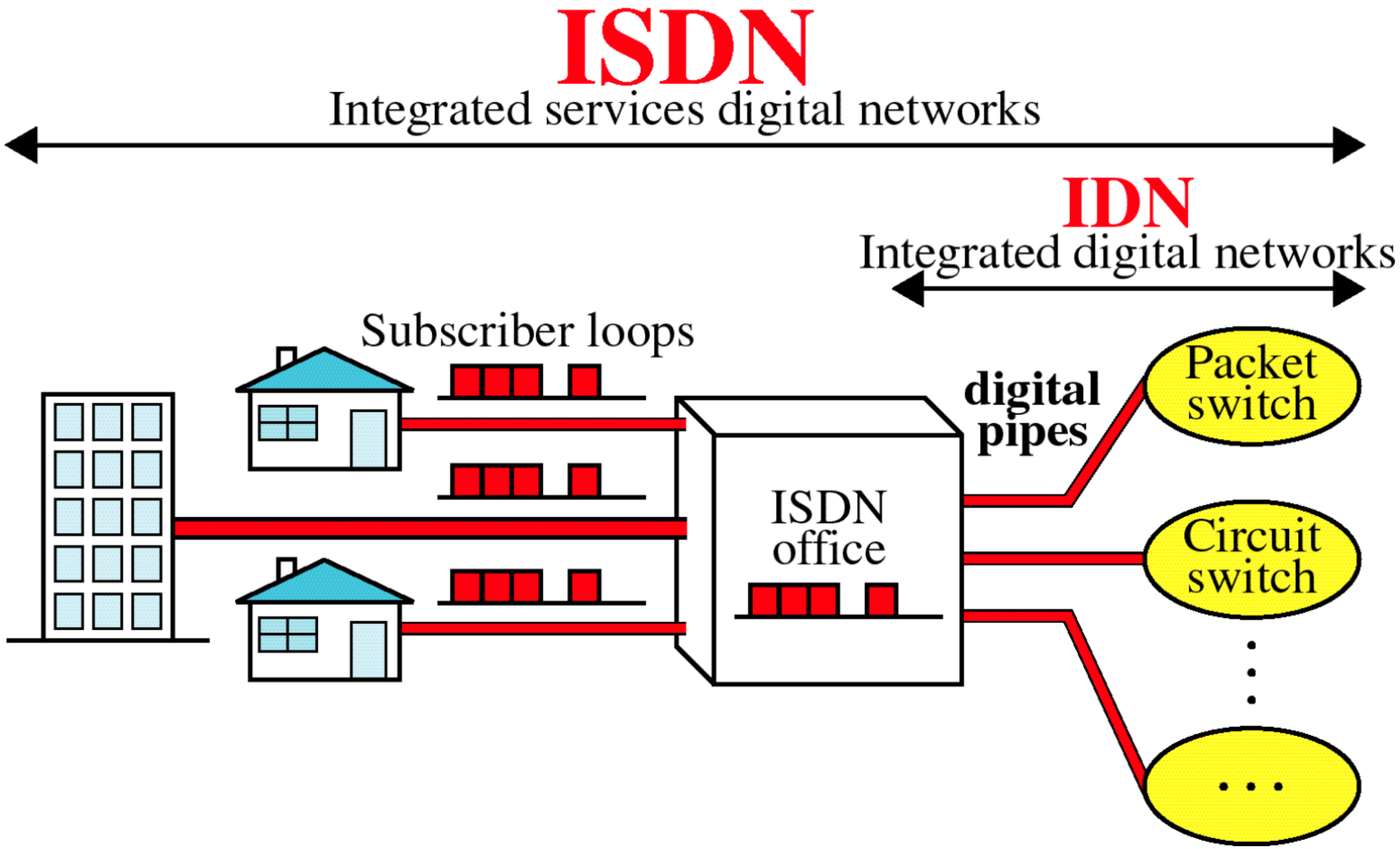


Packet Switched Network

- **Transition from analog local loop (end-user connection) to telephone network was inefficient. Need for new fully digital system**

ISDN Digital Telephony

- **ISDN: Integrated Services Digital Network**
  - Proposed in 1984, standardization completed in 1988, commercially available since 1992



Digital Telephone System: ISDN

- **ISDN:** Carry digitized voice using standard telephony bandwidth

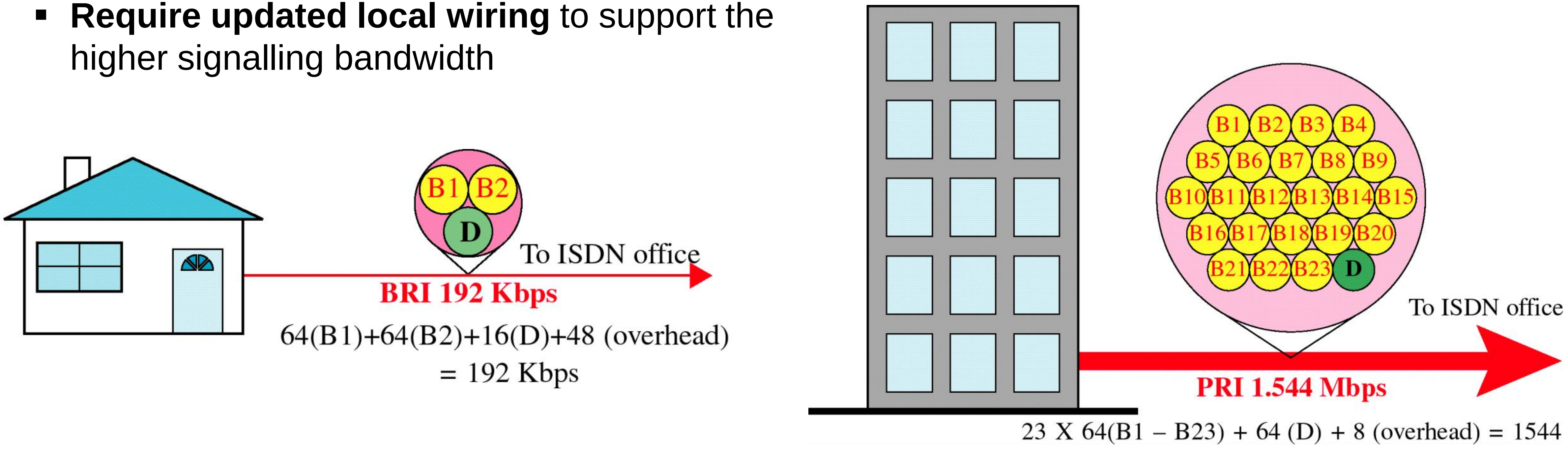
- **ISDN Voice Channels Bandwidth / Quantization:**
  - **Audio bandwidth**: ~300 Hz to 3400 Hz (speech-optimized passband)
  - **Nyquist rate**: To capture this bandwidth, the minimum sampling rate is 2 × 3400 = 6800 Hz
  - **Standard choice**: 8000 samples/second (8 kHz) is used in practice — the same rate as traditional PSTN systems.

  - Voice **quantization** with **8 bits/sample**
    - **Non-linear analog signal compression** based on A-law in Europe, µ-law in North America/Japan
    - No digital data compression

  - **64 kbps per voice channel** is referred to as a B-channel (Bearer channel) in ISDN
  - **Modulated using "2B1Q" (2 Binary 1 Quarternary) modulation (=4-PAM -3V,-1V,+1V,+3V)**

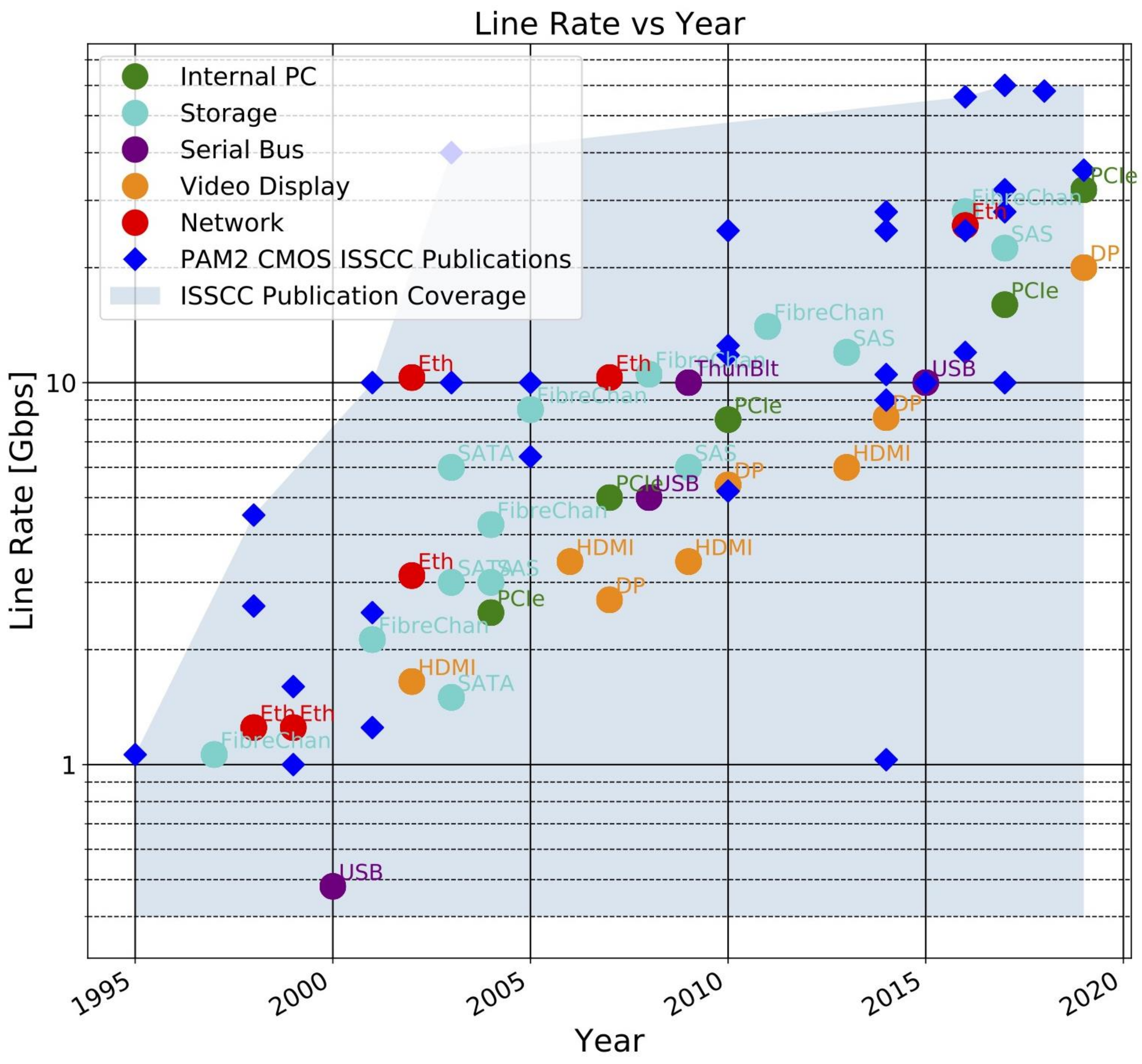
Digital Telephone System: ISDN

- **Basic Rate Interface (BRI):** residential and small-office customers
  - Two B-channels + one 16 Kbps D-channel (2B+D) and 48 Kbps of operating overhead
  - Can use the same **twisted-pair local loop** as analog network
- **Primary rate interface (PRI): business customers**
  - 23 B channels + one 64-kbps D channel and 8 kbps of overhead: 1.544 Mbps
  - **Require updated local wiring** to support the higher signalling bandwidth



SerDes High Speed Chip-to-Chip Links

- **High speed serial links are essential for many applications**
  - Optical Transmission: OC-192, OC-768, SONET
  - Internal PC: PCIe 1-5
  - Storage: Fibre Channel, SATA, SAS
  - Serial Bus: USB, Thunderbolt
  - Video Display: DisplayPort, HDMI



Ethernet

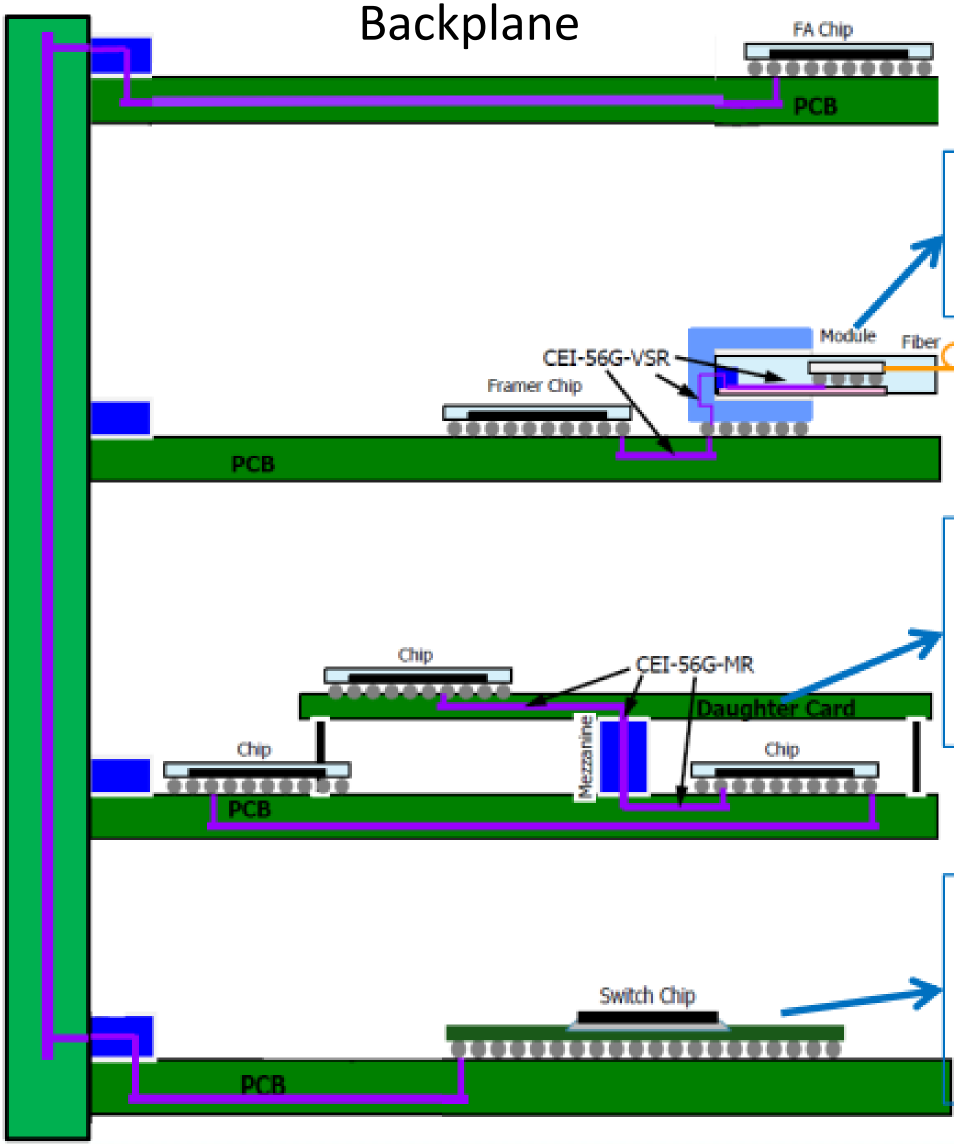
- **Purpose:** robust, and scalable method for local area communication (LAN)
  - Foundations developed in 1973
  - IEEE 802.3 standard since 1983 (10BASE5)

- **Evolution toward higher speed with** more parallel lanes, higher bandwidth, and higher order modulation

Standard	Data Rate	Modulation	Lanes	Baud rate	BW	Medium
10Base-T	10 Mbps	Machester	1	10	10 MHz	Twisted pair
100Base-T	100 Gbps	MLT-3 (PAM-3)	1	125	31.25 MHz	Cat 5
1000Base-T	1 Gbps	PAM-5	4	125	62.5 MHz	Cat 6
10GBase-T	10 Gbps	PAM-16	4	800	500 MHz	Cat 6A
25GBase-T	25 Gbps	PAM-16	4	2500	1250 MHz	Cat 8

Chip-to-Chip Links in the Gbps Regime

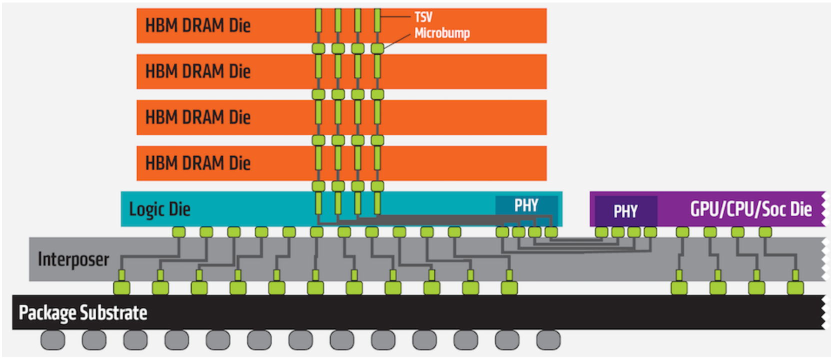
Backplane

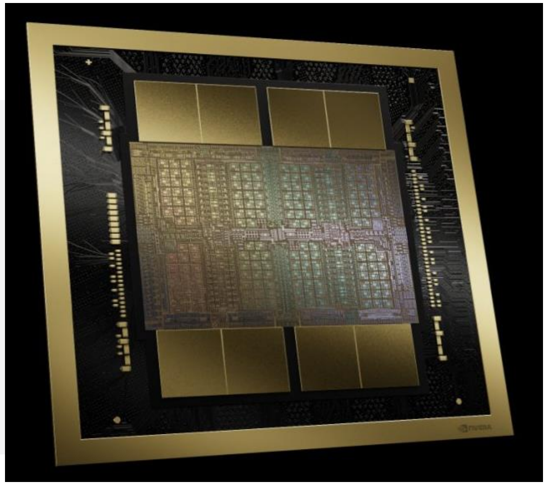


Memory Interfaces: moderate rate due to many parallel pins

Standard	Year	Baud Rate
DDR	1998	400 Mbaud
DDR 2	2003	400 – 1066 Mbaud
DDR 3	2007	800 – 2133 Mbaud
DDR 4	2014	1600 – 3200 Mbaud
DDR 5	2020	3200 – 6400 Mbaud

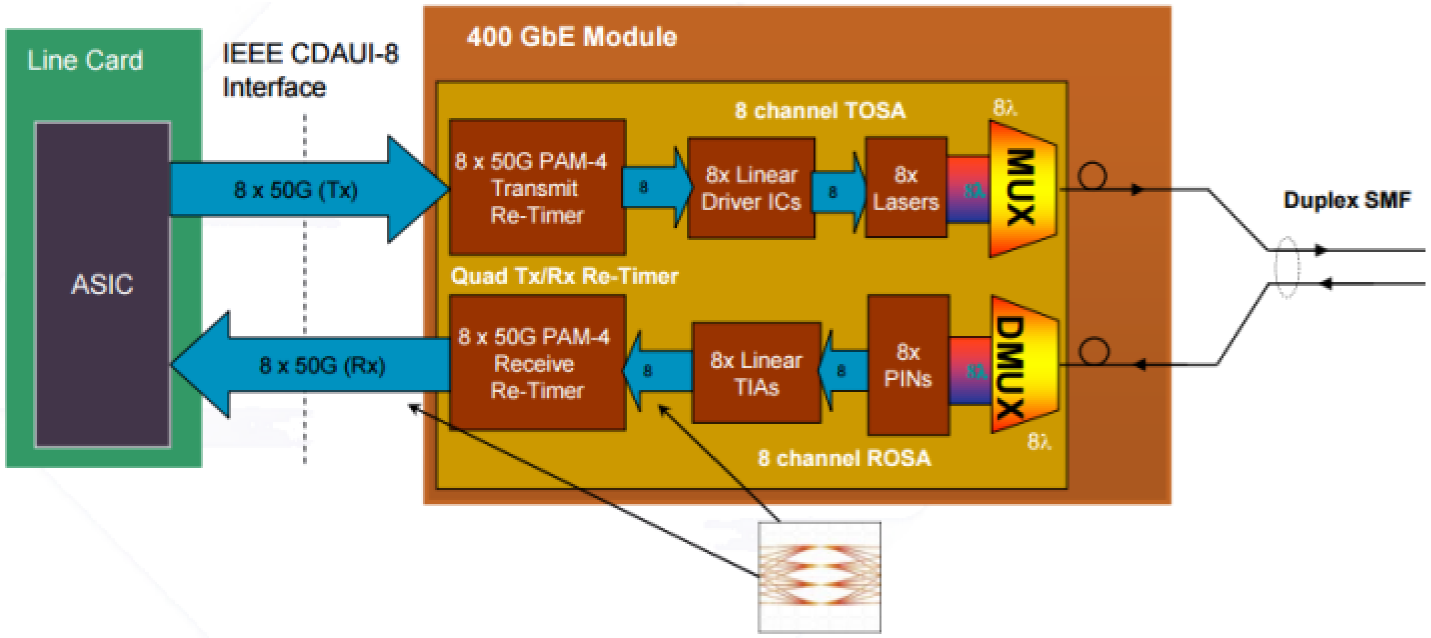
Memory Interfaces





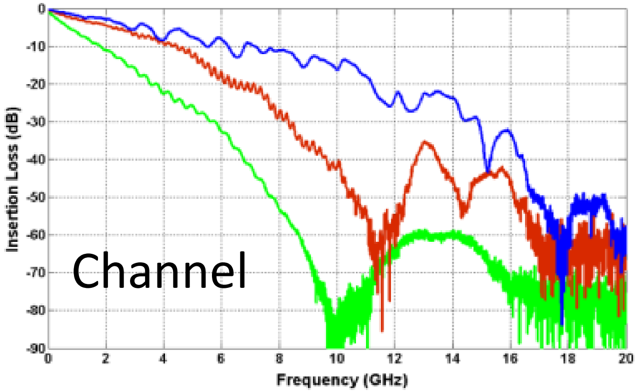
Line Card Interfaces

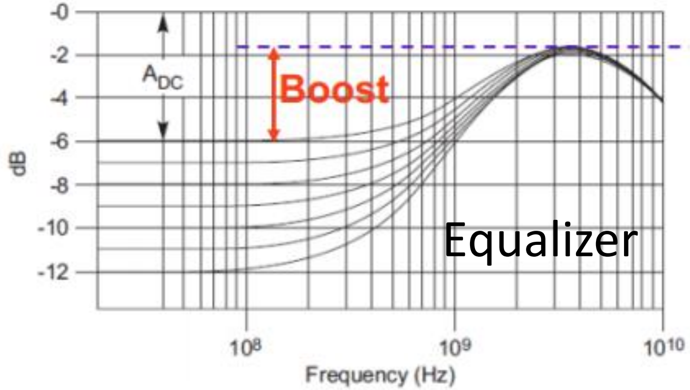
- **Interfaces to optical links are even more challenging**
  - Throughput in the 10s of Gbps per single pin

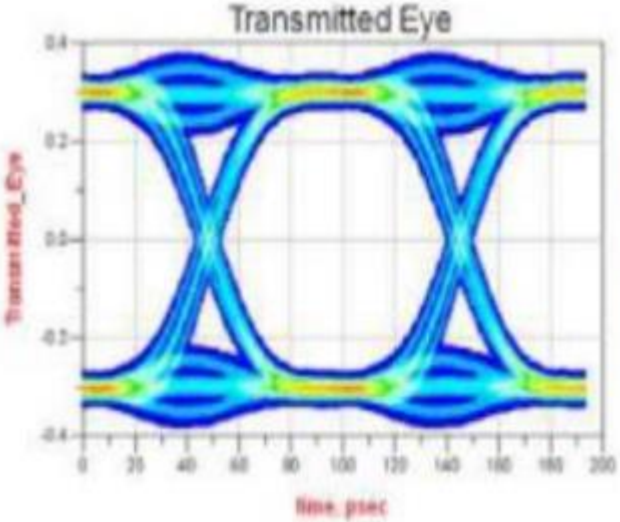


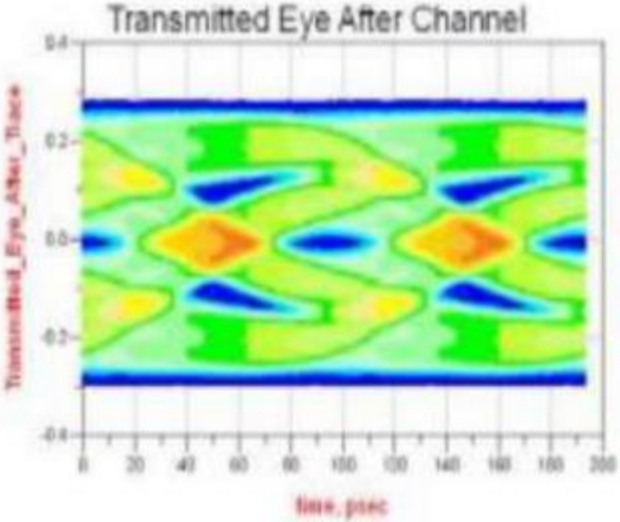
Serial Links Equalization

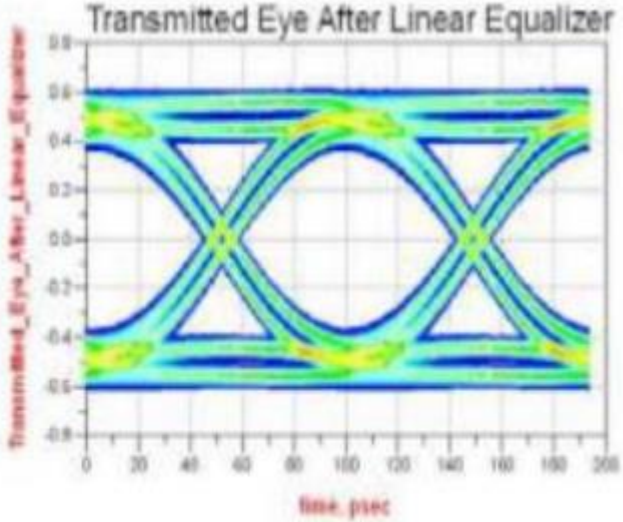
- **Chip-to-chip physical link has limited bandwidth: correction with equalization**





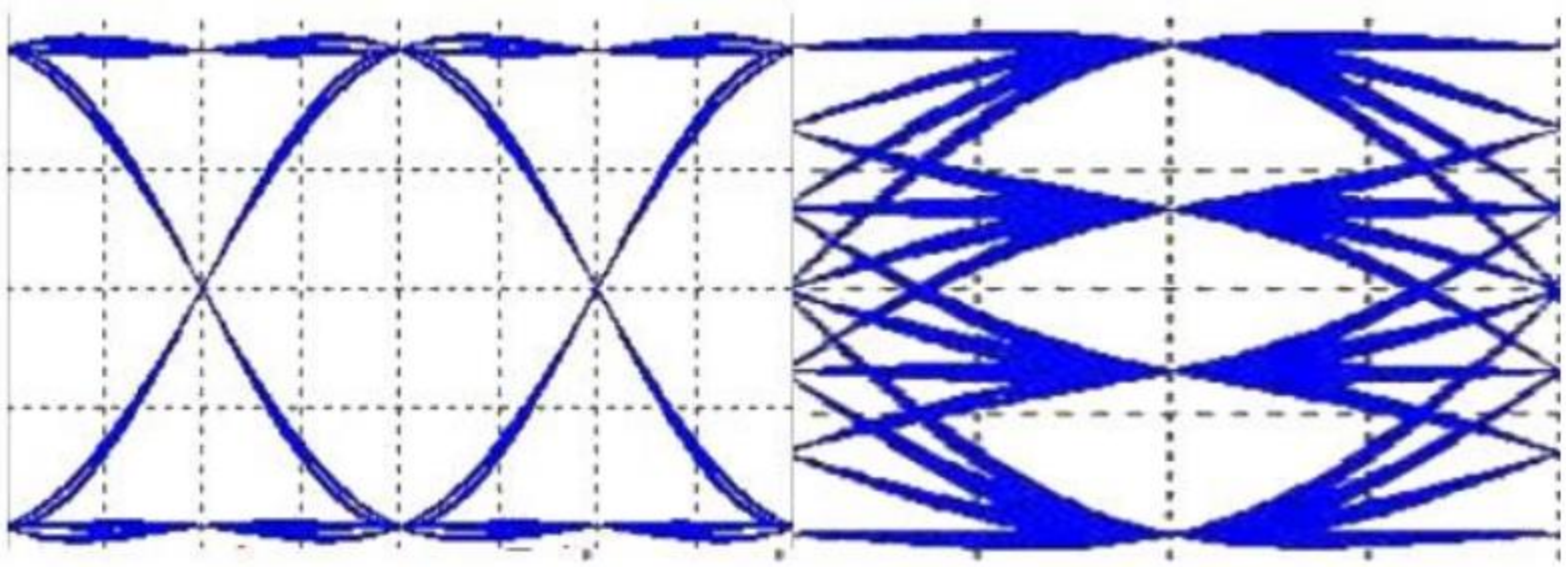






Serial Links with Higher PAM Order

- **Higher order PAM requires less bandwidth, but eye opening also reduces**

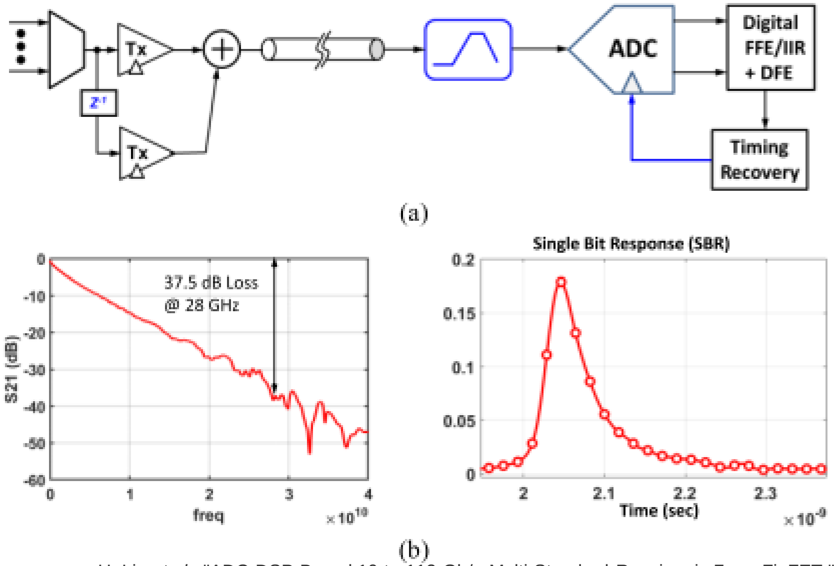


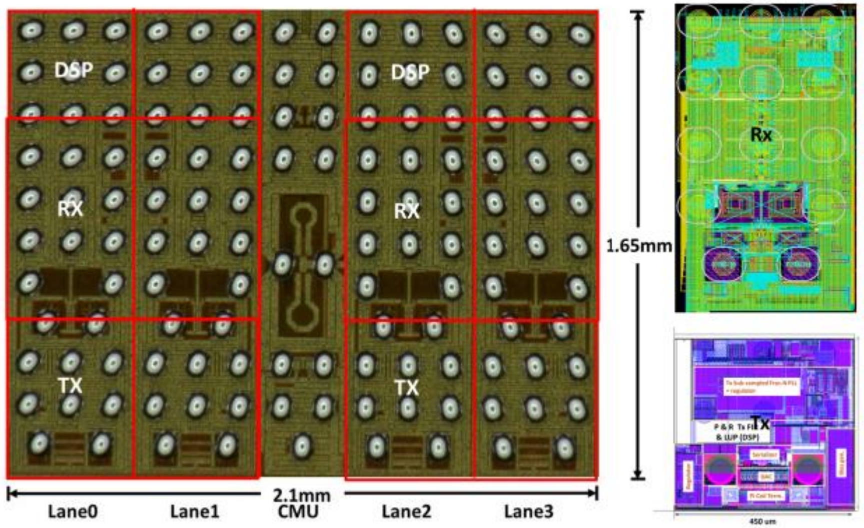
  - Eye height for PAM4 is 1/3 of that of PAM2, thus

$$\text{Loss in minimum distance} = 20 \cdot \log_{10}\left(\frac{1}{3}\right) \approx 9.5\ \text{dB}$$

Difficulty to Implement Advanced Receivers

- **Serial Links today can operate with Baud Rates in the 10s Gbps regime**
  - Signal processing mostly done in the analog domain
  - Digital signal processing only recently possible with very high-speed data converters



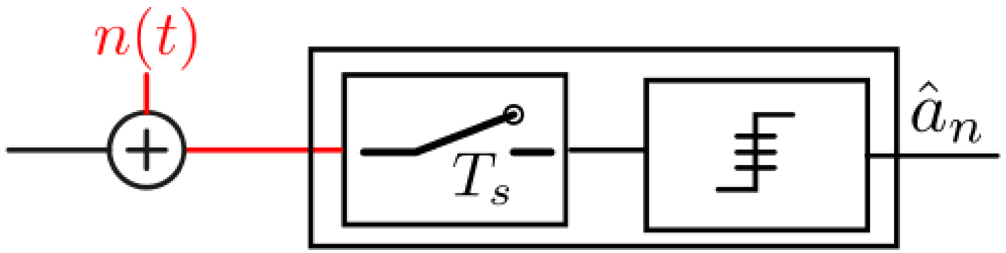


H. Lin *et al*., "ADC-DSP-Based 10-to-112-Gb/s Multi-Standard Receiver in 7-nm FinFET," in *IEEE Journal of Solid-State Circuits*, vol. 56, no. 4, pp. 1265-1277, April 2021, doi: 10.1109/JSSC.2021.3051109.

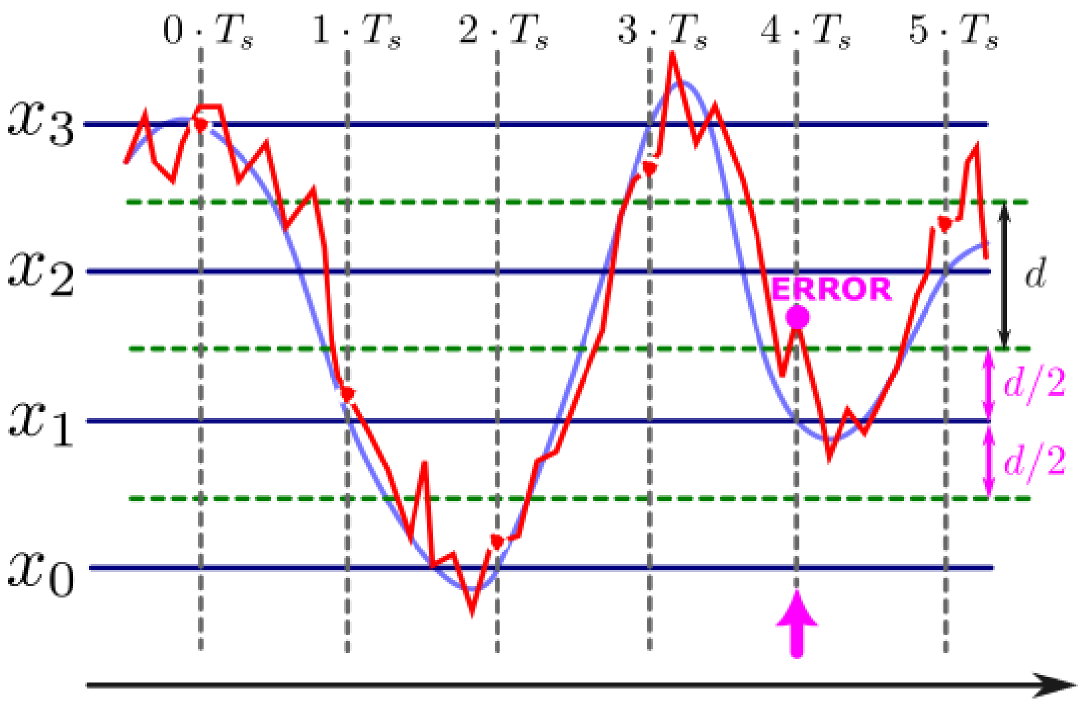
Week 7: Table of Contents

M-PAM Signal Reception with Noise

- **Consider a real M-PAM signal that is affected by additive noise $n(t), n_k$**
- **Noise can displace the signal at $t = k \cdot T_s$ "sufficiently" to lead to ERRORS**
  - An error occurs if the noise exceeds the margin* $d/2$ between adjacent constellation points



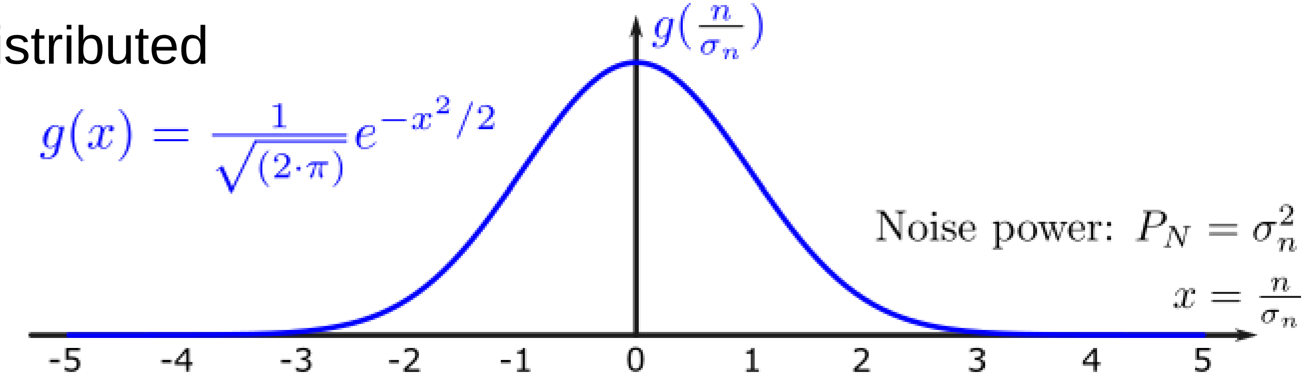
Transmitted symbol	Error if
x_0	$n_k > \frac{1}{2}d$
$x_1, \dots, x_{M-2}$	$\lvert n_k \rvert > \frac{1}{2}d$
x_0	$n_k < -\frac{1}{2}d$



* Assume equally spaced constellation points

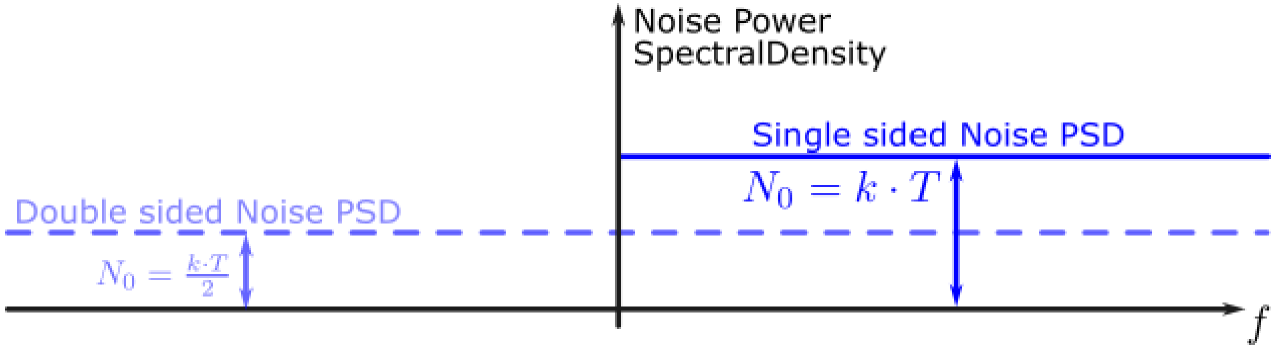
Thermal Noise Model

- **So far, we have specified the amount of noise by its variance σ^2 without further details.**
- **We consider Thermal Noise (random thermal motion of electrons in resistor)**
  - Thermal noise is Zero-Mean Gaussian distributed



  - Thermal noise has a flat power spectral density (for relevant frequency range $f \ll \frac{h}{kT} \approx 6\ \text{THz}$):

$$S_n(f) = |N(f)|^2 = \frac{2h|f|}{e^{\frac{h|f|}{kT}} - 1} \approx kT = N_0$$



M-PAM Error Probability with Gaussian Noise

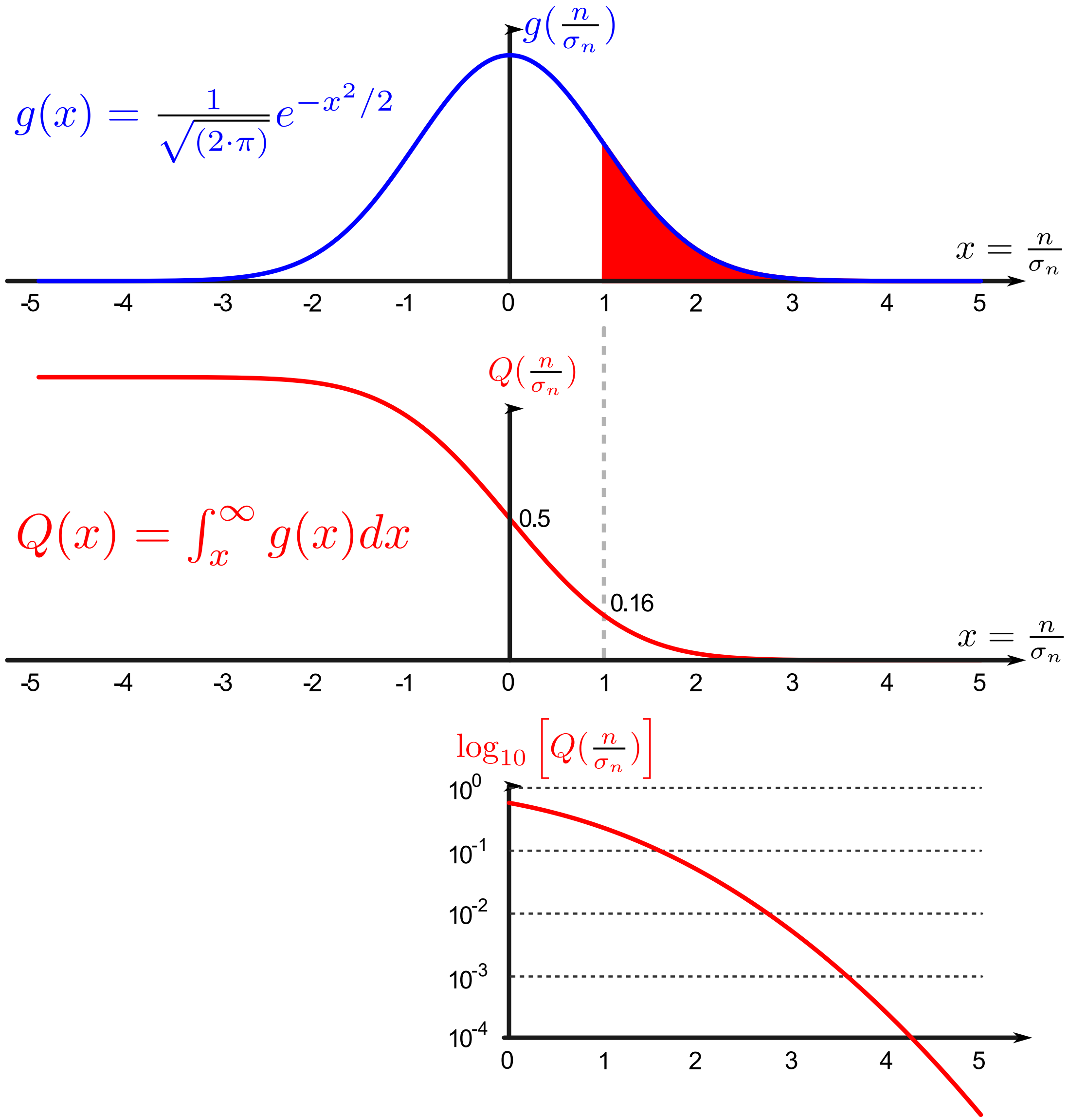
- **In practice the noise is often zero-mean Gaussian** (or we at least assume that for various reasons including for simplicity)
- **General Gaussian Random Variable n (zero-mean) with unit Variance σ_n^2**

Probability density function

$$g_{\sigma^2}(x) = g\left(\frac{x}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\cdot\pi}} e^{-\frac{1}{2}\cdot\frac{x^2}{\sigma^2}}$$

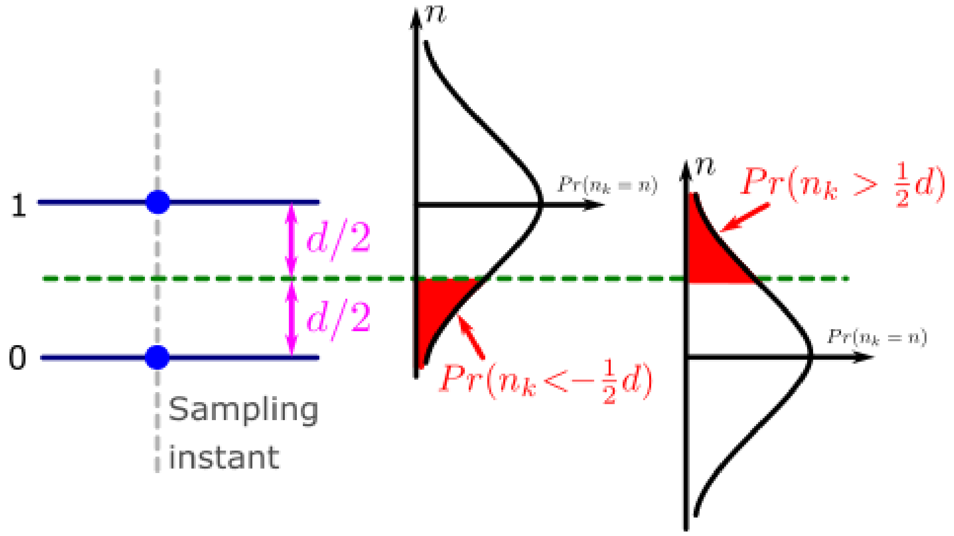
Cumulative distribution function:

$$Q_\sigma(x) = Q\left(\frac{x}{\sigma}\right) = \int_{\frac{x}{\sigma}}^{\infty} g\left(\frac{x}{\sigma}\right) dx$$



PCM Error Probability

- **Consider only the sampled signal (before the slicer)**
  - Constellation points with distance d



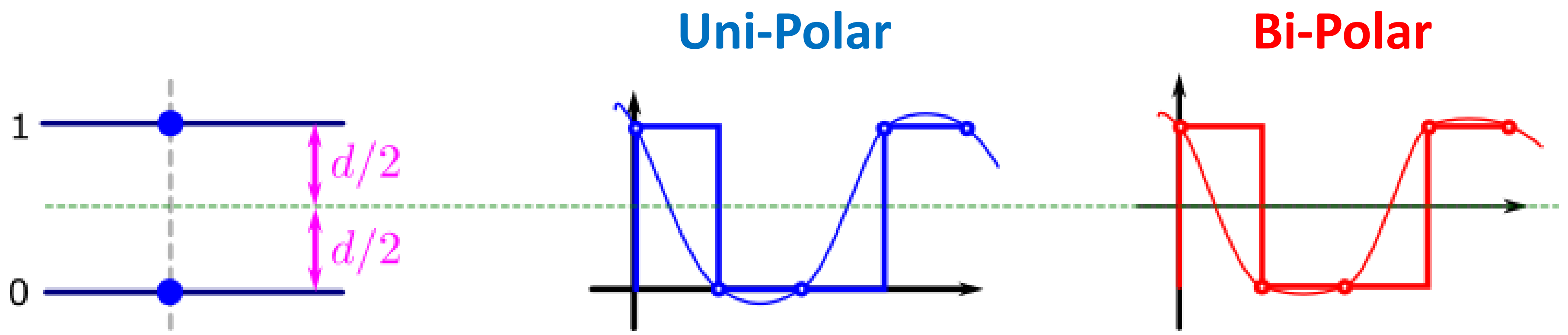
- **Expression for PCM error probability ε with distance d between symbols**
  - Two possible error events with equal probability

$$\varepsilon = \frac{1}{2}\mathrm{Pr}\left(n_k < -\frac{1}{2}d\right) + \frac{1}{2}\mathrm{Pr}\left(n_k > \frac{1}{2}d\right) =$$

$$= \mathrm{Pr}\left(n_k < \frac{1}{2}d\right)$$

PCM Error Probability: Uni-Polar/Bi-Polar

- **Consider two competing PCM signals: uni-polar and bi-polar**



- **We want to compare these in terms of error rate performance***
  - Define constellations with a given distance d
  - Express the signal power as a function of the distance d
  - Find the expression for the two respective error rates (should be equal for same distance)
  - Express the minimum distance as a function of the power for the two constellations separately
  - How are the power levels required for equal error probability related?
  - Which constellation is the better choice?

*See hand-written class notes

PCM Error Probability: Uni-Polar/Bi-Polar

Unipolar

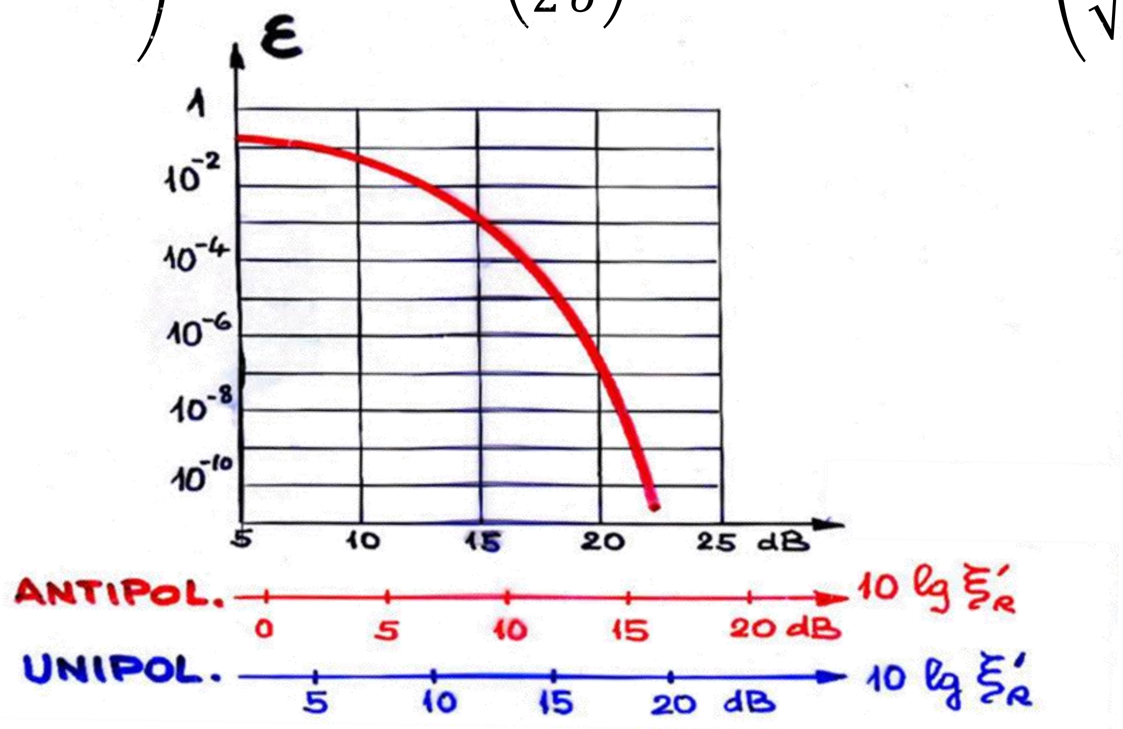
$$\varepsilon_U = Q\left(\sqrt{\frac{1}{2}\frac{\bar{P}_U}{\sigma^2}}\right) = Q\left(\sqrt{\frac{1}{2}SNR_U}\right)$$

General

$$\varepsilon = Q\left(\frac{1}{2}\frac{d}{\sigma}\right)$$

Bi-Polar

$$\varepsilon_B = Q\left(\sqrt{\frac{\bar{P}_B}{\sigma^2}}\right) = Q\left(\sqrt{SNR_B}\right)$$



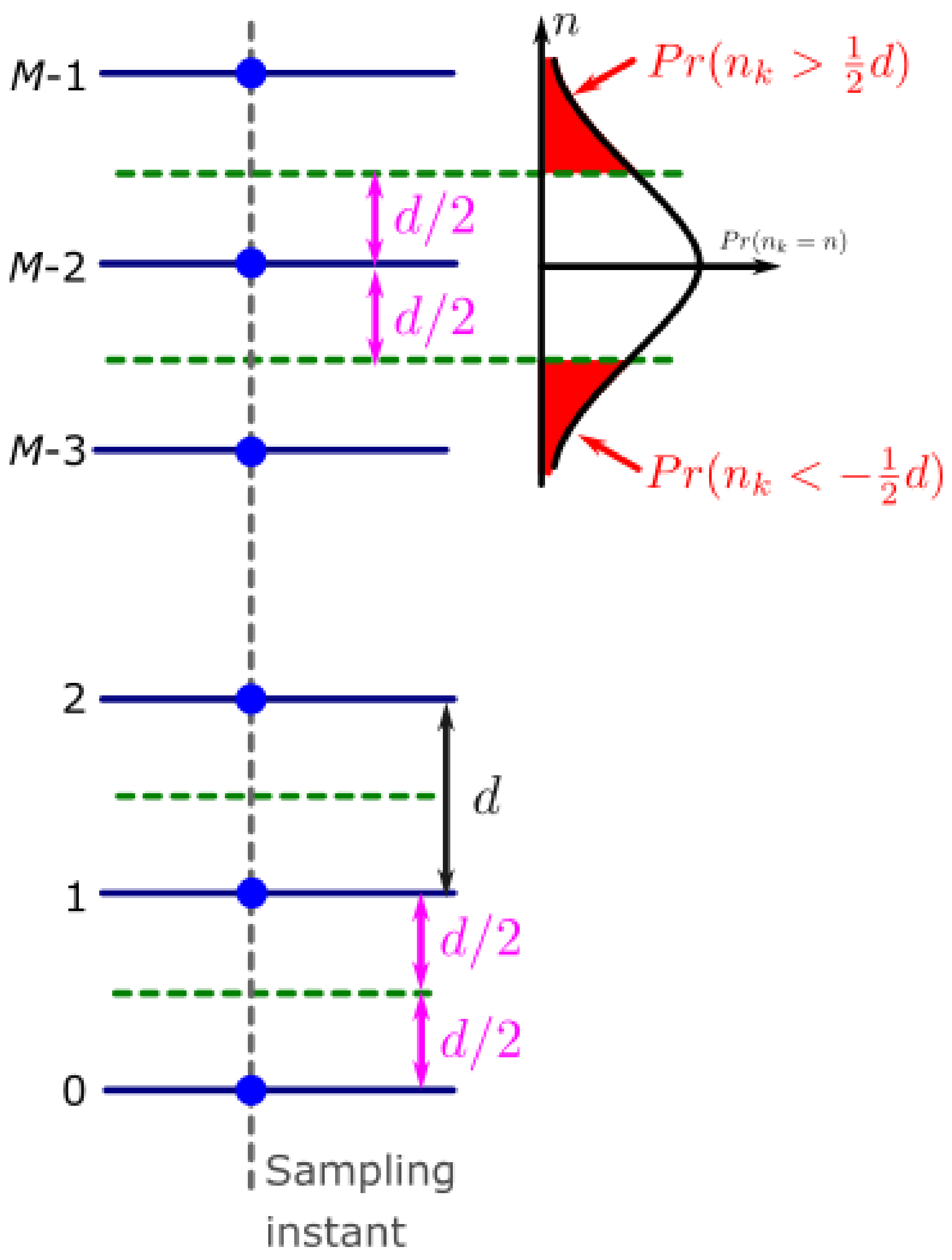
- **Bi-polar signalling provides a 3dB advantage over antipolar signalling for PCM (same error rate for $SNR_B = SNR_U/2$)**

M-PAM Error Probability (1)

- **Consider only the sampled signal (before the slicer)**

- **General expression for M-PAM error probability ε with**
  - Equal distance d between symbols: margin $\frac{d}{2}$

$$\varepsilon = \Pr(x_M) \cdot \Pr\left(n_k < -\frac{1}{2}d\right) + \sum_{m=1}^{M-2} \Pr(x_M) \cdot \Pr\left(|n_k| > \frac{1}{2}d\right) + \Pr(x_0) \cdot \Pr\left(n_k > \frac{1}{2}d\right)$$



M-PAM Error Probability (2)

- **Often we can make some simplifying assumptions (as we are often only interested in orders of magnitude):**

1. All symbol of the constellation are equally likely: $\Pr(x_m) = \frac{1}{M}$

$$\varepsilon = \frac{1}{M} \cdot \Pr\left(n_k < -\frac{1}{2}d\right) + \frac{M-2}{M} \cdot \Pr\left(|n_k| > \frac{1}{2}d\right) + \frac{1}{M} \cdot \Pr\left(n_k > \frac{1}{2}d\right) =$$

$$\frac{M-1}{M} \cdot \Pr\left(|n_k| > \frac{1}{2}d\right)$$

2. Noise is symmetric: $\Pr\left(n_k < -\frac{1}{2}d\right) = \Pr\left(n_k > \frac{1}{2}d\right)$

$$\varepsilon = 2 \cdot \frac{M-1}{M} \cdot \Pr\left(n_k > \frac{1}{2}d\right)$$

3. Ignore the special situation of the nodes on the edge of the constellation **(for large M only)**

$$\varepsilon \approx 2 \cdot \Pr\left(n_k > \frac{1}{2}d\right)$$

Error Probability: Symmetric M-PAM (fixed-BW/given SNR)

- **Consider a symmetric M-PAM constellation with distance d.**
  All symbols have equal probability and BW (SNR) is fixed
- **We are interested in the error rate as function of signal-to-noise ratio (SNR)**
  - Bandwidth is fixed: higher order M results in higher throughput

$$SNR_M = \bar{P}_{\mathcal{O}_M}/\sigma^2$$

  - Some useful expressions

$$\bar{P}_{\mathcal{O}_M} = \frac{M^2 - 1}{3}\left(\frac{d}{2}\right)^2$$

$$\varepsilon_M = 2 \cdot \frac{M-1}{M} \cdot \Pr\left(n_k > \frac{1}{2}d\right)$$

$$\Pr\left(n_k > \frac{1}{2}d\right) = Q\left(\frac{d}{2\sigma}\right)$$

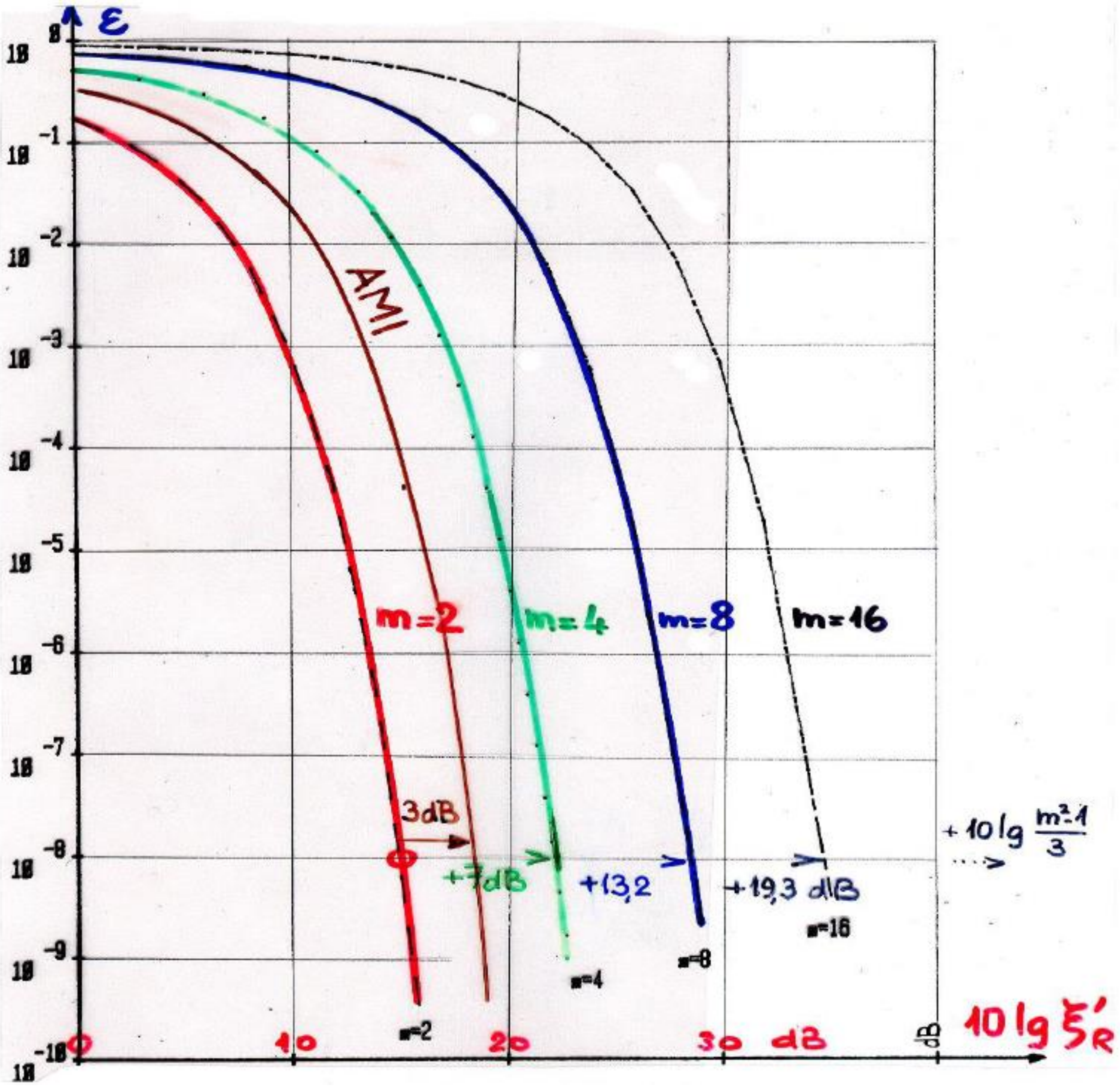
Error Probability: Scaling with Constellation size M

- Symmetric M-PAM (fixed-BW/given SNR)

$$\varepsilon_M = 2 \cdot \frac{M-1}{M} \cdot Q\left(\sqrt{\underbrace{\frac{3}{M^2-1} SNR_M}_{\zeta}}\right)$$

- SNR increase & bandwidth for given throughput and a given error rate

M	$\frac{3}{M^2-1}$	SNR_M	Bandwith $1/\log_2 \mathrm{M}$
2	1/1	1 = 0 dB	1
4	1/5	5 = +6.98 dB	½
8	1/21	21 = +13.2 dB	1/3
16	1/85	85 = +19.3 dB	1/4



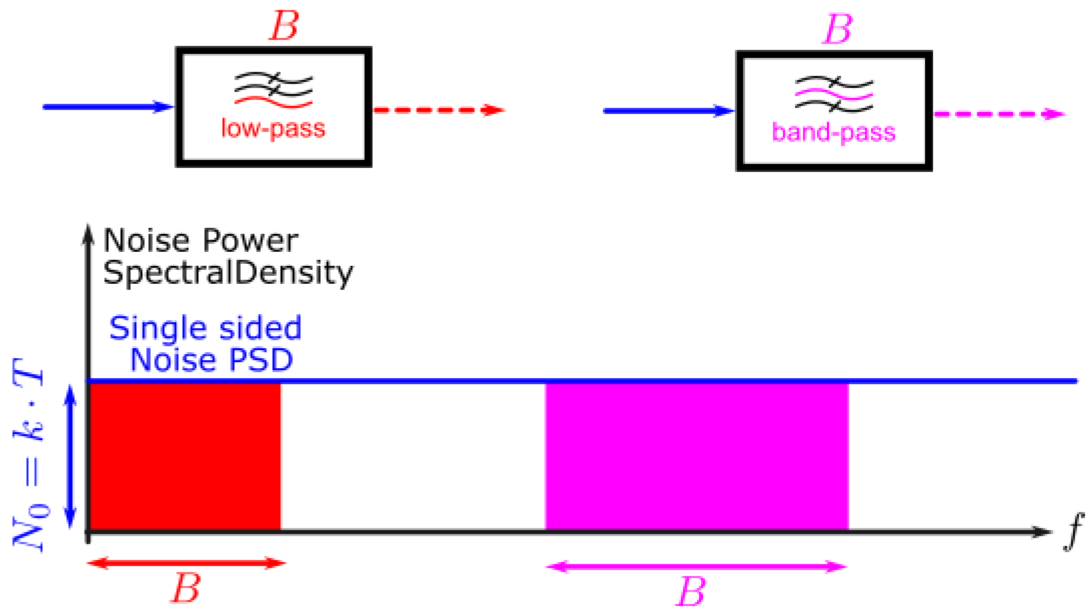
Relationship Between σ^2 and Noise PSD (N_0)

- **We define the SNR $SNR_M = \bar{P}_{\mathcal{O}_M}/\sigma^2$ relative to the noise variance.**
- **The Noise Variance σ^2 corresponds to the total amount of "relevant" noise**
  - As noise is zero mean ($E\{n(t)\} = 0$), we have $E\{|n(t)|^2\} = \sigma_n^2$

- **The "relevant" amount of noise depends on the signal bandwidth**
  - **Single sided:**

$$\sigma^2 = \int_{B_{start}}^{B_{stop}} S_n(f)df = k \cdot T \cdot B = N_0 \cdot B$$

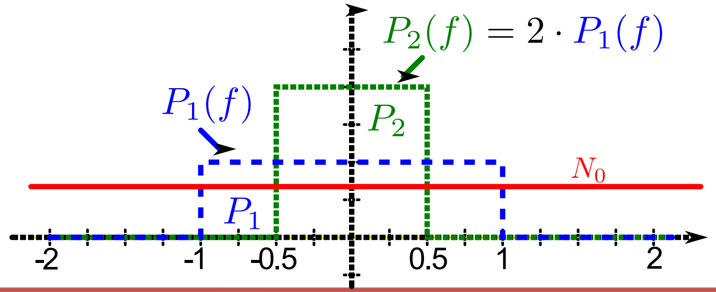
  - Double sided: integration on both positive and negative frequencies, but with a PSD of $N_0/2$

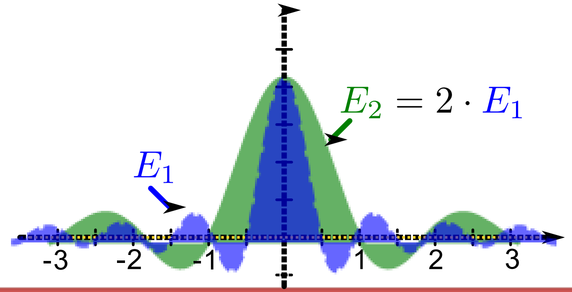


Energy/Power: Scaling with Symbol-Rate (inv. BW)

- **We scale the symbol duration and the used BW (inverse), for given constellation $\mathcal{O}_M$**

	Signal	Noise
Symbol duration	$T_s \rightarrow T_s' = \alpha \cdot T_s$	
Baud rate / BW	$B_s \rightarrow B_s' = \frac{B_s}{\alpha}$	
Energy/symbol	$E_s \rightarrow E_s' = \alpha \cdot E_s$	$E_N = T_s \cdot B_s \cdot N_0 \rightarrow$ $E_N' = \alpha \cdot T_s \cdot \frac{B_s}{\alpha} \cdot N_0 = E_N$
Power	$P_s = \frac{E_s}{T_s} \rightarrow P_s' = \frac{\alpha \cdot E_s}{\alpha \cdot T_s} = P_s$	$P_N = B_s \cdot N_0 \rightarrow P_N' = \frac{B_s}{\alpha} \cdot N_0 = \frac{P_N}{\alpha}$





SNR Scaling with Symbol-Rate (inv. prop. BW)

- **From the previous table, we notice the following:**
  - Signal energy (per symbol) scales with symbol rate, BUT noise energy (per symbol) does not
  - Signal power (per symbol) does not scale with symbol rate, but noise power does scale

- **Signal to noise ratio for scaling symbol duration $T_s \to T_s' = \alpha \cdot T_s$ if the constellation alphabet remains the same**

$$SNR' = \frac{E_s'}{E_N'} = \alpha \cdot \frac{E_s}{E_N} = \alpha \cdot \frac{P_s}{P_N}$$

  - Two interpretations:
    - Signal power remains the same, while noise power scales with bandwidth (inverse to symbol duration)
    - Signal energy increases, while noise energy remains the same (noise is correlated, as reflected by its reduced bandwidth)

Error Rate from an Energy Perspective

- **Longer symbols correspond to expending more power for each symbol**
  - Is there a “fair” way to account for the increase of power when extending symbol duration?

- **Idea: start from an energy-perspective**
  - Using the “energy per symbol” $E_s = P_s \cdot T_s$ allows to express error rate performance independently from the symbol rate, but still relating to the thermal noise constant PSD

  - Assume minimum bandwidth as $B_s = \frac{1}{2 \cdot T_s}$

$$SNR = \frac{P_s \cdot T_s}{T_s \cdot B_s \cdot N_0} = 2 \cdot \frac{E_s}{N_0}$$

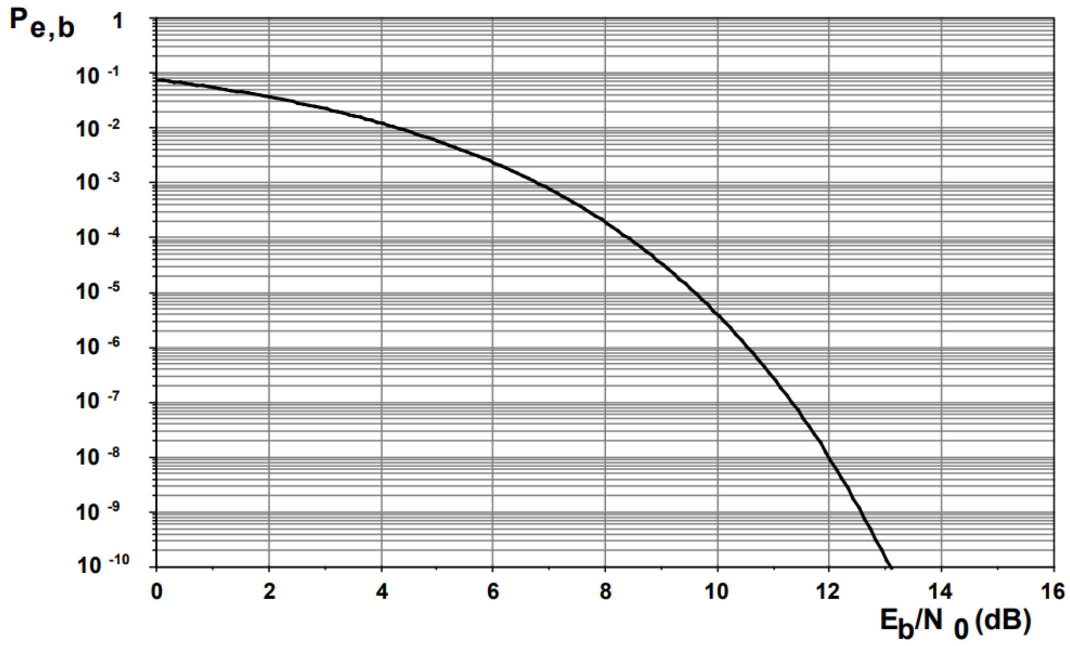
$$\varepsilon_2 = Q\left(\sqrt{2\frac{E_s}{N_0}}\right)$$

Example: Error rates based on Power and Symbol Duration

- **A typical example is a communication system, given the following**
  - Received power P
  - Bit rate f_{bit} and modulation order $2-PAM$ (Bi-Polar)
  - Ideal SINC pulse shape filter
  - Thermal noise PSD: $N_0 = -174\ dBm$

  - What is the error rate?

$$\varepsilon_{2-PAM} = Q\left(\sqrt{2\frac{E_b}{N_0}}\right) = Q\left(\sqrt{2\frac{P\cdot\frac{1}{f_{bit}}}{N_0}}\right)$$



Relationship between SNR, Es, and Eb

- **The three parameters SNR, Es, and Eb are closely related, but translating between them requires additional information**
  - The symbol duration is the link between transmitted power $\boldsymbol{P_s}$ and $\boldsymbol{E_s}$

$$\boldsymbol{E_s = P_s \cdot T_s}$$

  - The modulation order $\boldsymbol{M}$ links the energy-per-bit $\boldsymbol{E_b}$ and the energy per symbol $\boldsymbol{E_s}$

$$\boldsymbol{E_s = E_b \cdot \log_2 M}$$

  - The bandwidth links the noise power and the noise PSD

$$\boldsymbol{\sigma^2 = N_0 \cdot B}$$

$$\boldsymbol{SNR = \frac{P_s}{\sigma^2} = \frac{E_s}{N_0} \cdot \frac{1}{T_s \cdot BW} = \frac{E_b}{N_0} \cdot \frac{\log_2 M}{T_s \cdot B} = \frac{E_b}{N_0} \cdot \gamma}$$

Spectral efficiency $\boldsymbol{\gamma = \frac{\log_2 M}{T_s \cdot B}}$

SNR Scaling with Symbol-Rate (fixed BW)

- **In some systems, we may not have a proper filter to remove the noise outside of the bandwidth:**
  - Energy per symbol scales as $E_s = T_s \cdot P_s$
  - Noise energy is proportional to the fixed bandwidth B

$$SNR = \frac{P_s \cdot T_s}{T_s \cdot B \cdot N_0} = \frac{P_s}{B \cdot N_0}$$

Note: $B > \frac{f_s}{2}$

- **Scaling symbol duration without adapting the receive filter (before sampling) will not have an impact on SNR**

Bipolar M-PAM Error Probability with PSD

- **Increasing constellation order M allows to reduce the bandwidth (impact on noise as $\sigma^2 = N_0 \cdot B$**
- **Consider a M-PAM signal with a given bit rate f_{bit}, a pulse shaping filter with minimum spectrum occupation, and power $\bar{P}_s$**
- **Calculate the error rate ε_M as a function of the one-sided noise PSD N_0 and the bit rate f_{bit}**
  - Some useful expressions/hints

$$\varepsilon_M = 2 \cdot \frac{M-1}{M} \cdot Q\left(\sqrt{\frac{3}{M^2-1} SNR_M}\right)$$

Bandwidth: $\frac{f_{bit}}{2\log_2 M}$

Symbol-duration: $T_s = \frac{\log_2 M}{f_{bit}}$ Bit-duration: $T_{bit} = \frac{1}{f_{bit}}$

Bipolar M-PAM Error Probability with Fixed Throughput

- **First we obtain the Error Probability as a function of the symbol energy for the minimum required bandwidth**

$$\varepsilon_M = 2 \cdot \frac{M-1}{M} \cdot Q\left(\sqrt{\frac{3 \cdot 2}{M^2-1}\frac{E_s}{N_0}}\right)$$

- **Fixing the throughput** f_{bit} **implies that** $T_s = \frac{\log_2 M}{f_{bit}}$ and with $E_s = \bar{P}_s \cdot \frac{\log_2 M}{f_{bit}}$

$$\varepsilon_M = 2 \cdot \frac{M-1}{M} \cdot Q\left(\sqrt{\frac{3 \cdot 2}{M^2-1}\frac{\bar{P}_s}{N_0}\frac{\log_2 M}{f_{bit}}}\right)$$

  - Increasing the constellation implies a significant penalty in error rate performance
  - the increase in symbol energy when packing more bits with a fixed E_b provides only a slight compensation for the penalty

Energy per Bit

- **Similar to the "energy per symbol, we can define the "energy per bit"**

$$E_{\boldsymbol{b}} = \frac{\bar{P}_s \cdot \log_2 M}{f_{bit}} = E_s / \log_2 M$$

- E_b **provides another level of normalization that simplifies the comparison as it removes the dependency on the number of bits per symbol**

$$\varepsilon_M = 2 \cdot \frac{M-1}{M} \cdot Q\left(\sqrt{\frac{3 \cdot 2}{M^2 - 1} \frac{E_b}{N_0} \cdot \log_2 M}\right)$$

  - Constant "energy per bit" is equivalent to saying we pack more bits into a symbol and extend the symbol duration accordingly.