Telecommunications Systems Exercises 5

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5.1 Entropy

Consider a ternary symbol set $s \in \{a, b, c\}$. Two out of these three symbols have the same probability of occurrence $P_S(a) = p$ and $P_S(b) = p$.

- What is the probability of occurrence of the third symbol $P_S(c)$?
- Can you compute and plot the entropy $H(s)$ of S?
- Which probability p maximizes the entropy?

5.2 Entropy and Conditional Entropy

Suppose that X is a random variable whose entropy $H(X)$ is 8 bits. Suppose that $Y(X)$ is a deterministic function that takes on a different value for each value of X.

- What then is $H(Y)$, the entropy of Y ?
- What is $H(Y|X)$, the conditional entropy of Y given X?
- What is $H(X|Y)$, the conditional entropy of X given Y?
- What is $H(X, Y)$, the joint entropy of X and Y ?

5.3 BSC Mutual Information and Supported Rate

1. Consider a Binary Symmetric Channel as described in the slides with an error probability p. Calculate the maximum supported rate by this BSC as a function of p

2. Consider a Binary Erasure Channel as described in the slides with an erasure probability p_ϵ. Calculate the maximum supported rate by this BEC as a function of p_ϵ

5.4 Analyzing Channel Capacity and Spectral Efficiency

Given a communication system with:

- Power $P = 30dBm$
- Noise spectral density $N_0 = 10^{-9}W/Hz$

1. Compute the channel capacity C in bits per second for each of the given bandwidths:

$$B = 1\,\text{MHz}, \quad B = 5\,\text{MHz}, \quad B = 20\,\text{MHz}$$

2. For each case, calculate the **spectral efficiency** $\frac{C}{B}$ in bits/s/Hz.
3. Describe and explain the trend observed in both capacity and spectral efficiency as bandwidth increases.
4. Based on your results, explain what is meant by *"capacity saturates for large bandwidth"* using numerical evidence.

5.5 Power Increase and Channel Coding Implications

1. Consider two SNR scenarios:
   - Low SNR: 0 dB
   - High SNR: 20 dB

   Assume bandwidth $B = 1$ Hz and use the Shannon capacity formula:

$$C = B \cdot \log_2(1 + \text{SNR})$$

   (a) Compute the capacity for both SNR values.

   (b) Increase power by a factor of 4 (i.e., add +6 dB). Recalculate the new capacities and compare the rate of increase in each regime.

   (c) Interpret what the results suggest for system design under low vs. high SNR.

2. Consider the impact of channel coding using the capacity and uncoded modulation curves.

   (a) Why is there a large gap (approximately 9 dB) between the uncoded modulation and the Shannon capacity?

   (b) How does this motivate the use of channel coding in digital communications?

   (c) What is the minimum energy-per-bit to noise ratio $\left(\frac{E_b}{N_0}\right)_{\min}$ for reliable transmission, and what happens as the rate $R \to 0$?

5.6 Automatic Repeat Request

A file of size 200 kilobytes (1 kilobyte = 8,000 bits) is to be transmitted using an Automatic Repeat Request (ARQ) scheme. The bit-error probability after decoding is $P_e = 10^{-6}$.

1. Calculate the total number of bits in the file.
2. Calculate the block error probability assuming the whole file is transmitted as one block.
3. If the file is instead split into 100 blocks of equal size, calculate the new block size and the block error probability for each block.
4. Discuss the trade-off between using fewer large blocks versus more smaller blocks in the context of ARQ efficiency.

5.7 Error Detection and Correction Using Block Codes in BSC

Consider a block code used over a binary symmetric channel (BSC) with the following properties:

- The code is a block code with $k = 4$ information bits and $n = 7$ total bits.
- The code has a minimum Hamming distance $d_{\text{min}} = 3$.

Answer the following:

1. Compute the code rate R.
2. How many bit errors can this code detect?
3. How many bit errors can it correct?