Telecommunications Systems Exercise 2

Spring semester 2025

1 Stochastic signals: Noise in Wireless

In wireless communication, a received signal is often corrupted by **thermal noise** at the antenna, where the noise can be modeled as a **wide-sense stationary (WSS) process** and its spectral characteristics affect system performance. In many practical systems, due to some subsequent filters, the **impulse response** $h(t)$ causes the received noise to be colored (i.e., its power is not uniform across all frequencies). The received noise $Y(t)$ is related to the original white noise $X(t)$ through convolution:

$$Y(t) = X(t) * h(t)$$

where $h(t)$ is the impulse response of a **low-pass filter**.

1. The power spectral density (PSD) of the input noise $X(t)$ is flat across all frequencies (i.e., it is **white noise**). Given that the impulse response is modeled as an ideal **low-pass filter** with cutoff frequency B_c, sketch the approximate PSD $P_Y(f)$ of the output noise $Y(t)$. Explain why the noise is no longer white.

2. Assume the **autocorrelation function** of the filtered noise $Y(t)$ is given by:

   $$R_Y(\tau) = A\frac{\sin(2\pi B_c\tau)}{2\pi B_c\tau}$$

   Describe how the correlation between noise values at different times is affected by the bandwidth B_c of the filter.

2 AM signals: Square Wave Mixing

In practical RF CMOS circuits, instead of using a sinusoidal carrier for modulation, a **square wave** is often used in **IQ mixing** due to implementation constraints. A square wave with a fundamental frequency f_0 can be expressed as a **Fourier series**:

$$\mathrm{sq}(t) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{4}{\pi k}\sin(2\pi k f_0 t)$$

where f_0 is the **fundamental frequency** of the square wave. This means that in addition to the fundamental component at f_0, the square wave also contains **harmonics** at frequencies $3f_0, 5f_0, 7f_0, \ldots$. If a baseband signal $m(t)$ is up-converted using a square wave carrier, the modulated signal is:

$$s_{\text{mod}}(t) = m(t) \cdot \text{sq}(t)$$

This introduces multiple spectral components, making analysis important.

1. Assume an **ideal case** where the baseband message signal $m(t)$ is band-limited to B Hz and modulated using a **sinusoidal carrier** at frequency f_0

   $$s_{\text{mod}}(t) = m(t)\cos(2\pi f_0 t)$$

   Sketch the frequency domain representation $S_{\text{mod}}(f)$ of this signal.

2. Now, assume the modulation is performed using a **square wave** instead of a sinusoidal carrier

   $$s_{\text{mod}}(t) = m(t) \cdot \text{sq}(t)$$

   (a) Derive the **frequency components** introduced by the square wave using its Fourier series expansion.

   (b) Show that the modulated signal contains harmonics at $3f_0, 5f_0, \ldots$.

   (c) Sketch the spectrum of the modulated signal.

3. Explain why the additional harmonics at $3f_0, 5f_0, \ldots$ do not necessarily degrade communication performance.

4. What type of **filtering technique** can be used in an RF system to remove these unwanted harmonics?

5. In AM radio transmission, why does square wave mixing still work effectively despite the presence of these extra spectral components?

3 Digitalize your AM signal

As a good radio amateur, you sit on top of a hill and listen to some AM signals sent by another radio amateur on another hill somewhere. Because you missed on sport lately, you really did not want to carry all that analog AM hardware that is way to heavy. Instead, you made a small circuit board with an analog to digital converter that samples the received AM signal at sampling frequency f_s. But will it work ?
The other radio amateur emits an amplitude-modulated single tone $m(t)$ defined by:

$$m(t) = \cos(2\pi f_1 t) \cdot \sin(2\pi f_c t)$$

Consider $\tilde{m}(t)$, which is the signal $m(t)$, sampled at frequency f_s.

For the parameters:

- $f_c = 1\ MHz$
- $f_1 = 250\ kHz$
- $f_s = 2.1\ MHz$

1. Express $\tilde{M}(f)$, the Fourier transform of $\tilde{m}(t)$. You only need to find and express explicitly the terms for which $-\frac{f_s}{2} \leq f \leq \frac{f_s}{2}$.
2. Show that if $\tilde{m}(t)$ is used to reconstruct $m(t)$, the resulting signal will not correspond to the original $m(t)$ by finding the expression of the inverse Fourier transform of $\tilde{M}(f)$.
3. Explain why it does not correspond and how this could be prevented.

Hint: sketch the spectrum of the signals.

4 You can't have the cake and eat it too

Remember Heisenberg uncertainty principle ? Good ! But you wonder what does it have to do with telecommunications, fair enough. The principle says that for a given particle, one may not know its position with an arbitrarily high accuracy without loosing all knowledge on the velocity. This principle can be reformulated in many ways and the core of it is: you can't know it all. It applies for signals too, or as signal processing people would say, the principle of Heisenberg is a consequence of the Fourier transform. Here is why: a signal cannot be bounded in both time and spectrum. It means that a signal which starts at a moment in time and ends at another moment in time and occupy a finite band does not exist... you can't have the cake and eat it too. Then how can the TV signal that starts and ends in time be sampled by the receiver without an infinite sampling frequency ? Or how did we know at $t = -\infty$ before the Big Bang that we had to start broadcasting the signal that has a strictly limited bandwidth ? Be reassured, clever engineers have found some workaround so that you can enjoy 5G despite not eating the cake :)
Let's have a look at it. In your computer, the processor communicates with the RAM bars on copper lines along the motherboard. Let's assume for simplicity that 0 V means "0" and 5 V means "1". The processor wants to send a "1" and therefore transmits a perfect square pulse defined by the rectangle function :

$$s(t) = 5 \prod \left(\frac{t}{\tau} \right)$$

Where τ is the duration of the pulse.

1. Sketch the signal $s(t)$.
2. Express the spectrum $S(f)$ defined by the Fourier transform of $s(t)$ and sketch it (in magnitude only).
3. What shall be the sampling frequency f_s of the RAM bar to ensure that the message sent by the processor is received correctly (i.e. it can be perfectly reconstructed) ?

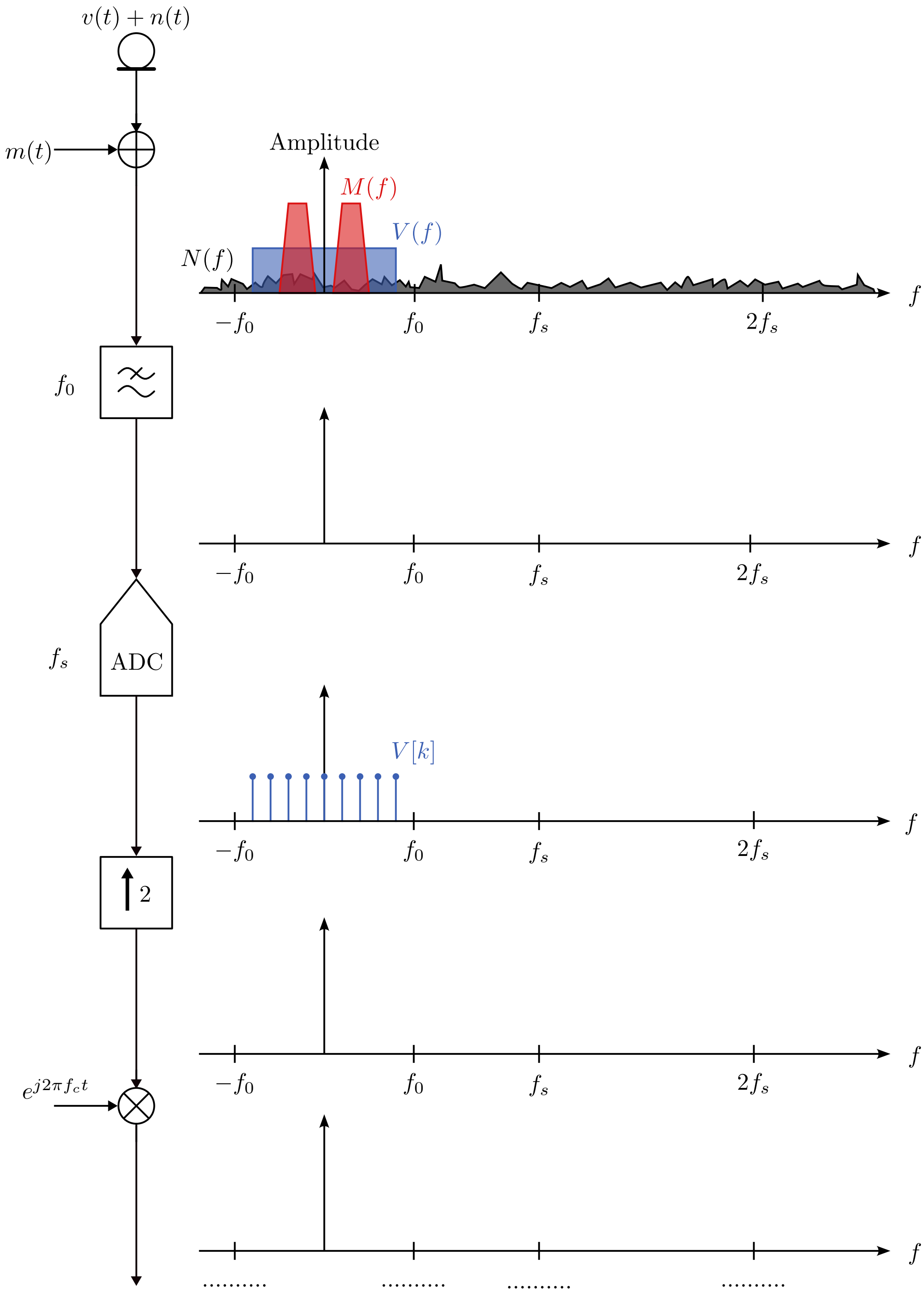
5 Time to sing

You want to play karaoke and ensure that everybody who can listen to the electromagnetic spectrum around you will enjoy it too. You found a small radio transmitter to do that. It only requires that you provide it with a digital signal that is already up-sampled and in the pass-band. You therefore set up a small system that includes:

- A microphone;
- An analog combiner;
- An ideal low-pass filter with cut-off frequency f_0;
- An Analog-to-Digital Converter (ADC) sampling at f_s;
- An up-sampler (low-pass filter included) that doubles the number of samples;
- A modulator that simply multiplies the signal by a pure complex carrier at frequency f_c.

The combiner merges (adds) your voice $v(t)$ and the music $m(t)$. Unfortunately, your microphone is not of the best quality and adds a lot of white noise $n(t)$ to your voice. The final signal is $v(t)+m(t)+n(t)$ and their spectra are illustrated down-below respectively in blue, red and dark.

Between each step of your system is given a set of axes. The spectrum is represented on the first step. Sketch the spectrum as it evolves through the system and fills the missing labels where necessary. Keep the color code for voice, music and noise. The spectrum of your voice after sampling is also represented to indicate the sampling rate. Note that for every step, f_s always refer to the initial sampling frequency, even after up-sampling.


$v(t) + n(t)$
$m(t)$
Amplitude
$M(f)$
$V(f)$
$N(f)$
f
$-f_0$
f_0
f_s
$2f_s$
f_0
ADC
f_s
$V[k]$
2
$e^{j2\pi f_c t}$