EPFL

Linear Transformations
EE-312

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Spring 2022

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$P_1(x), P_2(x)$ $\int_{-1}^{1} dx\, P_1(x)\, P_2(x)$

$a_0, \ldots, a_{n-1}$

$\hat{e}_0 = \frac{a_0}{\|a_0\|}$

$\hat{a}_1 = a_1 - \langle a_1, a_0 \rangle e_1$

$a_i(x)$ Gram-Schmidt

a_0

$a_1 - \langle a_1, a_0 \rangle a_0 = \tilde{a}_1$

$q_1(x) = \frac{\tilde{a}_1(x)}{\|\tilde{a}_1(x)\|}$

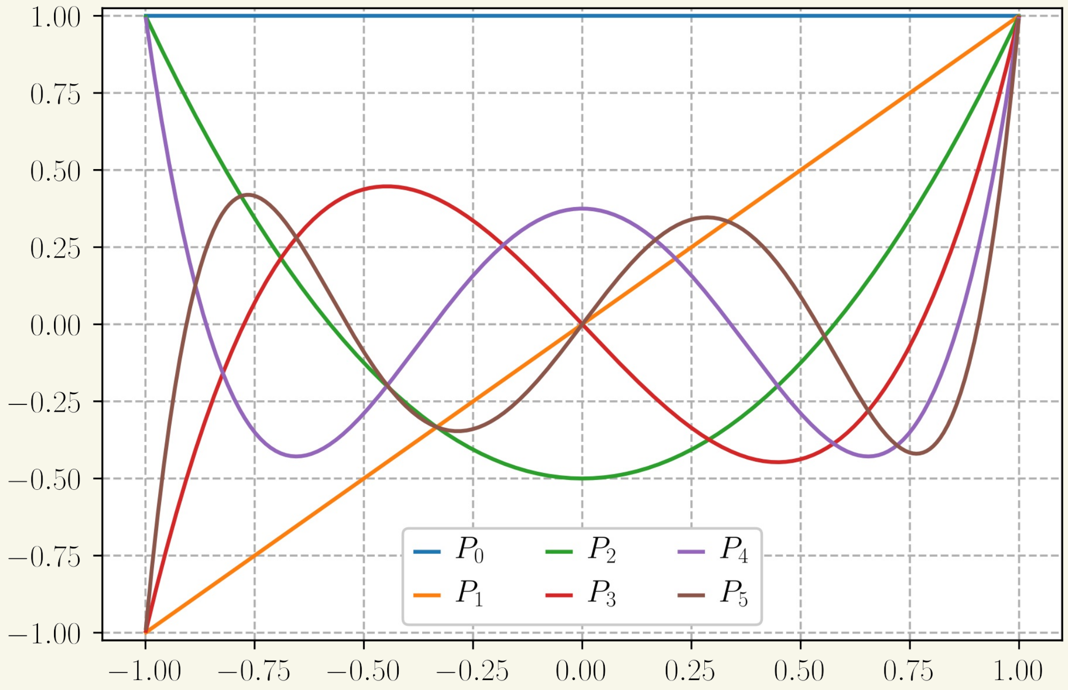
$\sqrt{\langle \tilde{a}_1, \tilde{a}_1 \rangle}$

$\langle \hat{a}_1, a_0 \rangle = \langle a_1, a_0 \rangle - \langle a_1, a_0 \rangle \langle a_0, a_0 \rangle$

1

$$\int_{-1}^{1} dx\, f\, g$$

$$\int_{-1}^{1} \frac{dx}{1-x^2}\, f\, g$$



Legendre.

Linear Transformations

This is probably the most intuitive view of linear algebra and among its most useful applications

A mapping between two vector spaces, that satisfies the axiom of linearity

$(\mathcal{V},\mathbb{F})$ and $(\mathcal{W},\mathbb{F})$ two vector spaces. $\mathcal{L}:\mathcal{V}\mapsto\mathcal{W}$ is a **linear transformation** IFF

$$\mathcal{L}(\alpha v_1+\beta v_2)=\alpha\mathcal{L}(v_1)+\beta\mathcal{L}(v_2)\,\forall\alpha,\ \beta\in\mathbb{F}\text{ and }v_1,v_2\in\mathcal{V}$$

Examples: $(\mathcal{V},\mathbb{F})=(\mathcal{P}^n,\mathbb{R}),\quad(\mathcal{W},\mathbb{F})=(\mathcal{P}^{n-1},\mathbb{R})$

$$\mathcal{L}(v)=v'$$

Most revealing is the case of a linear trans. between two <u>euclidean</u> vector spaces

$$L : V \longrightarrow W$$

$\{v_i\}_{i=1,\dots,n}$ ONB $\qquad \{w_j\}_{j=1,\dots,m}$ ONB

$$L v_i = \sum_{j=1}^{m} \beta_j w_j = \sum_{j=1}^{m} \underbrace{\langle w_j, L v_i \rangle}_{A[j,i]} w_j$$

$$f \in V \quad f = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \langle v_i, f \rangle v_i$$

$$g = Lf = \sum_{i=1}^{n} \alpha_i L v_i = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} A[j,i] w_j$$

$$\beta_k = \langle w_k, g \rangle = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} A[j,i] \langle w_k, w_j \rangle = \sum_i \alpha_i \sum_j A[j,i] \delta_{kj}$$

$$= \sum_i \alpha_i A[k,i] = (A\alpha)_k$$

$$f \longrightarrow \alpha$$

$$Lf \longrightarrow \beta$$

$$\beta = A\alpha$$

Linear Transformations

Most revealing is the case of a linear trans. between two euclidean vector spaces

Classic: a linear transformation of vectors, visualised as "arrows"

I strongly advise you check 3blue1brown's youtube channel
"Essence of Linear Algebra", for great pedagogical visualisations

"*Unfortunately, no one can be told what the Matrix is. You have to see it for yourself.*"
Morpheus

Linear Transformations

Linear System and Linear Dynamical System

$$\frac{d}{dt}x(t) = Ax(t),\ x \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n \times n}$$

Examples: $n = 1,\ \frac{d}{dt}x(t) = ax(t) \Rightarrow x(t) = x(0)e^{at}$ reminds you of something ?

$n = 2,$

$$\frac{d}{dt}x_1(t) = a_{11}x_1(t) + \boxed{a_{12}}x_2(t)$$

$$\frac{d}{dt}x_2(t) = a_{21}x_1(t) + \boxed{a_{22}}x_2(t)$$

predation

Linearised "predator-prey" model

Linear Transformations

A mapping between two vector spaces, that satisfies the axiom of linearity

$(\mathcal{V}, \mathbb{F})$ and $(\mathcal{W}, \mathbb{F})$ two vector spaces. $\mathcal{L} : \mathcal{V} \mapsto \mathcal{W}$ is a **linear transformation** IFF

$$\mathcal{L}(\alpha v_1 + \beta v_2) = \alpha \mathcal{L}(v_1) + \beta \mathcal{L}(v_2) \, \forall \alpha, \, \beta \in \mathbb{F} \text{ and } v_1, v_2 \in \mathcal{V}$$

If you decorate the two vector spaces with <u>bases V and W</u>

$$\forall u \in \mathcal{V} \quad u = \sum_i \alpha_i v_i \qquad \mathcal{L}u = \sum_i \alpha_i \mathcal{L} v_i = (\mathcal{L}V)\alpha$$

concatenate all vectors $\mathcal{L}v_i$ as matrix $\mathcal{L}V$

$$\mathcal{L}v_i \in \mathcal{W} \qquad \mathcal{L}v_i = \sum_j w_j \beta_{ji}$$

$$= WB_i$$

i-th column of B

$$\mathcal{L}u = (\mathcal{L}V)\alpha$$
$$= WB\alpha$$

holds $\forall \alpha$

$$\mathcal{L} = WB$$

Linear Transformations

In particular for two euclidian vector spaces $\mathcal{V} = \mathbb{R}^n$, $\mathcal{W} = \mathbb{R}^m$

$$\begin{aligned} \mathcal{L}u &= (\mathcal{L}V)\alpha \\ &= WB\alpha \end{aligned}$$

pick canonical bases $\mathcal{L} = B \in \mathbb{R}^{m \times n}$

any matrix is a linear transformation
between two vector euclidean spaces

what happens if we change basis ?

The four fundamental subspaces

$A : \mathcal{V} \to \mathcal{W}$ a linear transformation

The **range** (or the image) of A $\quad \mathcal{R}(A) = \{w \in \mathcal{W} : w = Av \text{ for some } v \in \mathcal{V}\}$

$$\mathcal{R}(A) = \{Av : v \in \mathcal{V}\}$$

$$\mathcal{R}(A) \subseteq \mathcal{W}$$

$Av = w$

$v' = v + v_0$

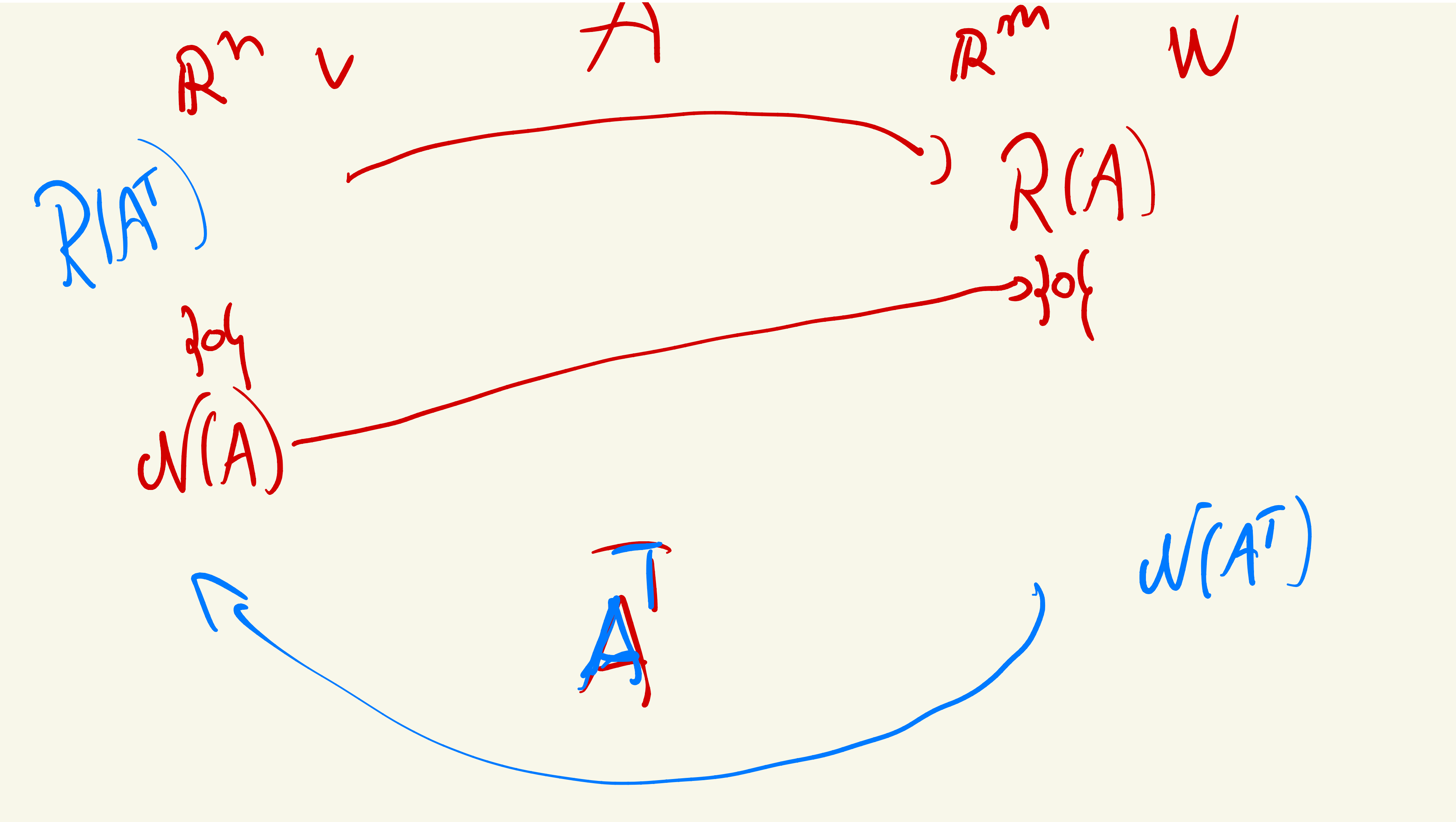
$v_0 \in \mathcal{N}(A)$

The **nullspace** (or the kernel) of A $\quad \mathcal{N}(A) = \{v \in \mathcal{V} : Av = 0\}$

$$\mathcal{N}(A) \subseteq \mathcal{V}$$

$Av' = w$

$$A \in \mathbb{R}^{m \times n}, \quad A = [a_1, \ldots, a_n] \Rightarrow \mathcal{R}(A) = \mathrm{Sp}(A)$$



$\mathbb{R}^n$ V
A
$\mathbb{R}^m$ W
$R(A^T)$
$R(A)$
$\{0\}$
$\{0\}$
$N(A)$
A^T
$N(A^T)$

The four fundamental subspaces

For any $x \in \mathcal{N}(A)$

$$\begin{aligned} Ax = 0 &\Rightarrow y^T A x = 0 \quad \forall y \in R^m \\ &\Rightarrow (A^T y)^T x = 0 \\ &\Rightarrow x \in \mathcal{R}(A^T)^\perp \end{aligned}$$

$\Longrightarrow$ $\mathcal{N}(A)^\perp = \mathcal{R}(A^T)$

For any $y \in \mathcal{N}(A^T)$

$$\begin{aligned} A^T y = 0 &\Rightarrow x^T A^T y = 0 \quad \forall x \in R^n \\ &\Rightarrow (Ax)^T y = 0 \\ &\Rightarrow y \in \mathcal{R}(A)^\perp \end{aligned}$$

$\Longrightarrow$ $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$

The four fundamental subspaces

Why are they so "fundamental" ?

They are linked to fundamental properties of A as a linear transformation

Let $A : \mathcal{V} \to \mathcal{W}$

A is **onto (or surjective)** if $\mathcal{R}(A) = \mathcal{W}$

A is **1-to-1 (or injective)** if $\mathcal{N}(A) = 0$

Equivalently:

$a)\, Av_1 = Av_2 \Rightarrow v_1 = v_2$

$b)\, v_1 \neq v_2 \Rightarrow Av_1 \neq Av_2$

Matrix Rank

Let $A : \mathcal{V} \to \mathcal{W}$

A is **onto (or surjective)** if $\mathcal{R}(A) = \mathcal{W}$

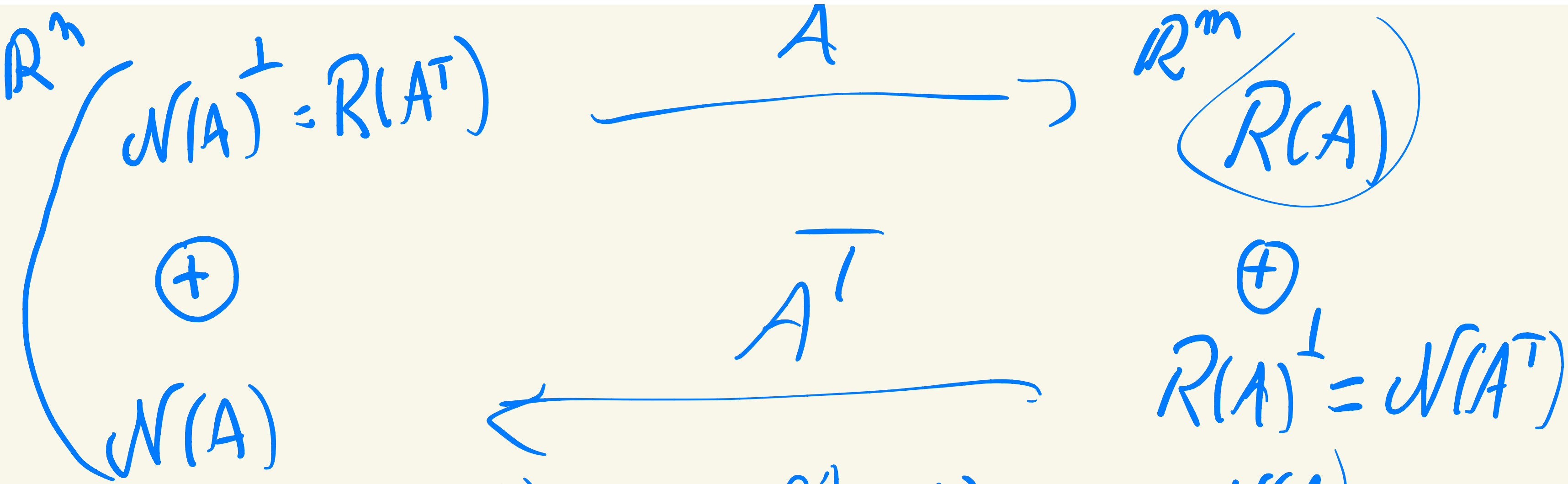
A is **1-to-1 (or injective)** if $\mathcal{N}(A) = 0$

The dimension of these subspaces is of particular significance !

$\text{rank}(A) = \dim(\mathcal{R}(A))$ and is the maximum number of independent columns **column rank!**

Note: $\dim(\mathcal{R}(A^T))$ is the maximum number of independent rows **row rank!**

$\text{nullity}(A) = \dim(\mathcal{N}(A))$



$$\text{rank}(A) = \dim(R(A)) \qquad \text{nullity}(A) = \dim N(A)$$

$$n = \dim N(A) + \dim R(A^T) = \text{nullity}(A) + \text{rank}(A^T)$$

Fabriquons $T : \mathbb{R}^n \to \mathbb{R}^m$

$\mathcal{N}(A)^\perp \to \mathcal{R}(A)$ $\quad T(x) = A(x)$

$Tx = Ax$

$\mathcal{N}(T) \overset{?}{=} \{0\}$ T est injective

$\mathcal{R}(T) \overset{?}{=} \mathcal{R}(A)$ T est surjective

Bijective de $\mathcal{N}(A)^\perp$ dans $\mathcal{R}(A)$

$T : U \to Q$

Prenons une base de $\mathcal{N}(A)^\perp$ v_i $i=1,\dots,n$

une base de $R(A)$? $\boxed{(Tv_i) \text{ sont une base de } R(A) \quad i=1,\dots,n}$

CONTRA

Supposons Tv_i $i=1,\dots,n$ sont <u>lin. dép</u>

$$Tv_1 = \sum_{k=2}^{n} \alpha_k Tv_k$$

$$T\left(v_1 - \sum_{k=2}^{n} \alpha_k v_k\right) = 0 \iff v_1 = \sum_{k=2}^{n} \alpha_k v_k$$

Impossible car $\{v_i\}$ sont une base !!

$$\Rightarrow \dim R(A) = n = \dim \mathcal{N}(A)^\perp = \dim R(A^T) = n$$

Matrix Rank

And all these characterisations are equivalent !

$A : \mathbb{R}^n \to \mathbb{R}^m \;\; \dim(\mathcal{R}(A)) = \dim(\mathcal{N}(A)^{\perp}) = \dim(\mathcal{R}(A^T))$

$\boxed{\text{row rank of } A = \text{column rank of } A = n \text{ - nullity}(A)}$

Second part of theorem already proved before.

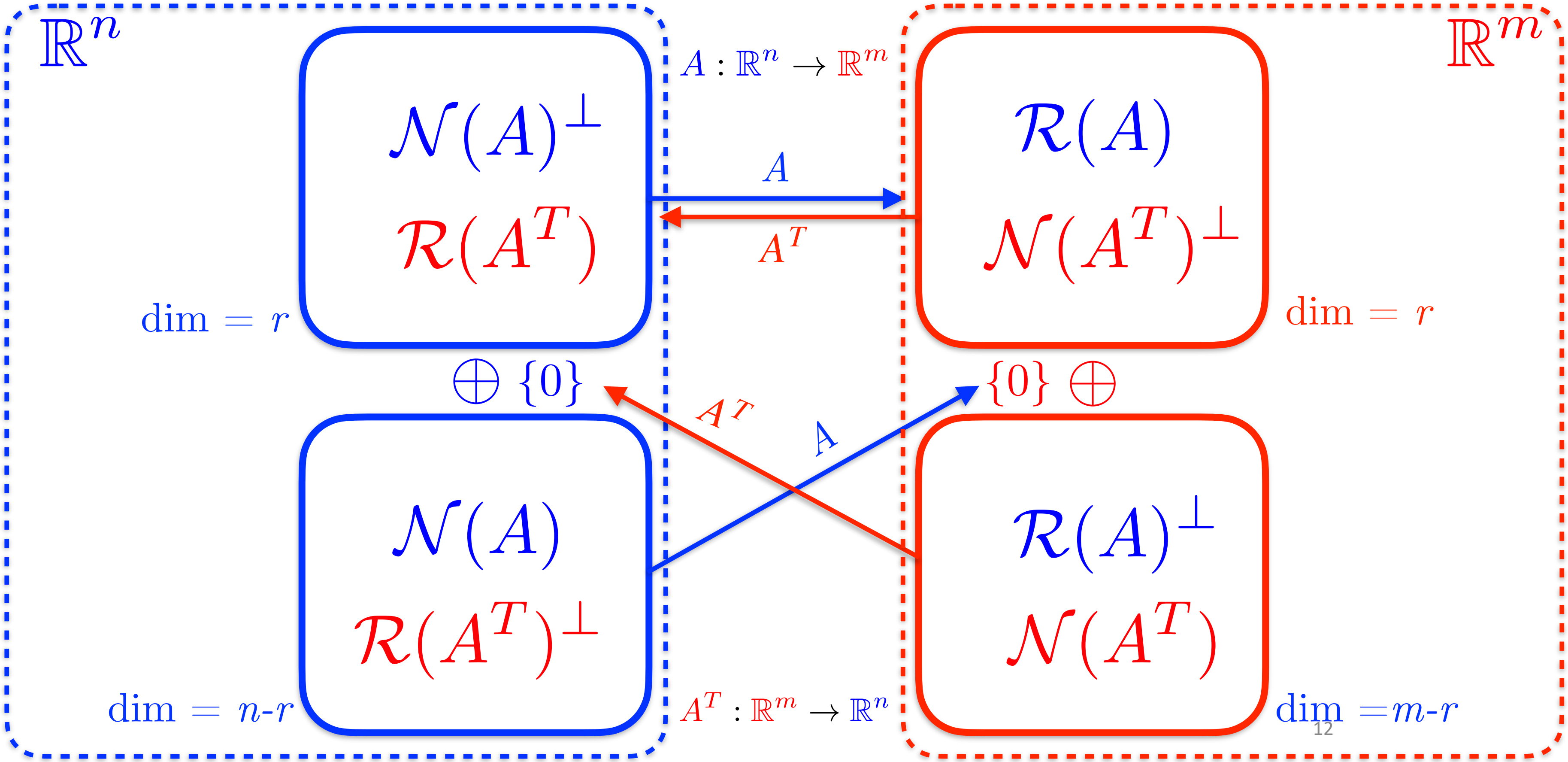
Consider the restriction: $T : \mathcal{N}(A)^{\perp} \to \mathcal{R}(A) \quad Tv = Av \quad \forall v \in \mathcal{N}(A)^{\perp}$

Show that T is <u>bijective</u>

Any basis of $\mathcal{N}(A)^{\perp}$ is mapped to a basis of $\mathcal{R}(A)$ by T

$\Rightarrow \dim(\mathcal{N}(A)^{\perp}) = \dim(\mathcal{R}(A))$

The four fundamental subspaces



A note on invertibility

A linear transformation is **invertible** if and only if it is **bijective** (1-1 and onto)

ex: $T : \mathcal{N}(A)^\perp \to \mathcal{R}(A)$ $\quad Tv = Av \quad \forall v \in \mathcal{N}(A)^\perp$

Consider $A : \mathbb{R}^n \to \mathbb{R}^n$ and suppose it has n linearly independent columns $\mathcal{R}(A) = \mathbb{R}^n$

full-rank

A is onto

$$\forall y \in \mathbb{R}^n \,\exists\, x_1, ..., x_n \text{ s.t. } y = a_1 x_1 + \cdots + a_n x_n$$

unique

A is 1-1

$$y = Ax$$

x → depends on A and y

$$x = A^{-1} y$$

linear

> $A : \mathcal{V} \to \mathcal{W}$ is **invertible** if and only if it is **bijective.**
> If A is invertible then $\dim(\mathcal{V}) = \dim(\mathcal{W})$
> $A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (non-singular) if and only if $\text{rank}(A) = n$

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