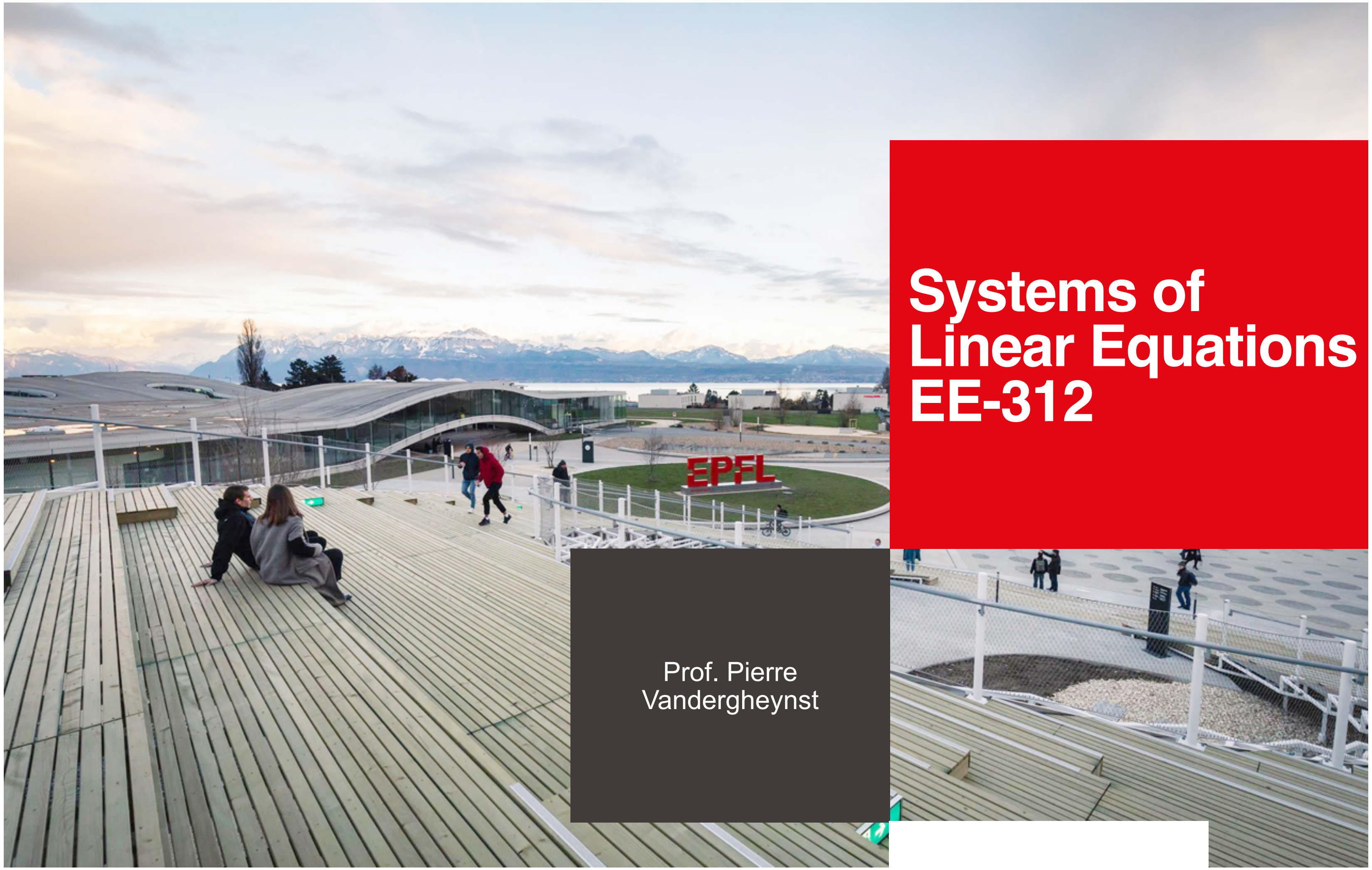

Systems of Linear Equations EE-312

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The unreasonable power of linear systems

$$Ax = b$$

Linear dynamics: $\dot{x}(t) = Ax(t)$ or $x_{k+1} = Ax_k$

Control: $x_{k+1} = Ax_k + Bu_k$

Data Science: $a_{i1}, \cdots, a_{in} \mapsto b_i$ $\qquad b_i = x_0 + a_{i1}x_1 + \cdots + a_{in}x_n$

features (predictors) characteristic Linear Model

Row & Column Views

A gentle reminder before we proceed

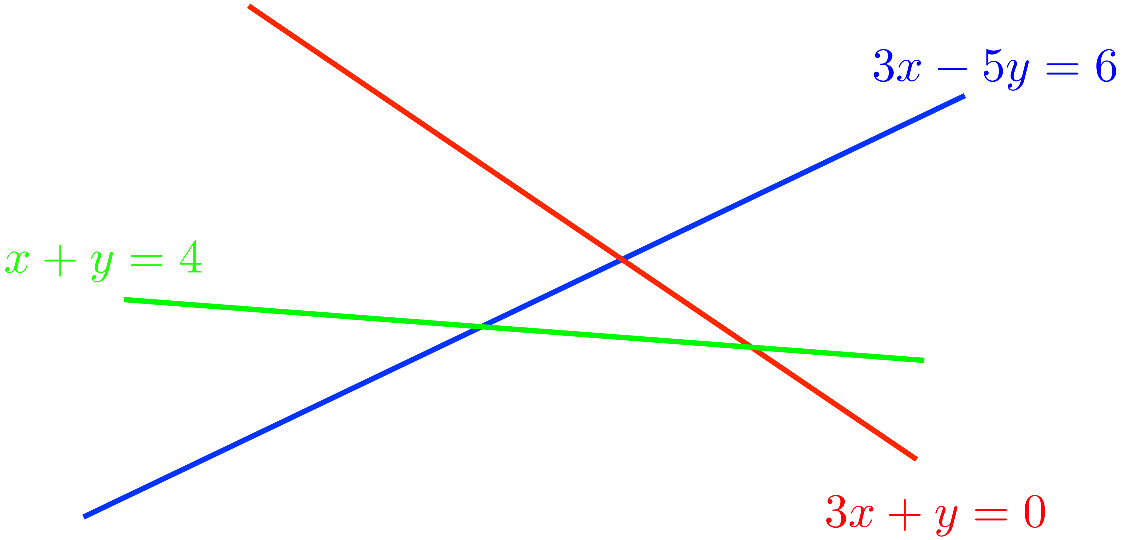
$$Ax = b,\ A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$$

Row view: each row of A defines a linear equation (matrix entries = coefficients)

m equations of the form $\displaystyle\sum_{j=1}^{n} a_{ij}x_j = b_i$ each equation is a line constrain that "carves out" the solution

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix} \qquad a_i^T x = b_i$$

Row View



Row & Column Views

A gentle reminder before we proceed

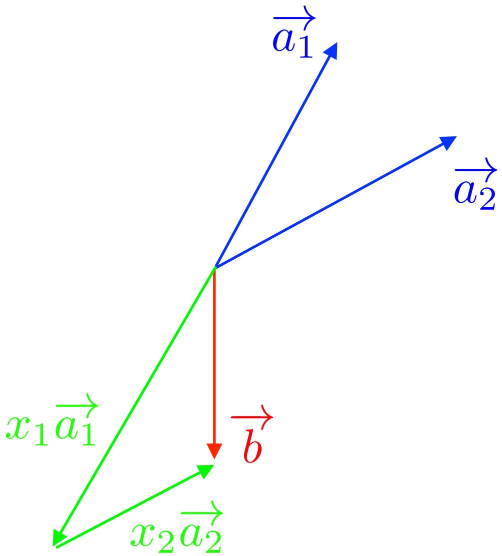
$$Ax = b,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$$

Column view: express b as a linear superposition of the columns of A

$$A = \begin{pmatrix} | & & | \\ a_1 & \dots & a_n \\ | & & | \end{pmatrix} \qquad \sum_{i=1}^{n} x_i a_i = b$$

A more geometric construction that relates well to fundamental subspaces

Column View


$\overrightarrow{a_1}$
$\overrightarrow{a_2}$
$x_1\overrightarrow{a_1}$
$\overrightarrow{b}$
$x_2\overrightarrow{a_2}$

Existence de solutions :

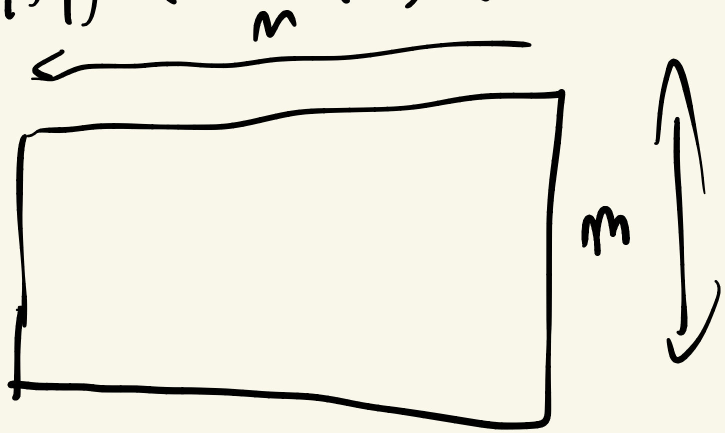
$Ax = b \Rightarrow \exists$ solutions ssi $b \in R(A)$

$\exists$ solutions $\forall b \in \mathbb{R}^m$ $\quad R(A) = \mathbb{R}^m \Rightarrow$ SURJECTIVITÉ

$m = \dim R(A) = \operatorname{rank}(A) \leq \min(m, n)$

$\Leftrightarrow \quad m \leq n$

moins équation que d'inconnues



Unicité des solutions

2 solution x, x' de $Ax = b$ $\quad b \in \mathbb{R}^m \quad x \neq x'$

$$b = Ax = Ax' \Rightarrow A\underbrace{(x - x')}_{\neq 0} = 0 \Leftrightarrow x - x' \in \mathcal{N}(A)$$

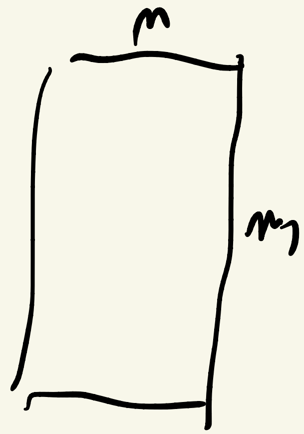
$\mathcal{N}(A) \neq \{0\}$

– Unicité ssi $\mathcal{N}(A) = \{0\}$ INJECTIVITÉ

$\exists$ au plus une solution à $Ax = b$ $\forall b \in \mathbb{R}^m$

ssi colonnes de A sont lin. indép.

$\Leftrightarrow n \leq m$



$\Leftrightarrow$ 1! solution ssi A est BIJECTIVE $n = m$

Les solutions de $Ax = b$ $\quad b \in R(A)$

on est tenté par $x^\# = A^+ b$

ⓐ A^+ est un g-inverse donc A^+b est une solution

ⓑ $R(A^+) = R(A^T) = N(A)^\perp$ $\quad A^+b \notin N(A)$

$AA^+b \neq 0$

ⓒ Si x s.t. $Ax = b$

$y \in N(A)$ $\quad A(x+y) = Ax + \cancel{Ay} = Ax = b$

$$\Rightarrow \quad \boxed{A^{+}b + \underbrace{\mathcal{N}(A)_{v}}_{\text{base de } \mathcal{N}(A)}} \equiv \underline{\text{Solutions}}$$

ou $P^{\perp}(\mathcal{N}(A)) = \mathbb{I} - A^{+}A$

DONC

$$x^{\#} = A^{+}b + (\mathbb{I} - A^{+}A)y \quad \forall y \in \mathbb{R}^{m}$$

<u>Toutes</u> les solutions sont de la forme précédente.

$z \in \mathbb{R}^n$ <u>EST</u> une solution $Az = b$

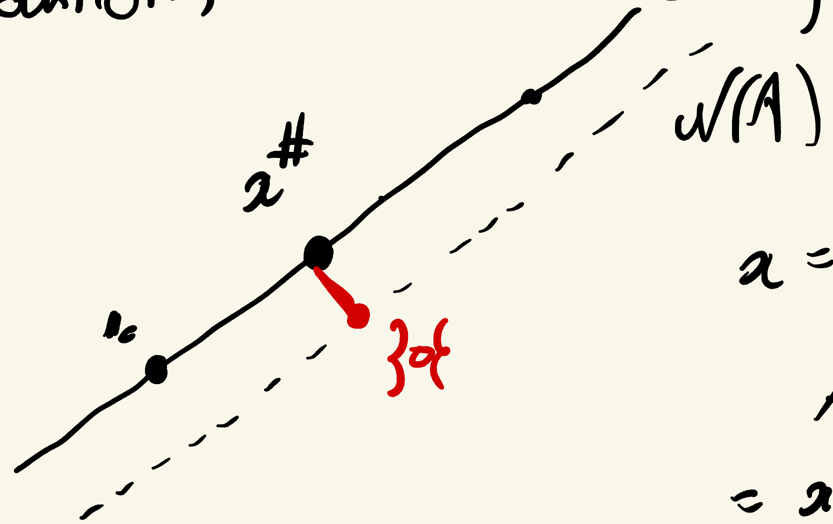
$$z = \underbrace{P^{\perp} z}_{\substack{\mathcal{N}(A) \\ (\mathbb{I} - A^{+}A) z}} + \underbrace{P^{\perp} z}_{\substack{\mathcal{N}(A)^{\perp} \\ A^{+}Az}}$$

$$z = A^{+}Az + (\mathbb{I} - A^{+}A) z$$

$$\boxed{z = A^{+}b + (\mathbb{I} - A^{+}A) z}$$

Les solutions

$S = \{x \text{ tq } Ax = b\}$



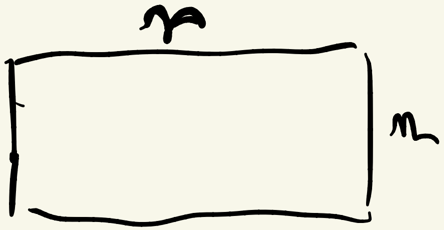
$$x = \underbrace{x_0}_{A^+b \,=\, x^\#} + y \quad \forall y \in \mathcal{N}(A)$$

$$x^\#_R = \min_x \|x\|_2^2 \text{ tq } Ax = b$$

Au sujet de la solution particulière $x^\# = A^+ b$

- dans le cas où il $\exists$ solution $\forall b \in \mathbb{R}^m$ cad $\operatorname{rank}(A) = m$

surjective



- $A^+ = A_R^{-1} = A^T (A A^T)^{-1} \Rightarrow x_R^\# = A^+ b = A_R^{-1} b$

à droite

Soit $x^\#$ une autre solution de $Ax = b$

$$(x^\# - x^\#_R)^T \underbrace{x^\#_R}_{} = (x^\# - x^\#_R)^T A^T (AA^T)^{-1} b$$

$$= \left(\left[A(x^\# - x^\#_R)\right]^T (AA^T)^{-1} b\right.$$

$$= \underbrace{\left(Ax^\# - Ax^\#_R\right)}_{0}^T (AA^T)^{-1} b$$

$$= 0$$

$x^\# - x^\#_R \perp x^\#_R \quad \forall x^\#$ solution

$$\|x^\#\|_2^2 = \|\underbrace{x^\# - x^\#_R}_{} + \underbrace{x^\#_R}_{}\|_2^2$$

$$= \underbrace{\|x^\# - x^\#_R\|_2^2}_{\geq 0} + \|x^\#_R\|_2^2$$

$$\|x^\#\|_2^2 \geq \|x^\#_R\|^2$$

General result for vector linear systems

$$Ax = b,\ A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$$

there exists a solution IFF $b \in \mathcal{R}(A)$

there exists a solution $\forall b \in \mathbb{R}^m$ IFF $\mathcal{R}(A) = \mathbb{R}^m$ $\quad$ A is onto

$\operatorname{rank}([A, b]) = \operatorname{rank}(A)$

$m \leqslant n$ $\quad$ More unknowns than equations

General result for vector linear systems

$Ax = b,\ A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$

A solution is unique IFF $\mathcal{N}(A) = 0$ A is 1-1

there exisits a unique solution $\forall b \in \mathbb{R}^m$ IFF A is 1-1 and onto, i.e non-singular

equiv. $A \in \mathbb{R}^{m\times m}$ and A has no zero singular nor eigenvalue

there exists at most a solution $\forall b \in \mathbb{R}^m$ IFF the columns of A are linearly independent, i.e $\mathcal{N}(A) = 0$ only possible if $m \geq n$

More equations than unknowns

there exists a non-trivial solution to $Ax = 0$ IFF $\operatorname{rank}(A) < n$ $\mathcal{N}(A)$ non-trivial

General result for vector linear systems

Now we would like to compute solutions

If A is non-singular, clearly $b = A^{-1}x$

In the general case, shall we try $b = A^{+}x$

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m}$ Then any x of the form

$$x = A^{+}b + \left(\mathbb{I} - A^{+}A\right) y \text{ where } y \in \mathbb{R}^{n} \text{ is arbitrary,}$$

is a solution of $Ax = b$.

Matrix Linear Equations

$$AX = B,\ A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times k},\ X \in \mathbb{R}^{n\times k}$$

<u>Existence:</u> The matrix linear equation $AX = B$ has a solution IFF $\mathcal{R}(B) \subseteq \mathcal{R}(A)$

Equivalently a solution exists IFF $AA^{+}B = B$

<u>Characterisation:</u> Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$ and suppose that $AA^{+}B = B$.

Then any matrix of the form

$$x = A^{+}B + \left(\mathbb{I} - A^{+}A\right)Y \text{ where } Y \in \mathbb{R}^{n\times k} \text{ is arbitrary,}$$

is a solution of $AX = B$.