EPFL


# Background Material
## EE-312

Prof Pierre Vandergheynst

Spring 2025

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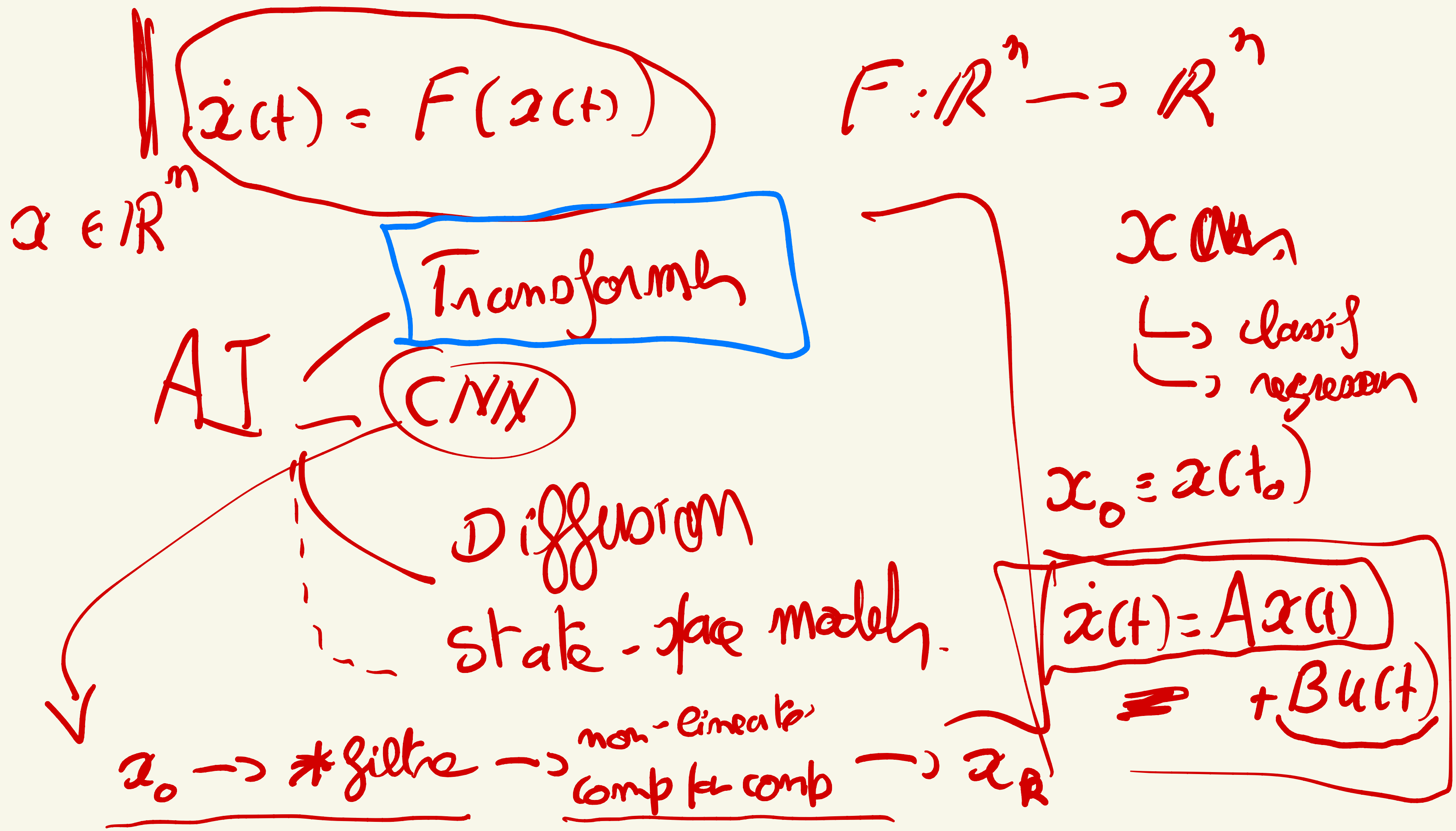
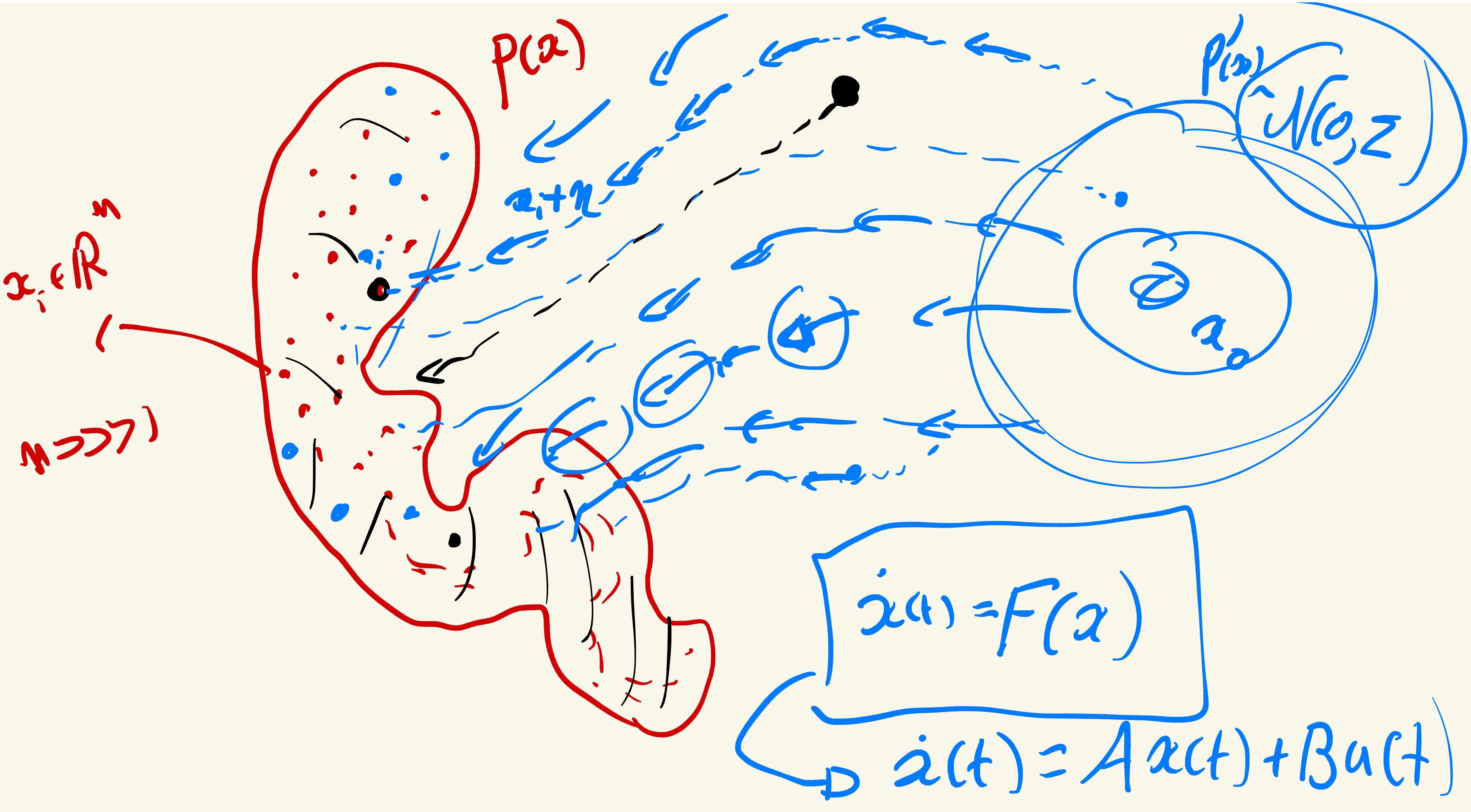


ẋ(t) = F(x(t))
x ∈ ℝ^n
F : ℝ^n → ℝ^n
AI
Transformer
CNN
Diffusion
state-space model
classif
regression
x_0 = x(t_0)
ẋ(t) = Ax(t) + Bu(t)
x_0 → * filtre → comp
non-lineaire
comp → x_R

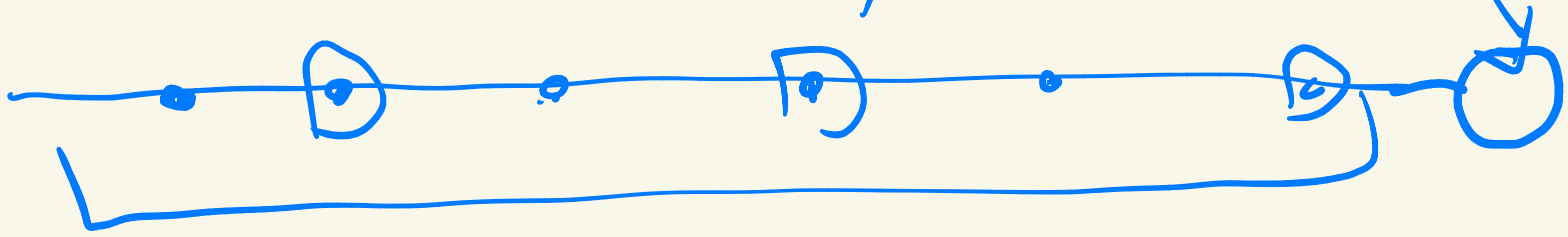

P(x)
P'(x)
~N(0,Σ)
x_i ∈ ℝ^n
n >>> 1
x_i + η
x_0
ẋ(t) = F(x)
ẋ(t) = Ax(t) + Bu(t)

Attention:

$\dot{x}(t) = F(x(t))$

$t \neq \delta(t)$

$\dot{x}(t) = J_F \, x$



linéarisation → $\dot{x}[k] = Ax[k] + Bu[k]$

# A bit of bookkeeping

Weekly: 2hrs lecture and 2hrs exercises

Slides will be available prior to the lecture on Moodle

There will be blackboard examples

Exercises will be (mostly) numerical, using python

We will use notebooks on the EPFL Noto platform

There will be a small introduction to all this !

Beyond basic python and Numpy, you hardly need anything else

Each week we will either provide a notebook to experiment with or give instructions to code your own notebook

There will be a mid-term and a final exam, counting 40-60

# Notations

x[i]

x[k]

$\mathbb{R}^n$ the set of n-tuples or <u>column</u> vectors

$$x \in \mathbb{R}^n \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{pmatrix} \qquad x_i \in \mathbb{R}$$

vectors are always column vectors and <u>row</u> vectors are transposed, i.e $y^T$

$\mathbb{C}^n$ likewise with complex-valued entries

$\overline{x}$ is the vector of complex conjugates $\qquad x^H$ is the transposed and conjugate vector

# Notations

$\mathbb{R}^{m,n}$ or $\mathbb{R}^{m\times n}$ is the set of real-valued $m$-by-$n$ matrices ($m$ rows, $n$ columns)

$A \in \mathbb{R}^{m,n}, \quad (A)_{ij} = a_{ij} \in \mathbb{R} \qquad (A^T)_{ij} = a_{ji}$

$\mathbb{C}^{m,n}$ likewise with complex-valued entries

$$(A^H)_{ij} = \overline{a_{ji}} \in \mathbb{C}$$

$A = A^T$ symmetric matrix

$A = A^H$ hermitian matrix

# Multiplications: matrix-matrix, matrix-vector, vector-vector

$A$ and $B$ are compatible for multiplication $A \in \mathbb{R}^{m\times p} \quad B \in \mathbb{R}^{p\times n} \quad AB \in \mathbb{R}^{m\times n}$

Everything is matrix-matrix! But some useful particular cases

**Real and Hermitian dot products, inner products or scalar products**

$x,\, y \in \mathbb{R}^n \to x^T y = y^T x \in \mathbb{R}$ $\qquad$ $x,\, y \in \mathbb{C}^n \to x^H y = \overline{y^H x} \in \mathbb{C}$

**Outer products**

$x \in \mathbb{R}^m \; y \in \mathbb{R}^n \to xy^T \in \mathbb{R}^{m\times n}$ $\qquad$ $x \in \mathbb{C}^m \; y \in \mathbb{C}^n \to xy^H \in \mathbb{C}^{m\times n}$

both "products" will play fundamental roles later on !

# Two views of matrix-vector mult

$A \in \mathbb{R}^{m\times n}$ $\quad$ $x \in \mathbb{R}^n$ $\quad$ $\|y - Ax\|^2 \to$ least square $\quad$ $y = Ax$

$A^{-1}y = \hat{x}$

**Column view:** $A = [a_1, \ldots, a_n]$, $\quad a_i \in \mathbb{R}^m$ the **columns** of $A$

$$Ax = \begin{pmatrix} | & \cdots & | \\ a_1 & \cdots & a_n \\ | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i a_i \in \mathbb{R}^m$$

linear combination of columns

$\sum_j a_j' x_j$

**Row view:** $A = [{a^1}^T \ldots {a^m}^T]$, $\quad {a^i}^T \in \mathbb{R}^n$ the **rows** of $A$ (seen as transposed vectors!)

$$Ax = \begin{pmatrix} - & {a^1}^T & - \\ \vdots & \vdots & \vdots \\ - & {a^m}^T & - \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} (a^1)^T x \\ \vdots \\ (a^m)^T x \end{pmatrix}$$

$\to$ Système d'équation en $x_j$

m-tuple of products of rows of $A$ with $x$

# Scalar (inner) products, orthogonality

$x = (1, j)$

$x^T x = 0$

$x^H x = 2$

$x,\ y \in \mathbb{R}^n$

$$\langle x, y \rangle \equiv x^T y = \sum_{i=1}^{n} x_i y_i$$

$$\langle x, x \rangle = 0 \text{ iff } x = 0$$

$x,\ y \in \mathbb{C}^n$

$$\langle x, y \rangle_H \equiv x^H y = \sum_{i=1}^{n} \overline{x_i} y_i$$

$$\langle x, x \rangle_H = 0 \text{ iff } x = 0$$

**Orthogonal vectors:** $\langle x, y \rangle = 0$ or $\langle x, y \rangle_H = 0$

**Orthogonal matrices:** $A \in \mathbb{R}^{n\times n}$ such that $A^T A = A A^T = I_n$

**Unitary matrices:** $A \in \mathbb{C}^{n\times n}$ such that $A^H A = A A^H = I_n$

# Vector Spaces

(essentially finite dimensional ones)

**Motivation:** abstraction of the more intuitive euclidean case, but allows to work with other interesting objects such as functions

A vector space over a field $\mathbb{F}$ is a set of *vectors* $\mathcal{V}$

two operations $+ : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$ vector addition

$\cdot : \mathbb{F} \times \mathcal{V} \mapsto \mathcal{V}$ mult. by a scalar

# Vector Spaces

A vector space over a field $\mathbb{F}$ is a set of *vectors* $\mathcal{V}$

two operations $+ : \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$

$\cdot : \mathbb{F} \times \mathcal{V} \mapsto \mathcal{V}$

$(\mathcal{V}, +)$ is an abelian group

$(\alpha\beta) \cdot u = \alpha(\beta \cdot u) \, \forall\alpha, \beta \in \mathbb{F}$ and $\forall u \in \mathcal{V}$

$(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u \quad \forall\alpha, \beta \in \mathbb{F}$ and $\forall u \in \mathcal{V}$

$\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \quad \forall\alpha \in \mathbb{F}$ and $\forall u, v \in \mathcal{V}$

$1 \cdot u = u, \quad \forall u \in \mathcal{V}$ and 1 is neutral element of product over $\mathbb{F}$

# Examples

$\mathcal{V} = \mathbb{R}^n$ and $\mathbb{F} = \mathbb{R}$ $\quad u + v = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \qquad \alpha \cdot u = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix}$

The set of polynomials of order $n$ with coefficients in $\mathbb{F}$

$\mathcal{V} = \mathbb{R}^{m \times n}$ and $\mathbb{F} = \mathbb{R}$ $\quad (A + B)_{ij} = a_{ij} + b_{ij}$ and $(\alpha \cdot A)_{ij} = \alpha a_{ij}$

Rem: when there is no risk of confusion we will save the product sign for other operations and simply write $\boldsymbol{\alpha u}$ or $\boldsymbol{\alpha A}$

# Subspaces

Part of a vector space that is closed under the natural operations.
Will play a major role when we discuss (in particular) linear applications.
More formally:

$(\mathcal{V}, \mathbb{F})$ a vector space. $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{W} \neq \emptyset$.

$(\mathcal{W}, \mathbb{F})$ is a subspace of $(\mathcal{V}, \mathbb{F})$ IFF

$$\alpha w_1 + \beta w_2 \in \mathcal{W} \quad \forall w_1,\, w_2 \in \mathcal{W} \text{ and } \forall \alpha,\, \beta \in \mathbb{F}$$

Rem: it is equivalent to saying it is itself a vector space

# Linear independence

Motivation: Somehow measure the "size" of a vector space or of a subspace

$X \equiv \{v_1, \ldots, v_k\}$, $v_i \in \mathcal{V}$ a collection of k vectors

Suppose there exists scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ <u>not all zeros</u> such that

$$\alpha_1 v_1 + \ldots + \alpha_k v_k = 0$$

For instance $\boldsymbol{\alpha_1 \neq 0}$ then: $v_1 = -\dfrac{\alpha_2}{\alpha_1} v_2 - \ldots - \dfrac{\alpha_k}{\alpha_1} v_k$

At least one vector in $X$ can be expressed as a linear combination of the others

$X$ is a linearly dependent set of vectors

# Linear independence

$X$ is a **linearly independent** set of vectors if the equation

$$\alpha_1 v_1 + \ldots + \alpha_k v_k = 0$$

can only be satisfied for $\alpha_1 = \ldots = \alpha_k = 0$

Examples: Pauli matrices

Monomials form a linearly independent family of vectors for the vector space of finite order polynomials

A homogeneous system of equations

# Pauli Matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_i^H = \sigma_i$$

$$M = \sum_{i=0}^{3} \alpha_i \sigma_i = 0 \quad \forall \alpha_i \neq 0 \quad \alpha_i \in \mathbb{R}$$

$$M = M^H \hookrightarrow \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - j\alpha_2 \\ \alpha_1 + j\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \alpha_i = 0$$

$$\alpha_0 + \alpha_3 = 0$$

$$\alpha_0 - \alpha_3 = 0$$

# Span, Basis, Dimension

Where linear independence gets us where we wanted

Let $X$ be a collection of vectors $v_i \in \mathcal{V}$

The **span** of $X$ is the set of all vectors that can be represented as lin. comb. of $X$

$$\mathrm{Sp}(X) = \{v : v = \alpha_1 v_1 + \ldots + \alpha_k v_k,\ \alpha_i \in \mathbb{F}\}$$

X is a **basis** for $\mathcal{V}$ IFF

$X$ is a linearly independent set, and

$\mathrm{Sp}(X) = \mathcal{V}$

# Span, Basis, Dimension

X is a **basis** for $\mathcal{V}$ IFF

$X$ is a linearly independent set, and

$\mathrm{Sp}(X) = \mathcal{V}$

This means you can write $v = X\alpha, \quad \forall v \in \mathcal{V}$ with <u>unique</u> coefficients $\boldsymbol{\alpha}$

The number of elements in a basis is independent of the basis
It is therefore a characteristic of the vector space spanned by the basis
called its **dimension** (note the space can be infinite dimensional)

# Sums, intersections of subspaces

Let $\mathcal{R}$, $\mathcal{S} \in (\mathcal{V}, \mathbb{F})$

$r_1 + s_1 = r_2 + s_2$

$(r_1 - r_2) + (s_1 - s_2) = 0$

$(r_1 - r_2) = -(s_1 - s_2)$

$\mathcal{R} + \mathcal{S} = \{r + s : r \in \mathcal{R},\ s \in \mathcal{S}\}$

$\mathcal{R} \cap \mathcal{S} = \{v : v \in \mathcal{R} \text{ and } v \in \mathcal{S}\}$

Subspaces! $\mathcal{R} \cup \mathcal{S}$ not necessarily subspace

$$\mathcal{R}_1 + \cdots + \mathcal{R}_k \subseteq \mathcal{V} \qquad \bigcap_{k \in A} \mathcal{R}_k \subseteq \mathcal{V}$$

The direct sum of two subspaces is a subspace $\quad \mathcal{T} = \mathcal{R} \oplus \mathcal{S}$

$$\begin{aligned} \mathcal{R} \cap \mathcal{S} &= 0 \\ \mathcal{R} + \mathcal{S} &= \mathcal{T} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &t = r + s \text{ uniquely } \forall t \in \mathcal{T},\ r \in \mathcal{R},\ s \in \mathcal{S} \\ &\dim(\mathcal{T}) = \dim(\mathcal{R}) + \dim(\mathcal{S}) \end{aligned}$$

# Inner product, orthogonality

Vector spaces over $\mathbb{R}$ or $\mathbb{C}$ are sometimes endowed with an inner product

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F} \qquad \forall u, v, w \in \mathcal{V} \text{ and } \alpha \in \mathbb{F}$$

$$\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$$

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

$$\langle u, u \rangle \geqslant 0 \quad \langle u, u \rangle = 0 \Rightarrow u = 0$$

**Important example: Euclidean "dot" product**

$$u, v \in \mathbb{R}^n \quad \langle u, v \rangle = u^T v = \sum_{i=1}^{n} u_i v_i$$

A set of non-zero vectors $\{v_1, \ldots, v_k\}$ is **orthogonal** if $\langle v_i, v_j \rangle = 0$ for $i \neq j$

It is **orthonormal** if $\langle v_i, v_j \rangle = \delta_{ij}$

# Orthogonal complements

Let the set $\mathcal{S} \subseteq \mathcal{V}$. The **orthogonal complement** is defined as:

$$\mathcal{S}^{\perp} = \{v \in \mathcal{V} \,:\, \langle v, s \rangle = 0 \quad \forall s \in \mathcal{S}\}$$

Some properties for $\mathcal{R}$, $\mathcal{S} \subseteq \mathcal{V}$

$\mathcal{S}^{\perp} \subseteq \mathcal{V}$

$\mathcal{S} \oplus \mathcal{S}^{\perp} = \mathcal{V}$

$\left(\mathcal{S}^{\perp}\right)^{\perp} = \mathcal{S}$

$\mathcal{R} \subseteq \mathcal{S}$ IFF $\mathcal{S}^{\perp} \subseteq \mathcal{R}^{\perp}$

$(\mathcal{R} + \mathcal{S})^{\perp} = \mathcal{R}^{\perp} \cap \mathcal{S}^{\perp}$

$(\mathcal{R} \cap \mathcal{S})^{\perp} = \mathcal{R}^{\perp} + \mathcal{S}^{\perp}$

# The Discrete Fourier Basis

$v_i^H v_j = \delta_{ij}$  $\frac{1}{\sqrt{N}}$  $v_i \in \mathbb{C}^N$

$i = 0, \dots, N-1$

The set of $k$-dimensional complex-valued vectors $v_k[n] = e^{j2\pi \frac{kn}{N}}$

$k, n \in 0, \cdots, N-1$

Is an orthogonal basis of $\mathbb{C}^N$ called the **Discrete Fourier Basis**

Change of basis: $f = \sum_{k=0}^{N-1} (v_k^H f) v_k$  $\alpha_k$  $f \in \mathbb{C}^N$

$$F[k] = v_k^H f = \sum_{n=0}^{N-1} e^{-j2\pi \frac{kn}{N}} f[n]$$ **Discrete Fourier Transform**