



The Moore-Penrose Pseudo-Inverse EE-312

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A note on invertibility

A linear transformation is **invertible** if and only if it is **bijective** (1-1 and onto)

ex: $T : \mathcal{N}(A)^\perp \rightarrow \mathcal{R}(A) \quad Tv = Av \quad \forall v \in \mathcal{N}(A)^\perp$

Consider $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and suppose it has n linearly independent columns $\mathcal{R}(A) = \mathbb{R}^n$
full-rank
A is onto

$$\forall y \in \mathbb{R}^n \exists x_1, \dots, x_n \text{ s.t. } y = a_1 x_1 + \dots + a_n x_n$$

unique

A is 1-1

$$y = Ax \rightarrow \text{depends on } A \text{ and } y$$

$$x = A^{-1}y$$

linear

$A : \mathcal{V} \rightarrow \mathcal{W}$ is **invertible** if and only if it is **bijective**.

If A is invertible then $\dim(\mathcal{V}) = \dim(\mathcal{W})$

$A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible (non-singular) if and only if $\text{rank}(A) = n$ 2

A note on invertibility

Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Some interesting square matrices from any matrix $A^T A \in \mathbb{R}^{n \times n}$ and $AA^T \in \mathbb{R}^{m \times m}$

A is onto iff $\text{rank}(A) = m$, equivalent to AA^T is non-singular

intuition: range of A is full, null-space of A^T is trivial

so the induced linear transformation from \mathbb{R}^m to itself is bijective

A is 1-1 iff $\text{rank}(A) = n$, equivalent to $A^T A$ is non-singular

intuition: range of A^T is full, null-space of A is trivial

so the induced linear transformation from \mathbb{R}^n to itself is bijective

These matrices play a crucial role for constructing “special inverses”

A note on invertibility

Even if A is not invertible, it is left (resp. right) invertible iff it is 1-1 (resp. onto)

$$A : \mathcal{V} \rightarrow \mathcal{W}$$

A is **right invertible** if there exists a linear transformation

$$A_R^{-1} : \mathcal{W} \rightarrow \mathcal{V} \quad \text{such that } AA_R^{-1} = \mathbb{I}_{\mathcal{W}}$$

A is right invertible IFF it is onto

A is **left invertible** if there exists a linear transformation

$$A_L^{-1} : \mathcal{V} \rightarrow \mathcal{W} \quad \text{such that } A_L^{-1}A = \mathbb{I}_{\mathcal{V}}$$

A is left invertible IFF it is 1-1

A note on invertibility

A is invertible IFF it is both left and right invertible, in which case $A_L^{-1} = A_R^{-1} = A^{-1}$

Now our special square matrices become useful

A is onto $\Rightarrow AA^T$ is nonsingular $\Rightarrow A_R^{-1} = A^T(AA^T)^{-1}$ is a right inverse

A is 1-1 $\Rightarrow A^T A$ is nonsingular $\Rightarrow A_L^{-1} = (A^T A)^{-1} A^T$ is a left inverse

Rem: $A : \mathcal{V} \rightarrow \mathcal{V}$

if there exists a *unique* left inverse, then A is invertible

if there exists a *unique* right inverse, then A is invertible

Generalized Inverses

A little motivation (we'll get back to it later)

$$Ax = b \quad A \in \mathbb{R}^{m \times m} \text{ and non-singular} \Rightarrow x = A^{-1}b$$

$$A \in \mathbb{R}^{n \times m} \text{ and } b \in \mathcal{R}(A)$$

Suppose there exists $G \in \mathbb{R}^{m \times n}$ s.t. $AGy = y \quad \forall y \in \mathcal{R}(A)$

$x = Gb$ is a solution and $AGA = A$

Generalized Inverses

$$\forall A \in \mathbb{R}^{m \times n} \exists G \in \mathbb{R}^{n \times m} \text{ s.t. } AGA = A$$

G = Generalized Inverse (always exists but not necessarily unique)

If the inverse of $A \in \mathbb{R}^{n \times n}$ exists, it is a generalised inverse (and there is only one)

$$A(A^{-1})A = A$$

$$\begin{aligned} G &= A^{-1}(AGA)A^{-1} \text{ (inverse exists)} \\ &= A^{-1}(A)A^{-1} \text{ (generalized inverse)} \\ &= A^{-1} \end{aligned}$$

Generalized Inverses

An explicit form for G :

$$A \in \mathbb{R}_r^{m \times n} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{with } A_{11} \text{ is } r\text{-by-}r \text{ and invertible}$$

$$\text{Then: } G = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{is a generalised inverse}$$

Any m -by- n matrix of rank r can be put in that form by performing rows and columns permutations.

This shows any matrix has a generalised inverse. Can you see why ?

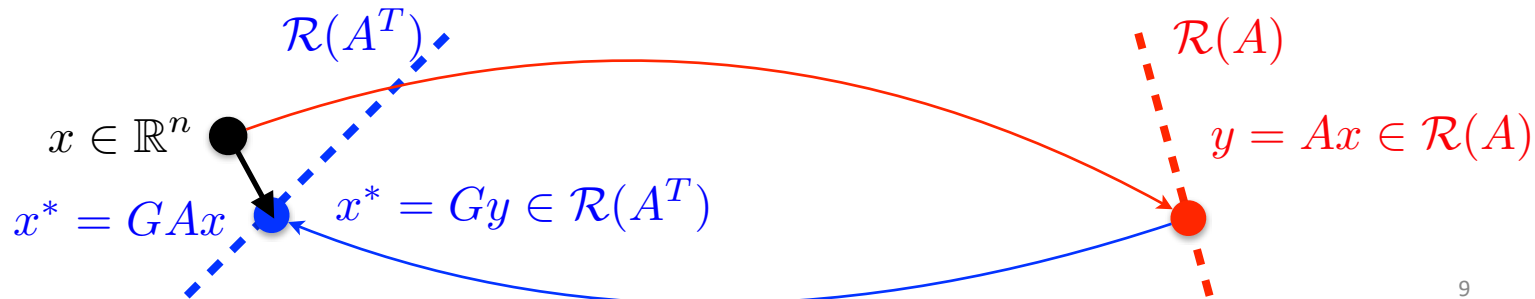
Generalised Inverses

A generalised inverse computes a projection !

$A \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times m}$ a g-inverse

AG is a projection onto $\mathcal{R}(A)$ column space

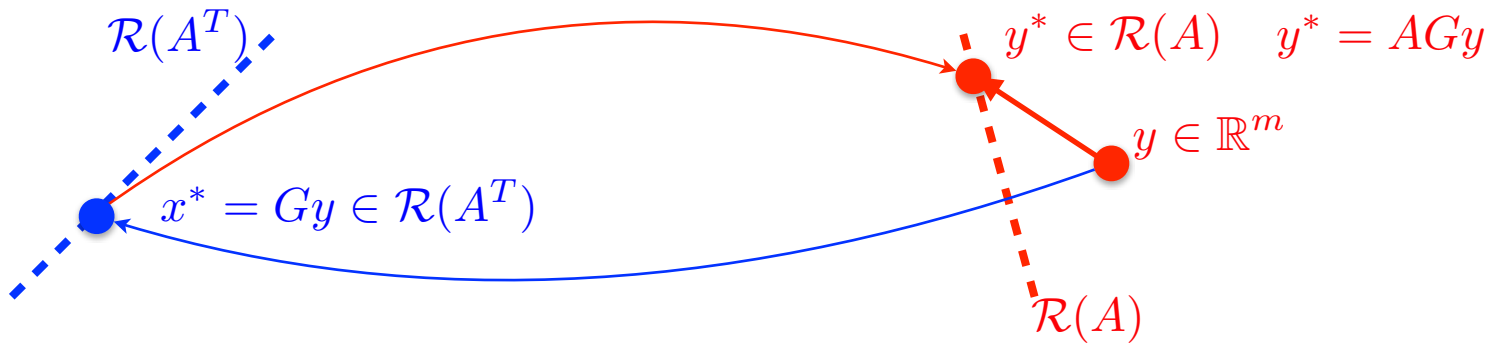
GA is a projection onto $\mathcal{R}(A^T)$ row space



Generalised Inverses

Can this be useful to solve tougher problems ?

$A \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times m}$ a g-inverse $Ax = y$ but $y \notin \mathcal{R}(A)$???



What would be the best possible y^* ?

The Moore-Penrose Pseudoinverse

Goal: show the existence and main properties of a generalised inverse for arbitrary matrices. Computational aspects and applications to systems of linear equations and least squares problem in next lectures

$A : \mathcal{X} \rightarrow \mathcal{Y}$ arbitrary linear transformation between finite dimensional vector spaces

$$T : \mathcal{N}(A)^\perp \rightarrow \mathcal{R}(A) \quad Tx = Ax \quad \forall x \in \mathcal{N}(A)^\perp \quad \textbf{T is bijective}$$

Define $A^+ : \mathcal{Y} \rightarrow \mathcal{X}$ $A^+y = T^{-1}y_1$ where $y = y_1 + y_2$ with $y_1 \in \mathcal{R}(A)$ and $y_2 \in \mathcal{R}(A)^\perp$

A^+ is the Moore-Penrose pseudoinverse of A

The Moore-Penrose Pseudoinverse

A^+ always exists and is unique! In particular for any rank r matrix $A \in \mathbb{R}_r^{m \times n}$

Some properties that characterise any $G = A^+$ **if and only if**

$$(P1) \quad AGA = A$$

$$(P2) \quad GAG = G$$

$$(P3) \quad (AG)^T = AG$$

$$(P4) \quad (GA)^T = GA$$

Rem: any nonsingular matrix satisfies (P1-4)

any left or right inverse satisfies at least 3 of these properties 12

The Moore-Penrose Pseudoinverse

Another characterisation

Let $A \in \mathbb{R}_r^{m \times n}$ then

$$\begin{aligned} A^+ &= \lim_{\delta \rightarrow 0} (A^T A + \delta^2 \mathbb{I})^{-1} A^T \\ &= \lim_{\delta \rightarrow 0} A^T (A A^T + \delta^2 \mathbb{I})^{-1} \end{aligned}$$

Interestingly:

if A is onto (independent rows) then $A^+ = A^T (A A^T)^{-1}$ (right inverse)

if A is 1-to-1 (independent columns) then $A^+ = (A^T A)^{-1} A^T$ (left inverse)

The Moore-Penrose Pseudoinverse

It is a generalised inverse with more constraints!

It always exists (like g-inverse) and is unique (unlike most g-inverse)

$$(P1) \quad AGA = A$$

G is a generalised inverse of A !

$$(P2) \quad GAG = G$$

A is a generalised inverse of G !

$$(P3) \quad (AG)^T = AG$$

AG is symmetric (and we know it is a projection ...)

$$(P4) \quad (GA)^T = GA$$

GA is symmetric (and we know it is a projection ...)

$$G = A^+ \quad (A^+)^+ = A$$

AA^+ is the orthogonal projection onto $\mathcal{R}(A)$

A^+A is the orthogonal projection onto $\mathcal{R}(A^T)$

The Moore-Penrose Pseudoinverse

$A \in \mathbb{R}^{m \times n}$ is a tall matrix $m \geq n$, full column rank n

Ex: more equations than unknowns

$$A^+ = (A^T A)^{-1} A^T \quad \begin{array}{l} \text{we know this is invertible (full column rank)} \\ \text{and that the orthogonal projection on the range of } A \text{ is:} \end{array} \quad \begin{array}{l} P_{\mathcal{R}(A)} = A(A^T A)^{-1} A^T \\ = AA^+ \end{array}$$

