Matrix Analysis - Midterm Exam - April 17th 2025

- You are NOT allowed to use any electronic device (phone, tablet, calculator, ...) during the exam.
- You may not use books, lecture slides, or any other paper document except for two A4 sheets (recto-verso) prepared before the exam.
- You can use directly the results proved during the exercise sessions.
- **Answer directly on the subject**, in French or English. If the available space for a question is too small, there are extra empty pages at the end of the exam.
- Read each problem **entirely** before beginning to answer the questions. Make sure you **justify** your answers appropriately.

Problem I

Suppose that $\mathcal{S}$ is a subspace of $\mathbb{R}^n$ with orthogonal complement $\mathcal{S}^\perp$. Let P be a projection matrix onto $\mathcal{S}$ and Q a projection matrix onto $\mathcal{S}^\perp$.

1) What are $P + Q$ and PQ ?

2) Show that $P - Q$ is its own inverse.

Let A be a $n \times m$ matrix.

3) What is the orthogonal projector P_R on the row space of A, and what is the condition required for it to exist ?

4) What is the relation between P_R and the orthogonal projector P_N on the nullspace of A ?

Problem II

Let A be a $n \times m$ real matrix.

1) Prove that rank(A)=1 iff. there exist column vectors vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ such that $A = vw^T$.

2) Prove that the pseudo-inverse of $A = vw^T$ is $\frac{1}{\|v\|^2\|w\|^2}A^T$.

3) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. What is x^* such that $\|Ax^* - b\|$ is minimal ?

4) Let us introduce a small perturbation in A denoted by ε like follows:

$$A_1 = \begin{pmatrix} 1 & \varepsilon \\ 0 & 0 \end{pmatrix}, \varepsilon > 0.$$

What is the solution x_1^* that minimizes $\|A_1x_1^* - b\|$? What happens to $\|x_1^* - x^*\|$ when ε approaches 0 ?

5) Let us introduce another small perturbation in A denoted by ε like follows:

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \varepsilon > 0.$$

What is the solution x_2^* that minimizes $\|A_2x_2^* - b\|$? What happens to $\|x_2^* - x^*\|$ when ε approaches 0 ?

Problem III

Consider a set of n countries, each producing and trading a single good (e.g., steel). We want to model the trade dynamics among these countries using a linear discrete dynamical system.

To simplify matters, we consider $n = 2$.

Let $x_j[k]$ denote the amount of steel country j has at discrete time $k \in \mathbb{N}$. We assume that each country' steel supply evolves according to the following recursive equation (each country trades the same fraction of their home production):

$$x_1[k+1] = \alpha x_1[k] + (1-\alpha)x_2[k]$$
$$x_2[k+1] = \alpha x_2[k] + (1-\alpha)x_1[k],$$

where $1-\alpha$, with $0 \leq \alpha \leq 1$, is the proportion of steel exported. We can represent this system more compactly using matrix notation:

$$x[k+1] = Ax[k]$$

where:

- $x(k)$ is the vector of steel supplies for both countries at discrete time k,
- A is an 2×2 matrix.

The matrix A represents the trade network among countries.

1) Compute the eigenvalues and eigenvectors of A as a function of the parameter α.

2) Find a regime where stability occurs and compute the the supply vector when $k \to +\infty$. Discuss the long-term stability of the system for choices of α. How do initial conditions $x[0]$ affect this solution ?

Let's introduce two new parameters, τ_1, τ_2 with $0 \leq \tau_1, \tau_2 \leq 1$ representing the trade barriers (for instance tariffs) imposed by countries on steel imports.

The modified recursive equation for steel supply becomes:

$$x_1[k+1] = \alpha x_1[k] + \tau_1(1-\alpha)x_2[k] \quad (1)$$
$$x_2[k+1] = \alpha x_2[k] + \tau_2(1-\alpha)x_1[k], \quad (2)$$

or, in matrix form,

$$x[k+1] = A_T x[k].$$

3) Compute the eigenvalues of the new matrix A_T. What are the differences with those of A ?

4) By analyzing (stability, solution at $k \to +\infty$) the modified system, discuss how trade barrier shave affected the steel supply in each country. You can set values for α, τ_1 and τ_2 if necessary, but provide a rationale for your choice.

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Do not return the exam subject before being instructed to do so.