EPFL

Linear Transformations
EE-312

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Linear Transformations

This is probably the most intuitive view of linear algebra and among its most useful applications

A mapping between two vector spaces that satisfies the axiom of linearity

$(\mathcal{V}, \mathbb{F})$ and $(\mathcal{W}, \mathbb{F})$ two vector spaces over the same field:

$\mathcal{L} : \mathcal{V} \mapsto \mathcal{W}$ is a **linear transformation** IFF

$$\mathcal{L}(\alpha v_1 + \beta v_2) = \alpha\mathcal{L}(v_1) + \beta\mathcal{L}(v_2)\, \forall \alpha,\, \beta \in \mathbb{F} \text{ and } v_1, v_2 \in \mathcal{V}$$

Examples: $(\mathcal{V}, \mathbb{F}) = (\mathcal{P}^n, \mathbb{R}), \quad (\mathcal{W}, \mathbb{F}) = (\mathcal{P}^{n-1}, \mathbb{R})$

$$\mathcal{L}(v) = v'$$

Most revealing is the case of a linear trans. between two <u>euclidean</u> vector spaces

Linear Transformations

Most revealing is the case of a linear trans. between two euclidean vector spaces

Classic: a linear transformation of vectors, visualised as “arrows”

I strongly advise you check 3blue1brown’s youtube channel
“Essence of Linear Algebra”, for great pedagogical visualisations

“Unfortunately, no one can be told what the Matrix is.
You have to see it for yourself.”
Morpheus

Linear Transformations

Linear System and Linear Dynamical System

$$\frac{d}{dt}x(t) = Ax(t),\ x \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n\times n}$$

Examples: $n = 1,\ \frac{d}{dt}x(t) = ax(t) \Rightarrow x(t) = x(0)e^{at}$ reminds you of something ?

$\text{n} = 2,$

$$\frac{d}{dt}x_1(t) = a_{11}x_1(t) + a_{12}x_2(t)$$

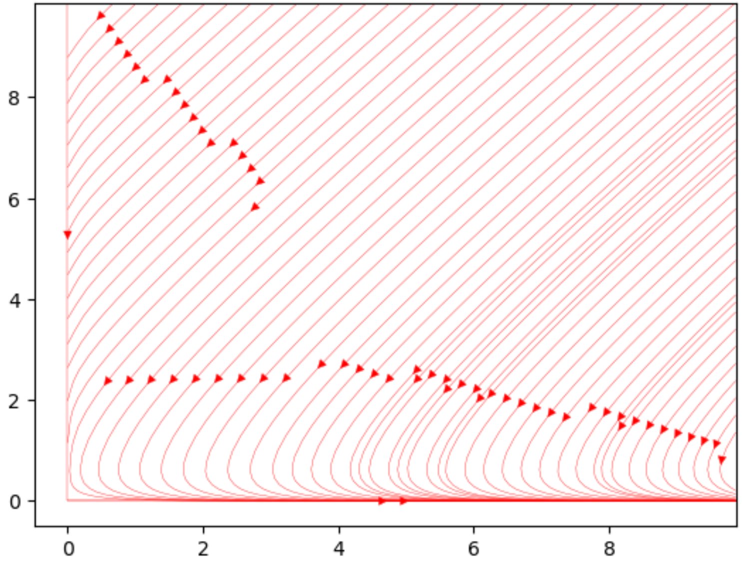
$$\frac{d}{dt}x_2(t) = a_{21}x_1(t) + a_{22}x_2(t)$$

predation

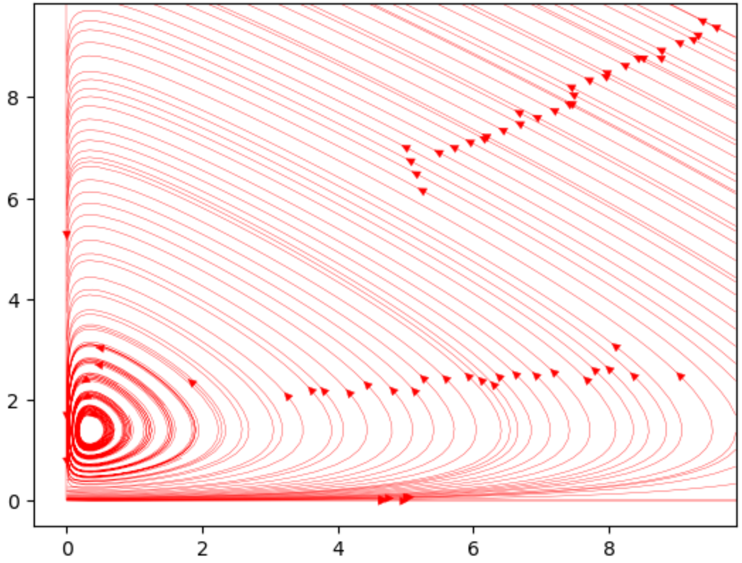
Linearised "predator-prey" model

Lotka-Volterra dynamics

$$\dot{x}(t) = f(x_1, x_2) = \begin{pmatrix} \alpha x_1 - \beta x_1 x_2 \\ -\gamma x_2 + \delta x_1 x_2 \end{pmatrix}$$



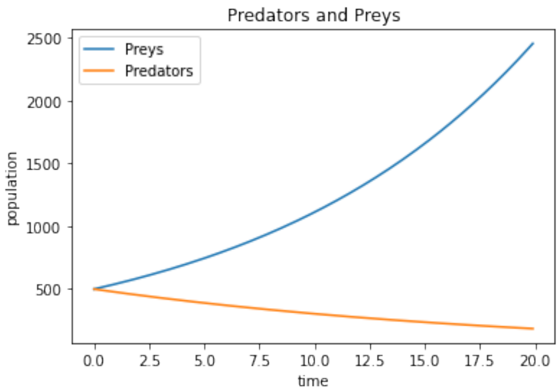
Random parameters



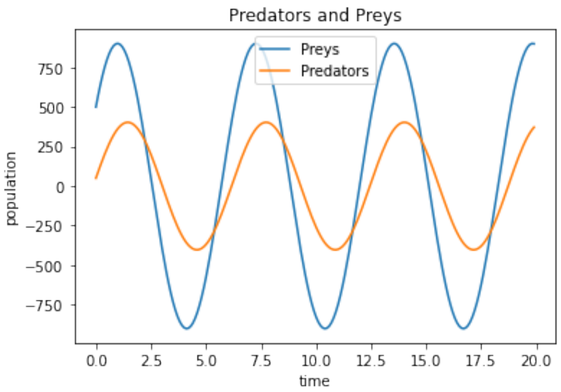
"Well chosen" parameters

Examples

$$A = \begin{pmatrix} 0.08 & 0 \\ 0 & -0.05 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$$



Linear Transformations

A mapping between two vector spaces, that satisfies the axiom of linearity

$(\mathcal{V}, \mathbb{F})$ and $(\mathcal{W}, \mathbb{F})$ two vector spaces. $\mathcal{L} : \mathcal{V} \mapsto \mathcal{W}$ is a **linear transformation** IFF

$$\mathcal{L}(\alpha v_1 + \beta v_2) = \alpha \mathcal{L}(v_1) + \beta \mathcal{L}(v_2) \, \forall \alpha, \, \beta \in \mathbb{F} \text{ and } v_1, v_2 \in \mathcal{V}$$

If you decorate the two vector spaces with bases V and W

$$\forall u \in \mathcal{V} \quad u = \sum_i \alpha_i v_i \qquad \mathcal{L}u = \sum_i \alpha_i \mathcal{L} v_i = (\mathcal{L}V)\alpha$$

concatenate all vectors $\mathcal{L}v_i$ as matrix $\mathcal{L}V$

$$\mathcal{L}v_i \in \mathcal{W} \qquad \mathcal{L}v_i = \sum_j w_j \beta_{ji}$$

$$= WB_i$$

i-th column of B

$$\mathcal{L}u = (\mathcal{L}V)\alpha$$
$$= WB\alpha$$

holds $\forall \alpha$

$$\mathcal{L} = WB$$

Linear Transformations

In particular for two euclidian vector spaces $\mathcal{V} = \mathbb{R}^n$, $\mathcal{W} = \mathbb{R}^m$

$$\begin{aligned} \mathcal{L}u &= (\mathcal{L}V)\alpha \\ &= WB\alpha \end{aligned}$$

pick canonical bases $\mathcal{L} = B \in \mathbb{R}^{m\times n}$

any matrix is a linear transformation between two vector euclidean spaces

what happens if we change basis ?

The four fundamental subspaces

$A : \mathcal{V} \to \mathcal{W}$ a linear transformation

The **range** (or the image) of A $\mathcal{R}(A) = \{w \in \mathcal{W} : w = Av \text{ for some } v \in \mathcal{V}\}$

$$\mathcal{R}(A) = \{Av : v \in \mathcal{V}\}$$

$$\mathcal{R}(A) \subseteq \mathcal{W}$$

The **nullspace** (or the kernel) of A $\mathcal{N}(A) = \{v \in \mathcal{V} : Av = 0\}$

$$\mathcal{N}(A) \subseteq \mathcal{V}$$

$$A \in \mathbb{R}^{m \times n}, \quad A = [a_1, \ldots, a_n] \Rightarrow \mathcal{R}(A) = \mathrm{Sp}(A)$$

The four fundamental subspaces

For any $x \in \mathcal{N}(A)$

$$Ax = 0 \Rightarrow y^T A x = 0 \quad \forall y \in R^m$$
$$\Rightarrow (A^T y)^T x = 0$$
$$\Rightarrow x \in \mathcal{R}(A^T)^\perp$$

$$\Longrightarrow \quad \mathcal{N}(A)^\perp = \mathcal{R}(A^T)$$

For any $y \in \mathcal{N}(A^T)$

$$A^T y = 0 \Rightarrow x^T A^T y = 0 \quad \forall x \in R^n$$
$$\Rightarrow (Ax)^T y = 0$$
$$\Rightarrow y \in \mathcal{R}(A)^\perp$$

$$\Longrightarrow \quad \mathcal{R}(A)^\perp = \mathcal{N}(A^T)$$

The four fundamental subspaces

Why are they so "fundamental" ?

They are linked to fundamental properties of A as a linear transformation

Let $A : \mathcal{V} \rightarrow \mathcal{W}$

A is **onto (or surjective)** if $\mathcal{R}(A) = \mathcal{W}$

A is **1-to-1 (or injective)** if $\mathcal{N}(A) = 0$

Equivalently:

$a)\, Av_1 = Av_2 \Rightarrow v_1 = v_2$

$b)\, v_1 \neq v_2 \Rightarrow Av_1 \neq Av_2$

Matrix Rank

Let $A : \mathcal{V} \to \mathcal{W}$

A is **onto (or surjective)** if $\mathcal{R}(A) = \mathcal{W}$

A is **1-to-1 (or injective)** if $\mathcal{N}(A) = 0$

The dimension of these subspaces is of particular significance !

$\text{rank}(A) = \dim(\mathcal{R}(A))$ and is the maximum number of independent columns

column rank!

Note: $\dim(\mathcal{R}(A^T))$ is the maximum number of independent rows **row rank!**

$\text{nullity}(A) = \dim(\mathcal{N}(A))$

Matrix Rank

And all these characterisations are equivalent !

$A : \mathbb{R}^n \rightarrow \mathbb{R}^m \;\; \dim(\mathcal{R}(A)) = \dim(\mathcal{N}(A)^{\perp}) = \dim(\mathcal{R}(A^T))$

row rank of A = column rank of A = n - nullity(A)

Second part of theorem already proved before.

Consider the restriction: $T : \mathcal{N}(A)^{\perp} \rightarrow \mathcal{R}(A) \quad Tv = Av \quad \forall v \in \mathcal{N}(A)^{\perp}$

Show thatT is <u>bijective</u>

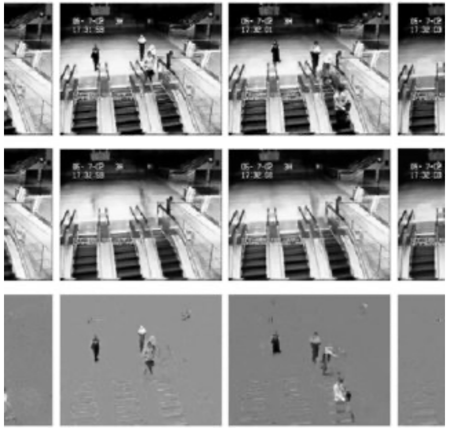
Any basis of $\mathcal{N}(A)^{\perp}$ is mapped to a basis of $\mathcal{R}(A)$ by T

$\Rightarrow \dim(\mathcal{N}(A)^{\perp}) = \dim(\mathcal{R}(A))$

Matrix Rank

The rank is not just a theoretical object !

Fundamental for the solutions of linear systems of equations (full-rank is nice!)



A measure of the "information" content of a data matrix (low-rank is nice!)

A low-rank matrix has a small number of degrees of freedom (ex: rank 1)

It is very useful to approximate a matrix by a low-rank one

Matrix Rank

The rank is not just a theoretical object !

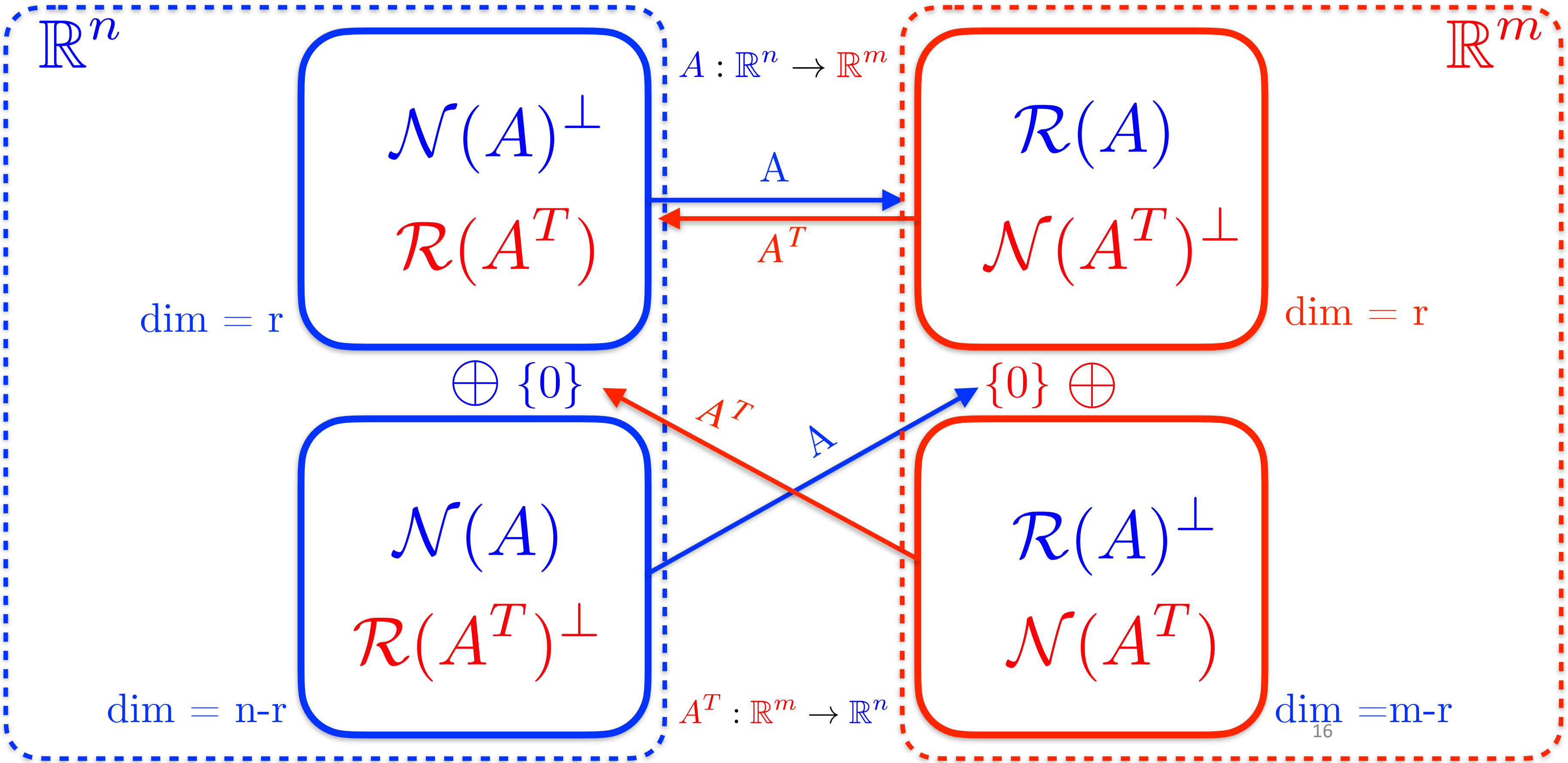
Application example: controllability

The state x_k of a discrete dynamical system evolves due to its own dynamics and a sequence of control inputs (commands) u_k :

$$x_{k+1} = Ax_k + Bu_k$$

Question: how can we guarantee that we can reach any final state x_K from any initial state x_0 in K steps via proper control inputs ????

The four fundamental subspaces



A note on invertibility

A linear transformation is **invertible** if and only if it is **bijective** (1-1 and onto)

ex: $T : \mathcal{N}(A)^\perp \to \mathcal{R}(A) \qquad Tv = Av \quad \forall v \in \mathcal{N}(A)^\perp$

Consider $A : \mathbb{R}^n \to \mathbb{R}^n$ and suppose it has n linearly independent columns $\mathcal{R}(A) = \mathbb{R}^n$

full-rank

A is onto

$$\forall y \in \mathbb{R}^n \, \exists \, x_1, \ldots, x_n \text{ s.t. } y = a_1 x_1 + \cdots + a_n x_n$$

unique

A is 1-1

$$y = Ax$$

x → depends on A and y

$$x = A^{-1} y$$

linear

$A : \mathcal{V} \to \mathcal{W}$ is **invertible** if and only if it is **bijective.**

If A is invertible then $\dim(\mathcal{V}) = \dim(\mathcal{W})$

$A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (non-singular) if and only if rank(A) $=$ n