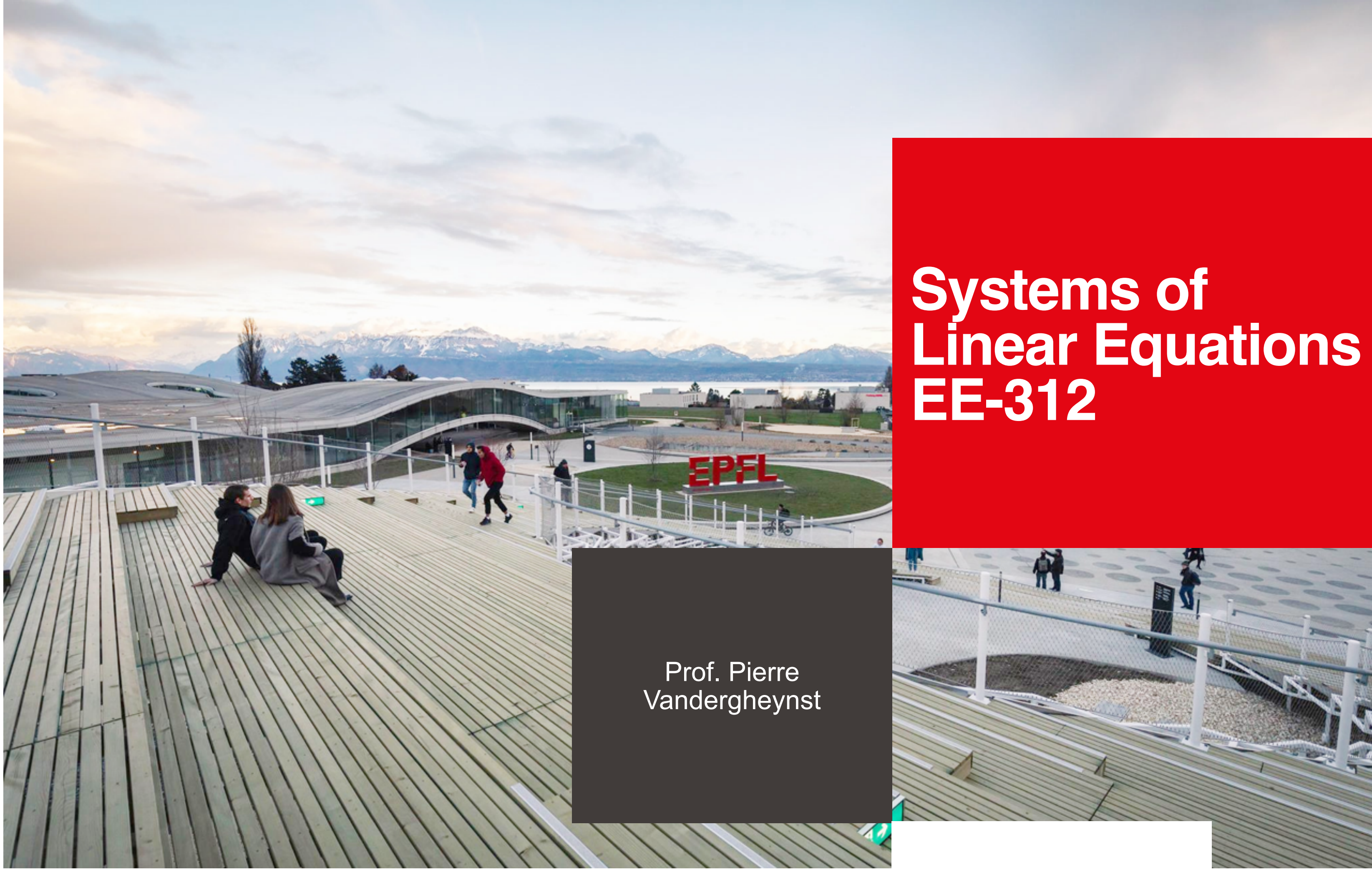
EPFL



Systems of Linear Equations EE-312

Prof. Pierre Vandergheynst

The unreasonable power of linear systems

$$Ax = b$$

Linear dynamics: $\dot{x}(t) = Ax(t)$ or $x_{k+1} = Ax_k$

Control: $x_{k+1} = Ax_k + Bu_k$

Data Science: $a_{i1}, \cdots, a_{in} \mapsto b_i$ $\qquad b_i = x_0 + a_{i1}x_1 + \cdots + a_{in}x_n$

features (predictors) characteristic Linear Model

Row & Column Views

A gentle reminder before we proceed

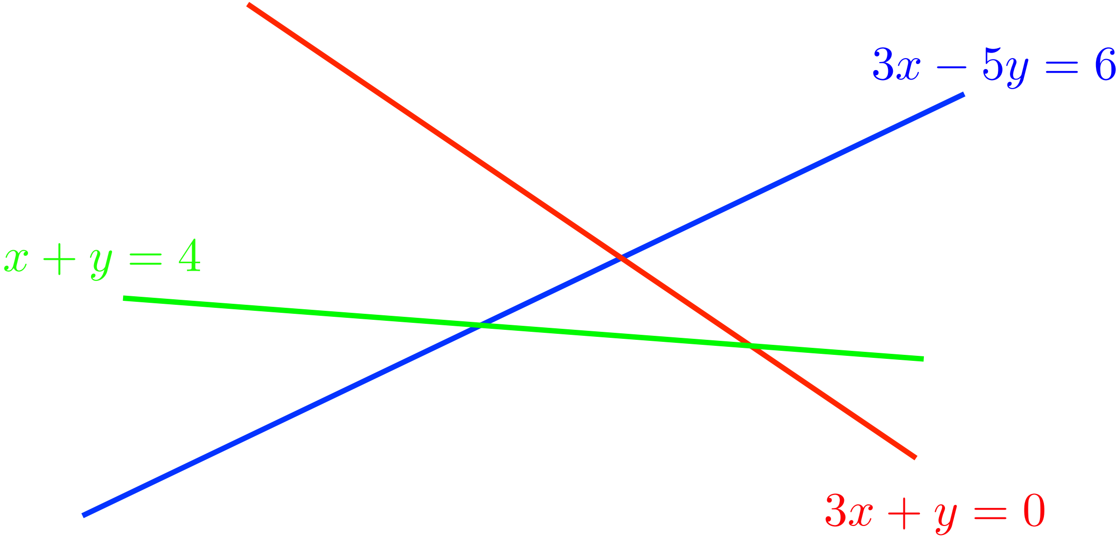
$$Ax = b,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$$

Row view: each row of A defines a linear equation (matrix entries = coefficients)

m equations of the form $\displaystyle\sum_{j=1}^{n} a_{ij} x_j = b_i$ each equation is a line constrain that "carves out" the solution

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix} \qquad a_i^T x = b_i$$

Row View



Row & Column Views

A gentle reminder before we proceed

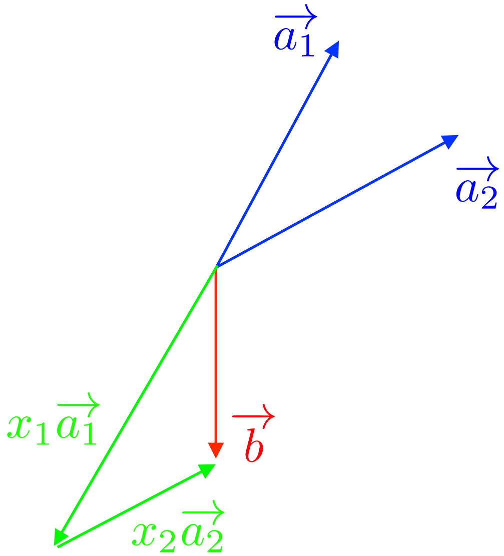
$$Ax = b,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^{m},\ x \in \mathbb{R}^{n}$$

Column view: express b as a linear superposition of the columns of A

$$A = \begin{pmatrix} | & & | \\ a_1 & \dots & a_n \\ | & & | \end{pmatrix} \qquad \sum_{i=1}^{n} x_i a_i = b$$

A more geometric construction that relates well to fundamental subspaces

Column View



$\overrightarrow{a_1}$
$\overrightarrow{a_2}$
$x_1\overrightarrow{a_1}$
$\overrightarrow{b}$
$x_2\overrightarrow{a_2}$

General result for vector linear systems

$$Ax = b,\ A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$$

there exists a solution IFF $b \in \mathcal{R}(A)$

there exists a solution $\forall b \in \mathbb{R}^m$ IFF $\mathcal{R}(A) = \mathbb{R}^m$ $\quad$ A is onto

$\text{rank}([A, b]) = \text{rank}(A)$

$m \leqslant n$ $\quad$ More unknowns than equations

General result for vector linear systems

$$Ax = b,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n$$

A solution is unique IFF $\mathcal{N}(A) = 0$ $\quad A$ is 1-1

there exisits a unique solution $\forall b \in \mathbb{R}^m$ IFF A is 1-1 and onto, i.e non-singular

equiv. $A \in \mathbb{R}^{m \times m}$ and A has no zero singular nor eigenvalue

there exists at most a solution $\forall b \in \mathbb{R}^m$ IFF the columns of A are linearly independent, i.e $\mathcal{N}(A) = 0$ only possible if $m \geq n$

More equations than unknowns

there exists a non-trivial solution to $Ax = 0$ IFF $\text{rank}(A) < n$ $\quad \mathcal{N}(A)$ non-trivial

General result for vector linear systems

Now we would like to compute solutions

If A is non-singular, clearly $b = A^{-1}x$

In the general case, shall we try $b = A^{+}x$

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^{m}$ Then any x of the form

$$x = A^{+}b + \left(\mathbb{I} - A^{+}A\right) y \text{ where } y \in \mathbb{R}^{n} \text{ is arbitrary,}$$

is a solution of $Ax = b$.

Matrix Linear Equations

$$AX = B,\ A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times k},\ X \in \mathbb{R}^{n\times k}$$

Existence: The matrix linear equation $AX = B$ has a solution IFF $\mathcal{R}(B) \subseteq \mathcal{R}(A)$

Equivalently a solution exists IFF $AA^{+}B = B$

Characterisation: Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$ and suppose that $AA^{+}B = B$.

Then any matrix of the form

$$x = A^{+}B + \left(\mathbb{I} - A^{+}A\right)Y \text{ where } Y \in \mathbb{R}^{n\times k} \text{ is arbitrary,}$$

is a solution of $AX = B$.