EPFL

Eigenvalues & Eigenvectors EE-312

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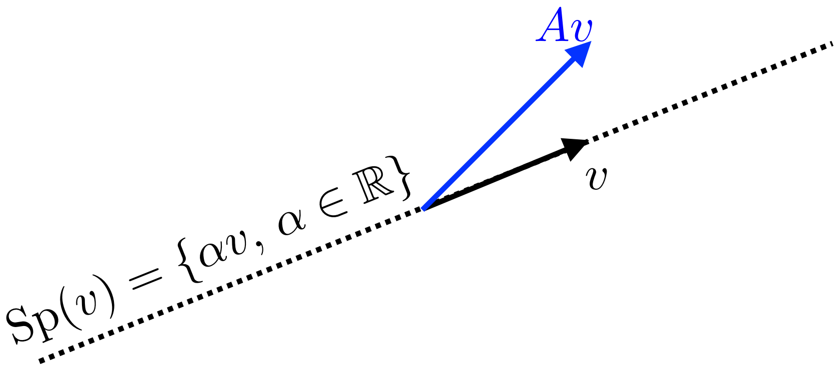
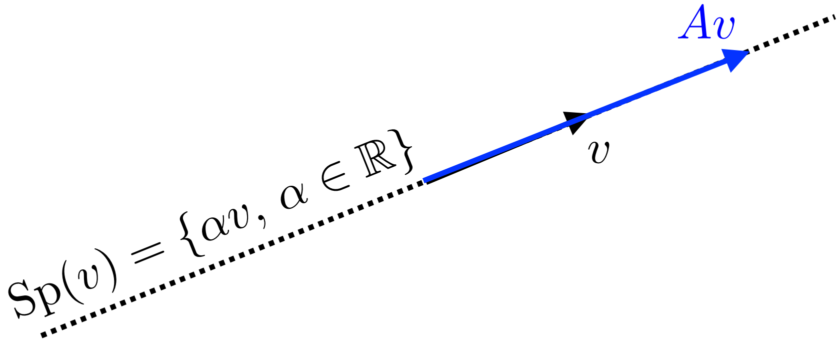
Once over, gently :)

Consider a linear transformation from $\mathbb{R}^2$ onto itself

Most vectors get "knocked off" their span





Sometimes, some vectors "remain" on their span

$$Av = \lambda v,\ \text{for some } \lambda \in \mathbb{R}$$

v is a (right) eigenvector of eigenvalue λ

Once over, gently :)

$$Av = \lambda v \Rightarrow Av = \lambda \mathbb{I} v \Rightarrow (A - \lambda \mathbb{I}) v = 0 \text{ for } v \neq 0$$

must be singular
(non-trivial null-space)

$$\det(A - \lambda \mathbb{I}) = 0$$

Once eigenvalues are identified you can solve for eigenvectors

Rem: imagine you have a full ortho basis of eigenvectors of A
the action of A becomes very simple !

Example: 3D rotation matrices

Definitions and Properties

A <u>nonzero</u> vector $x \in \mathbb{C}^n$ is a **right eigenvector** of $A \in \mathbb{C}^{n \times n}$

if there exists a scalar $\lambda \in \mathbb{C}$ called an **eigenvalue**, such that $Ax = \lambda x$

Similarly $y \in \mathbb{C}^n$ is **left eigenvector** corresponding to eigenvalue μ if $y^H A = \mu y^H$

Usually we normalise eigenvectors (so they have unit 2-norm)

$\pi(\lambda) = \det(A - \lambda \mathbb{I})$ is the **characteristic polynomial** of A.

For any $A \in \mathbb{C}^{n \times n}$, $\pi(A) = 0$ (Cayley-Hamilton)

Definitions and Properties

If $A \in \mathbb{C}^{n \times n}$ $\pi(\lambda)$ is a polynomial of degree n

$\pi(\lambda)$ has n roots, possibly with non-trivial multiplicity

The **spectrum** of $A \in \mathbb{C}^{n \times n}$ is the set of all eigenvalues of A, i.e all roots of $\pi(\lambda)$

$\Lambda(A)$ will denote the spectrum

A simple relationship between left/right eigenvectors of real square matrices

$A \in \mathbb{R}^{n \times n}$ $\quad Ay = \lambda y$ $\quad$ right eigenvector

$(Ay)^H = \boxed{\overline{\lambda} y^T = y^T A^T}$ $\quad$ left eigenvector of transposed matrix

Properties

$$\pi(\lambda) = \det(A - \lambda\mathbb{I}) = \prod_{i=1}^{n}(\lambda_i - \lambda) \quad \Rightarrow \pi(0) = \det(A) = \prod_{i=1}^{n}(\lambda_i)$$

$$\Lambda(A) = \Lambda(A^T)$$

if A is real valued, $\pi(A)$ has real coefficients, therefore its complex roots must appear in complex conjugate pairs $\Rightarrow \Lambda(A) = \overline{\Lambda(A)}$

Algebraic multiplicity of λ is the multiplicity of $\pi(\lambda)$

Geometric multiplicity of λ is the number of associated independent eigenvectors $\dim(\mathcal{N}(A - \lambda\mathbb{I}))$

Diagonalization - general case

Left/right eigenvectors are orthogonal

$A \in \mathbb{C}^{n\times n}$ $\quad Ax_i = \lambda_i x_i$ and $y_j^H A = \lambda_j y_j^H$ with $\lambda_i \neq \lambda_j$

Then $y_j^H x = 0$

... and form a linearly independent family

$A \in \mathbb{C}^{n\times n}$ with distinct eigenvalue $\lambda_1, \ldots, \lambda_n$

The corresponding n left (or right) eigenvectors are linearly independent

With proper normalisation, we get:

$$A = X\Lambda X^{-1} = X\Lambda Y^H = \sum_{i=1}^{n} \lambda_i x_i y_i^H$$

A is similar to the diagonal matrix Λ

Diagonalization: General Case

This points to an equivalent definition:

A complex matrix A is diagonalizable IFF it is similar to a diagonal matrix

There exists an invertible matrix X s.t. $A = X\Lambda X^{-1}$

Now let's recover previous objects and properties:

The column vectors of X are right eigenvectors (with corresponding eigenvalue in Λ)

Invertibility of X is equivalent to linear independence of eigenvectors.

The row vectors of X^{-1} are the left eigenvectors

Properties

Hermitian matrices ($A = A^H$) have real eigenvalues

$Ax = \lambda x \qquad A \in \mathbb{C}^{n \times n}$

$x^H A x = \lambda x^H x$

A is hermitian

$(x^H A x = \lambda x^H x)^H \Rightarrow x^H A^H x = \overline{\lambda} x^H x \Rightarrow x^H A x = \overline{\lambda} x^H x$

$x^H A x = \lambda x^H x = \overline{\lambda} x^H x$ and $x^H x \neq 0$ since x is an eigenvector

$$\Rightarrow \lambda = \overline{\lambda}$$

Properties

eigenvectors of a hermitian matrix corresponding to distinct eigenvalues are orthogonal

$$A \in \mathbb{C}^{n\times n} \quad A = A^H \qquad Ax = \lambda x \qquad Az = \mu z$$

$$z^H A x = \lambda z^H x$$

$$(z^H A x = \lambda z^H x)^H \Rightarrow x^H A^H z = \overline{\lambda} x^H z = x^H \Rightarrow x^H A z = \lambda x^H z$$

$$x^H A z = \mu x^H z \quad \Rightarrow \mu x^H z = \lambda x^H z \Rightarrow x^H z = 0 \text{ since } \mu \neq \lambda$$

The Spectral Theorem

If $A \in \mathbb{C}^{n \times n}$ is hermitian, there exists a orthonormal basis of $\mathbb{C}^n$ of eigenvectors of A

Main idea of the proof:

By the fundamental theorem of algebra applied to $\pi(\lambda)$, there is an eigenvalue λ_1 and a corresponding eigenvector v_1

Let W_1 be the orthogonal complement to v_1 in $\mathbb{C}^n$

$$\forall w \in W_1 : v_1^H A w = \overline{\lambda_1} v_1^H w = 0 \Rightarrow Aw \in W_1$$

A can be restricted to W_1 and we can proceed by induction

The Spectral Theorem

This decomposition expresses a nice factorization of hermitians matrices

$$X^{-1}AX = \Lambda \text{ or } A = X\Lambda X^{-1}$$

A is unitarily equivalent to the diagonal matrix Λ

where Λ is the diagonal matrix of eigenvalues (with multiplicity)

This also means you can write the linear transformation as a direct sum of orthogonal projections on the eigenspaces:

$$A = \sum_i \lambda_i P_{\lambda_i}$$

The Spectral Theorem

A fundamental reason the spectral theorem is so important :

Let f be an analytic function, i.e $f(t) = \sum_{k=0}^{+\infty} a_k t^k$ ex: sin, exp, ...

$A \in \mathbb{R}^{n \times n}$ such that $A = U \Lambda U^T$

Then $f(A) = U f(\Lambda) U^T$ where $f(\Lambda) = \text{diag}(f(\lambda_1), \cdots, f(\lambda_n))$

Proof: just think of eigendecomposition of A^k

This is crucial when solving differential equations $\frac{d}{dt} x(t) = A x(t)$