EPFL

Eigenvalues & Eigenvectors EE-312

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Diagonalization - general case

$A \in \mathbb{C}^{n\times n}$ **with distinct eigenvalues** $\lambda_1, \ldots, \lambda_n$

Corresponding right eigenvectors $X = [x_1, \cdots, x_n]$ form a linearly independent set

left eigenvectors $Y = [y_1, \cdots, y_n]$

No orthogonality ? No but "bi-orthogonality"

$Ax_i = \lambda_i x_i$ and $y_j^H A = \lambda_j y_j^H$ $\quad \lambda_i \neq \lambda_j$

Then $y_j^H x_i = 0$

x_i cannot be orthogonal to y_i as well (since they are independent)

So we can normalise the y_j's and/or the x_i's so that $y_i^H x_i = 1, \forall i = 1, \cdots n$

$$A = X\Lambda X^{-1} = X\Lambda Y^H = \sum_{i=1}^{n} \lambda_i x_i y_i^H$$

Diagonalization - special case

Hermitian matrices (symmetric in the real-valued case) : real eigenvalues

All eigenvectors of distinct eigenvalues are orthogonal
Moreover A has n linearly independent eigenvectors and it is always possible to find an orthonormal basis of eigenvectors of A
(even if some eigenvalues are degenerate)

If moreover A is positive definite $x^T A x > 0$, all eigenvalues are positive $\lambda_i > 0$
A is positive semi-definite $x^T A x \geq 0$, all eigenvalues are $\lambda_i \geq 0$

The Spectral Theorem

If $A \in \mathbb{C}^{n \times n}$ is hermitian, there exists a orthonormal basis of $\mathbb{C}^n$ of eigenvectors of A

<u>Main idea of the proof:</u>

By the fundamental theorem of algebra applied to $\pi(\lambda)$, there is an eigenvalue λ_1 and a corresponding eigenvector v_1

Let W_1 be the orthogonal complement to v_1 in $\mathbb{C}^n$

$$\forall w \in W_1 : v_1^H A w = \overline{\lambda_1} v_1^H w = 0 \Rightarrow Aw \in W_1$$

A can be restricted to W_1 and we can proceed by induction

The Spectral Theorem

This decomposition expresses a nice factorization of hermitians matrices

$$X^{-1}AX = \Lambda \text{ or } A = X\Lambda X^{-1}$$

A is unitarily equivalent to the diagonal matrix Λ

where Λ is the diagonal matrix of eigenvalues (with multiplicity)

This also means you can write the linear transformation as a direct sum of orthogonal projections on the eigenspaces:

$$A = \sum_i \lambda_i P_{\lambda_i}$$

The Spectral Theorem

A fundamental reason the spectral theorem is so important :

Let f be an analytic function, i.e $f(t) = \sum_{k=0}^{+\infty} a_k t^k$ ex: sin, exp, ...

$A \in \mathbb{R}^{n \times n}$ such that $A = U \Lambda U^T$

Then $f(A) = U f(\Lambda) U^T$ where $f(\Lambda) = \text{diag}(f(\lambda_1), \cdots, f(\lambda_n))$

Proof: just think of eigendecomposition of A^k

This is crucial when studying differential equations $\frac{d}{dt}x(t) = Ax(t)$

Solving linear homogeneous systems of ODEs

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0 \text{ with } x, x_0 \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n\times n}$$

Define the matrix exponential: $e^A = \sum_{k=0}^{+\infty} \frac{1}{k!} A^k$

Solution ($t \geq t_0$): $x(t) = e^{(t-t_0)A} x_0$

Proof: use the definition of the matrix exponential and differentiate under the sum

Higher Order Systems

$$x^{(k)}(t) = A_{k-1}x^{(k-1)}(t) + \cdots + A_0 x(t) \text{ where } A_i \in \mathbb{R}^{n\times n}$$

$$\text{and } x^{(k)}(t) = \frac{d^k}{dt^k}x(t)$$

Can be written as a larger order 1 ODE:

$$z(t) = \begin{bmatrix} x(t) \\ x^{(1)}(t) \\ \vdots \\ x^{(k-1)}(t) \end{bmatrix} \in \mathbb{R}^{nk} \qquad \dot{z}(t) = \begin{bmatrix} 0 & \mathbb{I} & 0 & \cdots & 0 \\ 0 & 0 & \mathbb{I} & \cdots & 0 \\ \vdots & & & & \vdots \\ A_0 & A_1 & A_2 & \cdots & A_{k-1} \end{bmatrix} z(t)$$

Modal Decomposition

Suppose now A is diagonalisable: $A = X\Lambda Y^H$

The matrix exponential can be written as: $e^A = \sum_{i=1}^{n} e^{\lambda_i} x_i y_i^H$

Therefore the solution of the homogeneous ODE becomes:

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i (t-t_0)} (y_i^H x_0) x_i$$

Modal directions are invariant

Modal velocities

Long-term behavior

$A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$

Consider the system: $\dot{x}(t) = Ax(t)$

Solutions involve only functions like: $t^k e^{\lambda_j t}$

Therefore the long-term evolution of solutions is governed by eigenvalue, i.e

$$\lim_{t\to+\infty} \|x(t)\| = \begin{cases} 0 & \text{if } \mathrm{Re}(\lambda_j) < 0 \quad \forall j \\ +\infty & \text{if } \mathrm{Re}(\lambda_j) > 0 \quad \text{for some } j \end{cases}$$

Example

Consider a damped pendulum: $mx^{(2)}(t) = -kx(t) - bx^{(1)}(t)$

$$z(t) = \begin{bmatrix} x(t) \\ x^{(1)}(t) \end{bmatrix} \qquad \dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} z(t)$$

You can now solve the system and check how $z(t)$ evolves as a function of the eigenvalues of the system.

Linear Homogeneous Systems of Difference Equations

Discrete-time equivalent to ODEs: $x[k+1] = Ax[k]$; $x[0]$ given

It is easy to see that: $x[k] = A^k x[0] \;\; k \geq 0$

And again, if A is diagonalizable this simplifies thanks to: $A^k = X\Lambda^k Y^H = \sum_{i=1}^{n} \lambda_i^k x_i y_i^H$

Asymptotic Behaviour of Linear Dynamical Systems

The eigendecomposition can also give us insights about the long-term evolution of the system

Example: discrete-time case $x[k+1] = Ax[k];\ x[0]$ given

We already know: $x[k] = A^k x[0] \;\; k \geq 0$

$$x[k] = A^k(c_1 x_1 + \cdots + c_n x_n)$$
$$= c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$$

If we have $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$

dominant eigenvalue

$$\lim_{k \to +\infty} x[k] \simeq \lambda_1^k c_1 x_1$$

system evolves in the direction of the dominant eigenvector

Inhomogeneous systems

$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_0) = x_0 \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m, \quad B \in \mathbb{R}^{n\times m}$ given

"Variation of parameters" formula $t \geq t_0$

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}Bu(s)ds$$

homogeneous case

At each time, a new solution with initial conditions is computed and added

Matrices and Eigendecompositions of Networks

Networks can be represented by matrices whose eigenvectors and eigenvalues have many interesting applications!

Undirected networks: adjacency matrix

$A[i, j] = 1$ IFF there is an edge linking nodes i and j

A is symmetric and the degree of node i is $d_i = \sum_j A[i, j]$

$D = \text{diag}(d_i)$ is the degree matrix of the network

Matrices and Eigendecompositions of Networks

Undirected networks:

The (combinatorial) Laplacian of the network is $L = D - A$

$L \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite, i.e $x^T L x \geqslant 0 \quad \forall x \in \mathbb{R}^n$

It has only real, non-negative eigenvalues and an ortho. basis of eigenvectors

The smallest eigenvalue is $\lambda_1 = 0$ and its multiplicity $= \{\#$ connected components$\}$

If a network is connected but has k clusters or communities there will typically be a gap between λ_k and λ_{k+1}. This is the basis of a famous algorithm...

Matrices and Eigendecompositions of Networks

Spectral Clustering

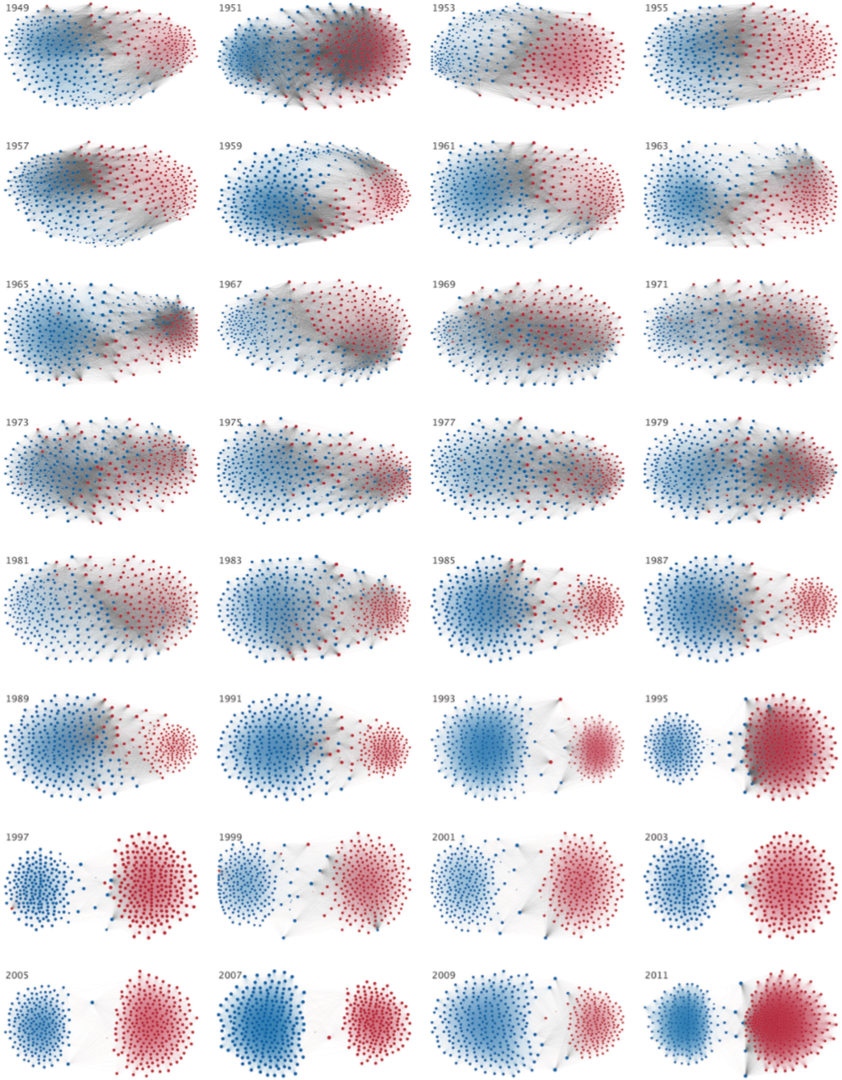
Input: adjacency matrix of graph with k communities or clusters

Compute: eigenvectors x_i corresponding to k smallest eigenvalues

$$X = \begin{pmatrix} | & \dots & | \\ x_1 & \dots & x_k \\ | & \dots & | \end{pmatrix}$$

k-dimensional graph features
will be similar for all nodes in same community

apply clustering to the feature matrix (group similar features)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Matrices and Eigendecompositions of Networks

Networks can be represented by matrices whose eigenvectors and eigenvalues have many interesting applications!

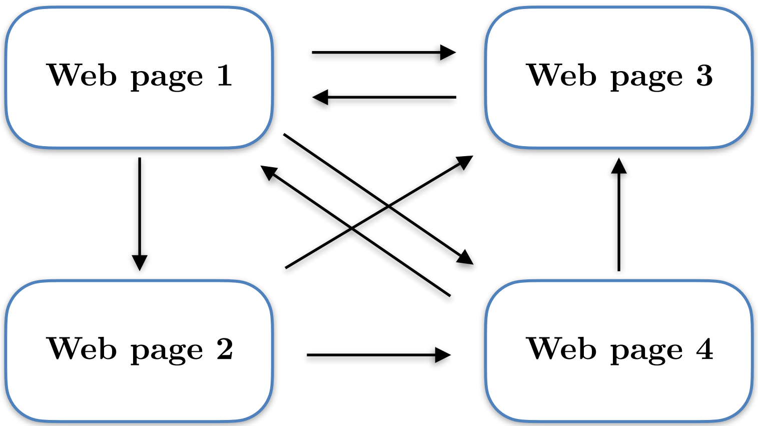
Directed networks: Directed adjacency matrix

$A[i,j] = 1$ IFF there is an edge from i to j

A is not symmetric, nodes have in and out degrees

PageRank:
The $1611 billion eigenvector

Web as a directed graph: nodes = web pages, edges = hyperlinks

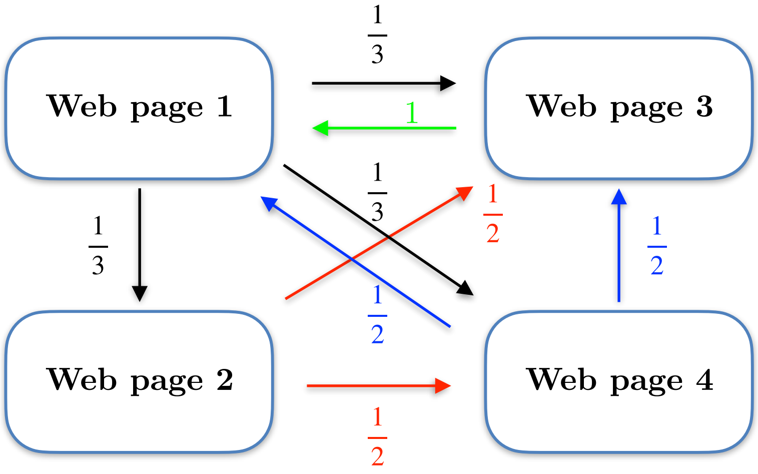


A web page is authoritative if many pages point to it
or if authoritative pages point to it

Challenge: Compute a ranking of web pages by authority

PageRank:
The $1611 billion eigenvector

Hyp. 1: Each page transfers its authority equally to pages it links to



Source page

$$A = \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Target page

Hyp. 2: Each link to a page increases its importance $v \mapsto Av$

PageRank: The \$1611 billion eigenvector

Source page

$$A = \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Target page

Dynamical systems view:

$$v[0] = \frac{1}{\text{numb. pages}}$$

$$v[k] = A^k v[0]$$

All matrices of this form ("column stochastic") have a dominant eigenvalue $= 1$

$$\lim_{k \to +\infty} v[k] = x_{\text{dominant}}$$

In our example: $x_{\text{dominant}} = \begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}$

Understanding Recurrent Neural Networks (RNNs)

Let's consider a state variable following a first order ODE: $\dot{h}(t) = f(h(t))$

$h(t) \in \mathbb{R}^n$ and $f()$ is a component-wise non-linearity

This can be discretised in time using the forward Euler method:

$$h_n = h_{n-1} + Tf(h_{n-1})$$

which has the form of a Recurrent Neural Net with no input

Understanding Recurrent Neural Networks (RNNs)

Consider the particular case: $\dot{h}(t) = f(Wh(t) + b)$ $W \in \mathbb{R}^{n \times n}$ Weight matrix

We will study how the solution $h(t)$ depends on initial conditions $h(0)$

$$\frac{d}{dt}\left(\frac{\partial h(t)}{\partial h(0)}\right) = J_F(h(t))\frac{\partial h(t)}{\partial h(0)}$$

$F(x) = f(Wx + b)$ Jacobian: $J_F(x)_{ij} = \frac{\partial F_i}{\partial x_j}(x)$

$J_F(x) = \operatorname{diag}\left(f'(Wx + b)\right) W$

Understanding Recurrent Neural Networks (RNNs)

$$\frac{d}{dt}\left(\frac{\partial h(t)}{\partial h(0)}\right) = J_F(h(t))\frac{\partial h(t)}{\partial h(0)}$$

as a function of t and initial conditions

$$\frac{d}{dt}A(t) = J_F(t)A(t)$$

$$A(0) = \mathbb{I}_n$$

$$A_{ij}(t) = \frac{\partial h_i(t)}{\partial h_j(0)}$$

If the Jacobian changes slowly with time, we therefore have:

$$A(t) = e^{tJ_F}$$

And we see that the eigenvalues of the Jacobian will play a crucial role!

Aside: stability of initial value problems

Stability (informal definition)**:**

Consider a solution $h(t)$ of the initial value problem $\dot{h}(t) = f(h(t)) \quad h(0) = h_0$

Let $z(t)$ be a solution with $z(0) = z_0$

Stable (Lyapunov stable): $\forall \delta > 0, \exists \epsilon > 0$ s.t

$$\|h_0 - z_0\| \leq \delta \Rightarrow \|h(t) - z(t)\| \leq \epsilon, \forall t$$

Asymptotically stable: $\exists \delta > 0$ s.t

$$\|h_0 - z_0\| \leq \delta \Rightarrow \lim_{t \to +\infty} \|h(t) - z(t)\| = 0$$

Another point of view: perturbation analysis

$$\dot{h}(t) = f(h(t))$$

$$z(t) = h(t) + \epsilon d(t)$$

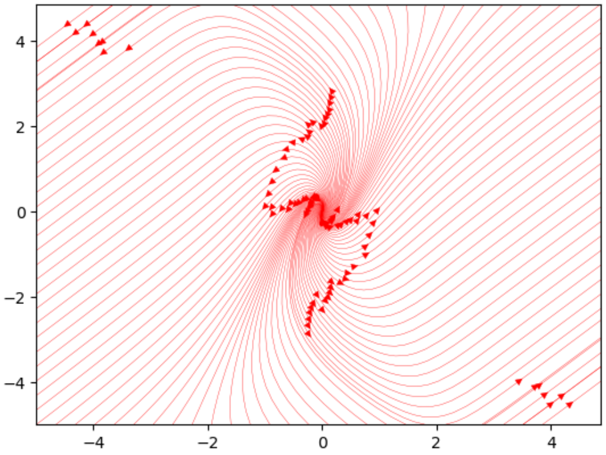
$$\begin{aligned} \dot{z}(t) &= \dot{h}(t) + \epsilon \dot{d}(t) \\ &= f\big((h(t) + \epsilon d(t)\big) \\ &= f\big((h(t))\big) + \epsilon J_f d(t) \end{aligned}$$

$$\dot{d}(t) = J_f d(t)$$

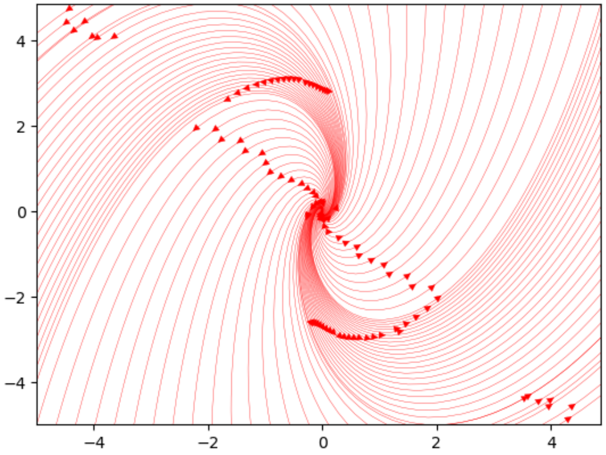
Again the spectrum of the Jacobian will inform us whether perturbations can grow, decay or sustain.

Aside: stability of initial value problems

Original RNN

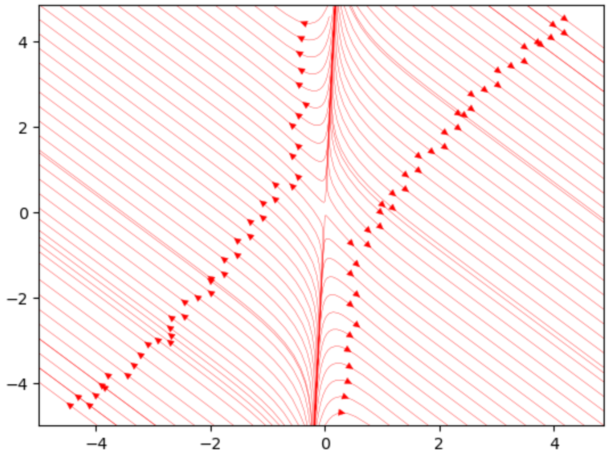


Dynamics linearized around (0,0)

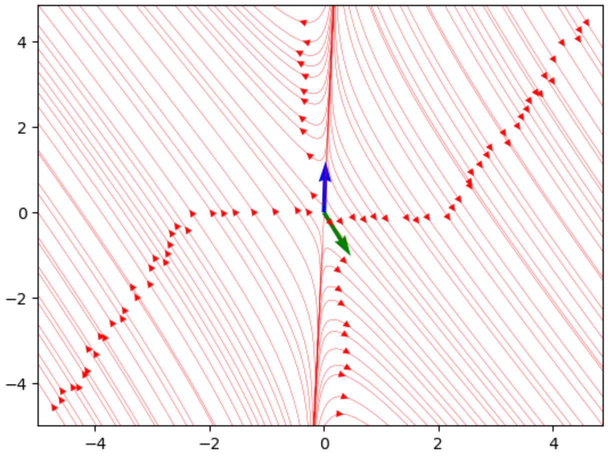



```
W=[ 0.64952629, -0.22683761],
        [ 1.40245917,  0.55148058]]
```

Original RNN

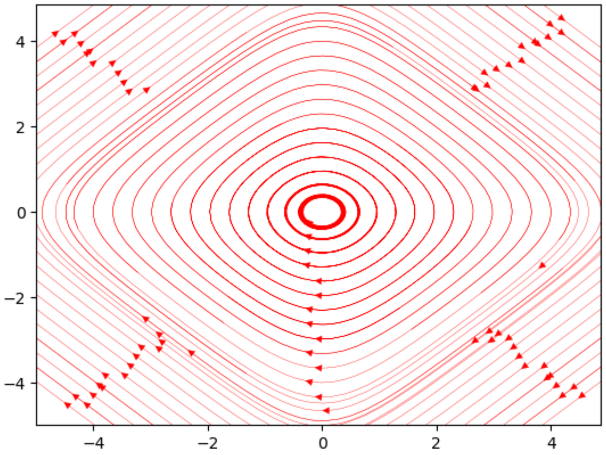


Dynamics linearized around (0,0)

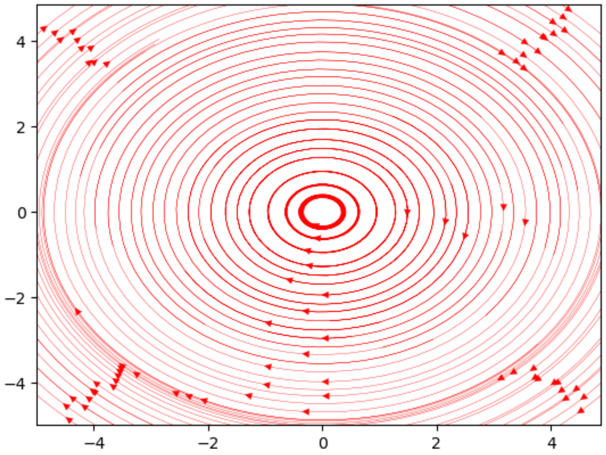



```
W= array([[ 1.20163856, -0.05038193],
          [-2.33322499, -0.14888851]])
```

Original RNN



Dynamics linearized around (0,0)




```
W= array([[ 0, 1],
          [-1, 0]])
```

Back to the discretization !

Question: when is Euler's approximation a stable discrete system ?

$$h_n = h_{n-1} + Tf(h_{n-1})$$

Again we know that with linearization/perturbation analysis, we can just look at how this will behave in the simplest linear case and in the Jacobian eigen-basis :

$$u_n = u_{n-1} + T\lambda u_{n-1}$$

$$u_{n+1} = (1 + T\lambda)u_n \rightarrow u_n = (1 + T\lambda)^n u_0$$

We therefore need: $|1 + T\lambda| < 1$