Signals & Systems
Shkel

Ecole Polytechnique Fédérale, Lausanne: Spring 2022
April 29, 2022

Midterm Exam

EXAM INFORMATION

General Note: In our view, the most important issue is to know how to address a particular problem. Therefore, there will be partial credit for good solution outlines even if not all the mathematical manipulations are completed.

*** Good Luck! ***

First Part *(Multiple Choice Questions)* 32 points

For each statement, select a single correct answer.

(Question 1) (4 Pts): Which of the following claims about the Fourier Transform is true?

- If $x(t) = \frac{1}{2}(u(t+1) - u(t-1))$, then $X(\omega) = \text{sinc}(w)$
- If $X(w) = 2\pi\delta(\omega - 2)$, then $x(t) = e^{-2t}u(t)$
- Signals $u(t)$ and $u(-t)$ have the same Fourier Transform
- If $x(t) = 2\sin(7t + \pi/3)$, then $X(w) = \frac{2\pi}{j}(\delta(w-7)e^{\frac{j\pi}{3}} - \delta(w+7)e^{-\frac{j\pi}{3}})$

Solution:

- (F) We can recognize that $x(t)$ is a box function from -1 to 1. Using the Appendix 4.B for the pairs, we find that $X(w) = sinc(\frac{w}{\pi})$.
- (F) Using the Appendix 4.B for the pairs we see that $x(t) = e^{2t}$
- (F) Using the Appendix 4.B for the pairs and property of time reversal
- (T) Using the Appendix 4.B for the pairs and the property of shifting in time. Specifically, define $z(t) = 2\sin 7t$ and observe that $x(t) = z(t + \frac{\pi}{3\cdot 7})$. Then $Z(\omega) = \frac{2\pi}{j}(\delta(w-7) - \delta(w+7))$ and

$$X(\omega) = e^{\frac{\pi}{3\cdot 7}\omega} Z(\omega) = \frac{2\pi}{j}(\delta(w-7)e^{\frac{j\pi}{3}} - \delta(w+7)e^{-\frac{j\pi}{3}}).$$

  The last equality comes from the fact that $f(\omega)\delta(\omega - \omega_0) = f(\omega_0)\delta(\omega - \omega_0)$. This problem could also be solved with Euler's formula and the harmonics pair from Appendix 4.B.

(Question 2) (4 Pts): Which of the following claims about the Fourier Transform is false?

- If $x(t)$ is an energy signal with non-zero energy, $X(\omega)$ must be a power signal
- If $x(t) = \int_{-\infty}^{\infty} y(\tau)z(t-\tau)d\tau$ then $X(\omega) = Y(\omega)Z(\omega)$
- If $X(w)$ is a Fourier transform of the signal $x(t)$, the Fourier transform of the signal $x(2t)$ is $\frac{1}{2}X(\frac{w}{2})$
- If $|X(w)|$ is an even function and $\arg(X(w))$ is an odd function, signal $x(t)$ with a Fourier transform $X(w)$ is a real function.

Solution:

- (F) Using Parseval's relation we see that if $x(t)$ has finite energy, then so does $X(\omega)$
- (T) This is the convolution in time property
- (T) This is a property of scaling in time and frequency: For $x(at)$, CTFT is given as $\frac{1}{|a|}X(\frac{w}{a})$

- (T) For real functions in time domain, property $X(w) = X^*(-w)$ holds. Representing $X(w) = |X(w)|e^{jarg(X(w)}$ and plugging it into the previous statement, we get $|X(w)| = |X(-w)|$ which is an even function, and $\arg(X(w)) = -\arg(X(-w))$ which is an odd function.

(Question 3) (4 Pts):

Which of the following claims about convolution properties is true for signals $x(t)$, $y(t)$ and $z(t) = x(t) * y(t)$?

- $x(t-1) * y(t-2) = x(t-2) * y(t-1)$
- $x(t) * y(-t) = z(-t)$
- $x(2t) * y(2t) = z(2t)$
- $\frac{dx(t)}{dt} * \frac{dy(t)}{dt} = \frac{dz(t)}{dt}$

Solution:

- (T) $x(t-1)*y(t-2) = x(t)*\delta(t-1)*y(t)*\delta(t-2) = x(t)*\delta(t-2)*y(t)*\delta(t-1) = x(t-2)*y(t-1)$
- (F) $x(-t) * y(-t) = z(-t)$ but $x(t) * y(-t) \neq z(-t)$ in general. A simple counter example is $x(t) = rect[2,3]$ and $y(t) = rect[0,1]$ where $rect[a,b] = u(t-a) - u(t-b)$
- (F) $x(2t) * y(2t) = 2z(2t)$ is actually the correct relation. You can obtain it by taking the Fourier Transform and use Convolution with Time/Frequency scaling properties. You could also obtain it by writing out the convolution integral and doing a change of variables.
- (F) The correct relation is $\frac{dx(t)}{dt} * \frac{dy(t)}{dt} = \frac{d^2z(t)}{dt^2}$. Again, you could obtain it wit ha Fourier Transform and differentiation in time property together with the convolution property

(Question 4) (4 Pts): The input-output relationship for an LTI system S satisfies $y[n] - \frac{3}{2}y[n-1] = x[n]$. Which of the following statements about the system is false?

- If this system is stable, then it is not invertible
- The system could be causal and have an impulse response $h[n] = \left(\frac{3}{2}\right)^n u[n]$
- The system could be stable and have the frequency response $H(e^{j\omega}) = \frac{1}{1-\frac{3}{2}e^{-j\omega}}$
- If this system is causal, then it is not stable

Solution:

- (F) The stable system is invertible and is given by $H(e^{j\omega}) = 1 - \frac{3}{2}e^{-j\omega}$
- (T) As seen in Problem Set 4.1, this impulse response could be found by assuming the condition of initial rest/causality

- (T) This frequency response could be found by applying the DTFT to both sides of the equation
- (T) This is true since the impulse response for a casual LTI system, $h[n] = \left(\frac{3}{2}\right)^n u[n]$, is not absolutely summable.

(Question 5) (4 Pts): Let the Discrete-Time Fourier Transform of the signal $x(t)$ be given by $X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$ where α is a real number with $|\alpha| < 1$. Which of the following claims is always true?

- The integral $\int_{-\infty}^{+\infty} |X(e^{j\omega})| d\omega$ exists (that is, it is not infinite)
- The shifted frequency version $X(e^{j(\omega-\omega_0)})$ is a Fourier Transform of a complex valued signal
- $\lim_{\omega\to\infty} X(e^{j\omega}) = 0$
- $X(e^{j\omega})$ is conjugate symmetric; that is, $X(e^{j\omega}) = X^*(e^{-j\omega})$

Solution:

- (F) Since the DTFT is 2π periodic, the integral of something always positive (since we take the absolute value), and never converging to 0, can not converge.
- (F) $X(e^{j(\omega+\omega_0)})$ represent the signal $e^{j\omega_0 n}x[n]$, where x_n is real due to the conjugate symmetric properties of the DTFT. Since the complex exponential is real when $\omega_0 = K\pi$, there are infinite conditions that will produce a real signal
- (F) The DTFT is a periodic signal and so it cannot converge to zero
- (T) This is a DTFT of a real signal, and so it is conjugate symmetric

(Question 6) (4 Pts): Let $x[n] = \delta[-n-1] - \delta[-n]$, then

- $X(e^{j\omega}) = e^{-j\omega} - 1$
- $X(e^{j\omega}) = e^{-j\omega} + 1$
- $X(e^{j\omega}) = e^{j\omega} - 1$
- $X(e^{j\omega}) = 1 - e^{-j\omega}$

Solution: The expression could also be simplified to $x[n] = \delta[n+1] - \delta[n]$ and $X(e^{j\omega}) = e^{j\omega} - 1$ is the correct answer.

(Question 7) (4 Pts): A discrete-time system is defined by an input-output relationship $y[n] = x[-n]$. Which of the following claims about the systems is true?

- The system is linear
- The system is time-invariant
- The system is causal
- The system is memoryless

Solution:

- (T) The system is linear since $\mathcal{H}\{ax_1[n] + bx_2[n]\} = ax_1[-n]+bx_2[-n] = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}$
- (F) The system is not time invariant since $\mathcal{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0] = x[-n+n_0]$.
- (F) The system is not causal since output at time $n < 0$ depend on future inputs.
- (F) The system is not memoryless. The outputs at time n depend on the inputs at time $-n$.

(Question 8) (4 Pts): Given an input $x(t) = e^{j3t}$, a stable LTI system produces an output $y(t) = e^{j3t}$. Based on this information, which of the following could NOT be the frequency response of the system?

- $H(\omega) = \delta(\omega)$
- $H(\omega) = 1$
- $H(\omega) = \frac{3j+1}{j\omega+1}$
- $H(\omega) = \begin{cases} 1, & \omega < \frac{3\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$

Solution:

- (T) We have that $y(t) = H(3)e^{j3t} = 0$ and so $H(\omega) = \delta(\omega)$ could not be the frequency response of the system
- (F) We have that $y(t) = H(3)e^{j3t} = e^{j3t}$ and so $H(\omega) = 1$ could be the frequency response of the system
- (F) We have that $y(t) = H(3)e^{j3t} = e^{j3t}$ and so $H(\omega) = \frac{3j+1}{j\omega+1}$ could be the frequency response of the system
- (F) We have that $y(t) = H(3)e^{j3t} = e^{j3t}$ and $H(\omega)$ could be the frequency response of the system

Problem 2 *(Difference Equations.)* 20 points

(a)(4pts) Find the frequency response of a stable LTI system characterized by

$$y(t) + 4\frac{d}{dt}y(t) + 3\frac{d^2}{dt^2}y(t) = 2\frac{d^2}{dt^2}x(t) \tag{1}$$

(b)(8pts) Find the impulse response of this system.

(c)(8pts) Suppose that an input $x(t) = e^{-t}u(t)$ is applied to the system. What is the output $y(t)$?

Solution: *(a)* Since the system is stable and LTI, applying Fourier transform we get

$$Y(\omega)[1 + 4j\omega + 3(j\omega)^2] = 2(j\omega)^2 X(w) \tag{2}$$

Thus, the frequence response of the LTI system is

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2(j\omega)^2}{1 + 4j\omega + 3(j\omega)^2} = \frac{2(j\omega)^2}{(1 + j\omega)(1 + 3j\omega)} = \frac{2}{3} - \frac{2}{3}\frac{1 + 4j\omega}{(1 + j\omega)(1 + 3j\omega)} \tag{3}$$

(b) Applying partial fractions by "everywhere but not there method"

$$H(\omega) = \frac{2}{3} - \frac{2}{3}\frac{1 + 4j\omega}{(1 + j\omega)(1 + 3j\omega)} = \frac{2}{3} - \frac{2}{3}\left[\frac{3/2}{1 + j\omega} + \frac{-1/2}{1 + 3j\omega}\right] \tag{4}$$

Thus, using the inverse Fourier transform table, we get

$$h(t) = \frac{2}{3}\delta(t) - e^{-t}u(t) + \frac{1}{9}e^{-t/3}u(t) \tag{5}$$

(c)

$$X(\omega) = \frac{1}{1 + j\omega} \tag{6}$$

$$\implies Y(\omega) = X(\omega).H(\omega) = \frac{1}{1 + j\omega}\left\{\frac{2}{3} - \frac{2}{3}\left[\frac{3/2}{1 + j\omega} + \frac{-1/2}{1 + 3j\omega}\right]\right\} \tag{7}$$

$$= \frac{2}{3}\frac{1}{1 + j\omega} - \frac{1}{(1 + j\omega)^2} + \frac{1}{3}\frac{1}{(1 + j\omega)(1 + 3j\omega)} \tag{8}$$

$$= \frac{2}{3}\frac{1}{1 + j\omega} - \frac{1}{(1 + j\omega)^2} + \frac{1}{3}\left[\frac{-1/2}{1 + j\omega} + \frac{3/2}{1 + 3j\omega}\right] \tag{9}$$

$$= \frac{1}{2}\frac{1}{1 + j\omega} - \frac{1}{(1 + j\omega)^2} + \frac{1}{2}\frac{1}{1 + 3j\omega} \tag{10}$$

Using Inverse Fourier transform tables,

$$y(t) = \frac{1}{2}e^{-t}u(t) - te^{-t}u(t) + \frac{1}{6}e^{-t/3}u(t) \tag{11}$$

Problem 3 *(System Composition)* 24 points

In this problem, we study the system composition illustrated in Figure 1 with input $x(t)$ and output $y(t)$. System $\mathcal{H}_1$ is described by the impulse response $h_1(t) = \delta(t-2)$, while system $\mathcal{H}_2$ is described as $y_2(t) = 2x_2(t-1)$.

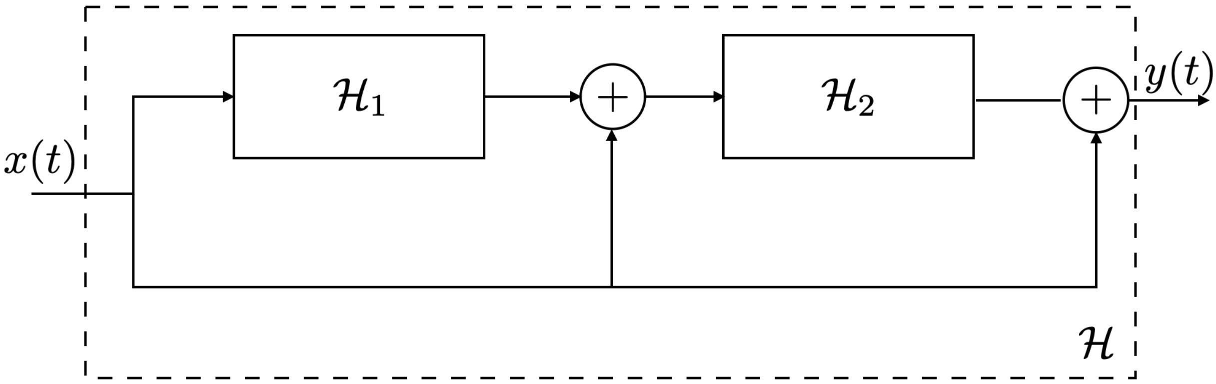


Figure 1: A composed system

(a)(8pts) Find the impulse response of the composed system $\mathcal{H}$.

(b)(8pts) Is the composed system $\mathcal{H}$ stable? Is it causal?

(c)(8pts) Signal $x(t)$ is given as

$$x(t) = \begin{cases} 2, -2 < t < 2 \\ 0, \text{otherwise} \end{cases} \tag{12}$$

Calculate and sketch the output of the system $y(t)$.

Solution

(a) We first need to calculate the impulse response of $\mathcal{H}_2$. By sending a Dirac impulse to the input of the system, impulse response of the system $\mathcal{H}_2$ is given as:

$$h_2(t) = 2\delta(t-1) \tag{13}$$

With these types of problems, it is convenient to define the intermediate signals.

Let $w(t)$ denote the signal entering $\mathcal{H}_2$. Then

$$w(t) = x(t) * h_1(t) + x(t) = x(t) * (\delta(t) + h_1(t)) \tag{14}$$

$y(t)$ is then calculated as

$$y(t) = w(t) * h_2(t) + x(t) \tag{15}$$

Combining these two:

$$y(t) = x(t) * (\delta(t) + h_1(t)) * h_2(t) + x(t) = x(t) * (h_2(t) + h_1(t) * h_2(t) + \delta(t)) \tag{16}$$

Impulse response of the system is given as

$$h(t) = h_2(t) + h_1(t) * h_2(t) + \delta(t) \tag{17}$$

For our $h_1(t)$ and $h_2(t)$, impulse response is

$$h(t) = 2\delta(t-1) + 2\delta(t-3) + \delta(t) \tag{18}$$

(b) For a system to be causal, its impulse response should be equal to zero for each $t < 0$. Since this is a case for our $h(t)$, the system is causal.

To inspect the stability, we can first rewrite our system in the form

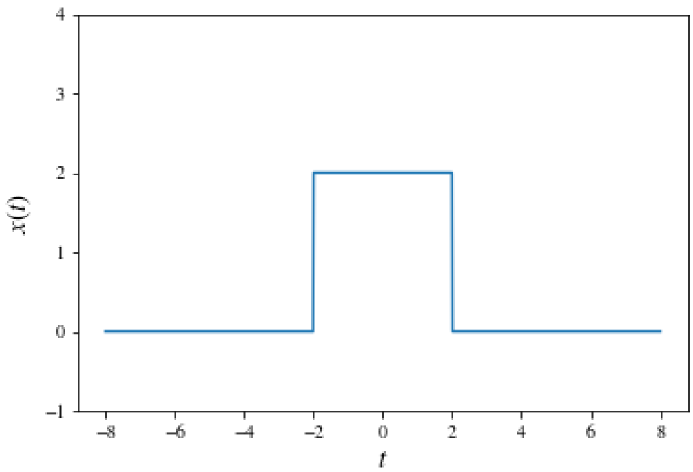
$$y(t) = x(t) + 2x(t-1) + 2x(t-3) \tag{19}$$

If $|x(t)| < B$ (in other words bounded), $|y(t|$ will be smaller than $5B$, which is bounded as well. This means that the system is stable.

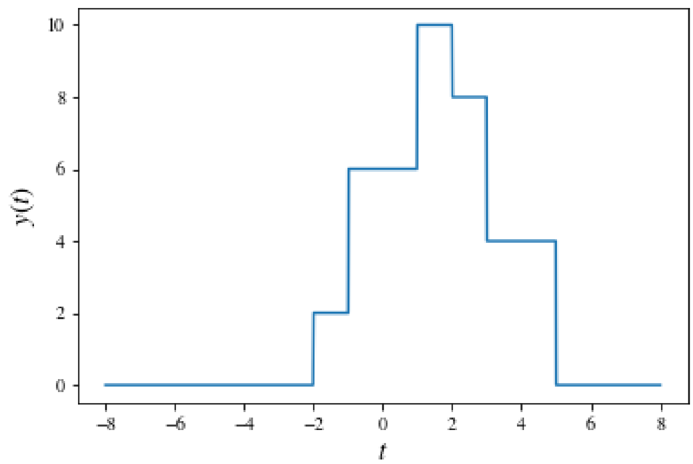
(c) As mentioned, our system has a form

$$y(t) = x(t) + 2x(t-1) + 2x(t-3) \tag{20}$$

$x(t)$ is a box function that can be sketched as follows:



By shifting $x(t)$ and summing the shifted signals, we get $y(t)$:



Problem 4 *(LTI Systems)* 24 points

(a)(7pts) First, let $a > 0$ be a real number and let us define an LTI system with the following impulse response:

$$h_a(t) = \begin{cases} \frac{1}{2a}, & |t| \leq \text{a} \\ 0, & \text{otherwise.} \end{cases}$$

A signal $x_1(t) = \sin(\omega_0 t)$ is applied to the system. Find the output $y_{1,a}(t)$ using convolution in the time domain.

(b)(7pts) A signal $x_2(t) = u(t)$ is applied to the system in part (a). Find the output $y_{2,a}(t)$ using convolution in the time domain.

(c)(6pts) Find the frequency response $H_a(\omega)$ of the system in part (a).

(d)(4pts) We now decide to push our system to its limit and make a very small. What should happen to our system as we do this? That is, what will happen to the frequency response and the impulse response as a goes to zero? What do you expect $y_{1,a}(t)$ and $y_{2,a}(t)$ to converge to as a goes to zero?

Solution: *(a)*

The convolution integral can be expressed as:

$$y_{1,a}(t) = \frac{1}{2a}\int_{-a+t}^{a+t} \sin(\omega_0\tau)\, d\tau = -\frac{1}{2a}\frac{1}{\omega_0}\cos(\omega_0\tau)\Big|_{-a+t}^{a+t} = \frac{\sin(a\omega_0)}{a\omega_0}\sin(\omega_0 t) = sinc(\frac{a}{\pi}\omega_0)\sin(\omega_0 t)$$

This solution can be derived expressing:

$$-\frac{1}{2a}\frac{1}{\omega_0}cos(\omega_0\tau)\Big|_{-a+t}^{a+t} = \frac{1}{2a\omega_0}(cos(\omega_0(t-a)) - cos(\omega_0(t+a)))$$

end either use the Werner formulae or decompose the cosine in complex exponential

(b)

$$h_a(t) * u(t) = \int_{-\infty}^{\infty} h_a(\tau)u(t-\tau)d\tau = \frac{1}{2a}\int_{-a}^{a} u(t-\tau)d\tau$$

For $t < -a$ we see that $u(t-\tau) = 0$ for all $-a < \tau < a$ and so the integral is zero.

For $-a < t < a$ we see that

$$\frac{1}{2a}\int_{-a}^{a} u(t-\tau)d\tau = \frac{1}{2a}\int_{-a}^{t} d\tau = \frac{t+a}{2a}.$$

Finally, for $t > a$ we see that

$$\frac{1}{2a}\int_{-a}^{a} u(t-\tau)d\tau = \frac{1}{2a}\int_{-a}^{a} d\tau = 1.$$

Thus, the overall answer is

$$h_a(t) * u(t) = \begin{cases} 0, & t < -a \\ \frac{t+a}{2a}, & -a < t < a \\ 1, & t > a. \end{cases}$$

(c)

The frequency response is the Fourier transform of $h(t)$. From Appendix 4.B this is

$$H(\omega) = \text{sinc}\left(\frac{a}{\pi}\omega\right).$$

(d)

Qualitatively, as a goes to zero, the impulse response will look more and more like a Dirac delta. Thus, the system will behave more like an identity system. The frequency response will also converge to 1. That is, we can see via the L'Hopital's rule that

$$\lim_{a\to 0} \text{sinc}\left(\frac{a\omega_0}{\pi}\right) = \lim_{a\to 0} \frac{\sin a\omega_0}{a\omega_0} = 1.$$

Likewise, we would expect $y_{1,a}(t)$ and $y_{2,a}(t)$ to converge to $x_1(t)$ and $x_2(t)$, respectively.