Signals & Systems
Spring 2022

Ecole Polytechnique Fédérale, Lausanne

Final Exam Solutions

Question 13

Find the transfer function for each system described below. Remember to specify the algebraic expression and the ROC. Justify your answer.

(Transfer Function) 12pts

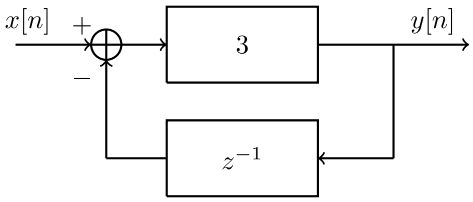
(a) An LTI system described by the input-output relationship

$$y(t) = \int_{-\infty}^{t} x(\tau - 3)d\tau$$

(b) A stable continuous-time LTI system with the frequency response $H(\omega) = \frac{1}{(j\omega)^2 - j\omega - 6}$

(c) A discrete-time LTI system with an impulse response $h[n] = \delta[n - n_0] + a_0\delta[n] + \delta[n + n_0]$ for some real constant $a_0 > 0$ and integer $n_0 > 0$

(d) A causal LTI system described by the following block diagram



where recall that the labels 3 and z^{-1} denote the transfer functions for the two composed systems

SOLUTION

(a) (3 pts) First, we can find the impulse response as

$$h(t) = \int_{-\infty}^{t} \delta(\tau - 3)d\tau = u(t - 3)$$

Taking the Laplace transform, and using time-shift property and unit step pair gives us

$$H(s) = \frac{e^{-3s}}{s}, \quad Re(s) > 0$$

It is also possible to take the Laplace transform of both sides of the input-output equation, apply integration in time property, and apply shift in time property to get

$$\begin{aligned} Y(s) &= \frac{1}{s}Z(s), \quad Re(s) > 0 \cap R \\ &= \frac{e^{-3s}}{s}X(s), \quad Re(s) > 0 \cap R \end{aligned}$$

where $z(t) = x(t - 3)$.

Note that this holds for all $x(t)$, including those for which the ROC is the whole complex plane. From this, the correct $H(s)$ follows as

$$H(s) = \frac{e^{-3s}}{s}, \quad Re(s) > 0$$

(b) (3 pts)

Since the frequency response is obtained by particularizing the transfer function to the real line, the algebraic expression must be of the form

$$H(s) = \frac{1}{s^2 - s - 6}$$

The system is stable and so the ROC must contain the imaginary axis and so

$$H(s) = \frac{1}{s^2 - s - 6} = \frac{1}{(s-3)(s+2)}, \quad -2 < Re(s) < 3.$$

(c) (3 pts)

Taking the z-transform of the impulse response we obtain,

$$H(z) = z^{-n_0} + a_0 + z^{n_0}, \quad |z| \neq 0 \text{ and } |z| \neq \infty$$

(d) (3 pts) This is a standard feedback composition of a one-sided exponential derived in lectures

$$H(z) = \frac{3}{1 + 3z^{-1}}, \quad |z| > 3.$$

The ROC is right-sided since the system is causal. We could also just apply the feedback composition equation with the forward system being 3 and the feedback system z^{-1}.

Question 14 *(Difference Equations)* 12 points

An LTI system is described by the following difference equation:

$$y[n+2] - 5y[n+1] + 6y[n] = x[n+1].$$

(a) Find the transfer function $H(z)$ and draw the pole-zero plot for it.

(b) If the system is known to be stable, determine the impulse response of the system. Could this system be causal?

(c) If the system is known to be causal, determine the impulse response of the system. Could this system be stable?

Solution:

(a) (4 pts) Taking Z-transformation of both sides and applying the property of shifting in time we get:

$$z^2Y(z) - 5zY(z) + 6Y(z) = zX(z) \tag{1}$$

Transfer function is then:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z^2 - 5z + 6} = \frac{z}{(z-2)(z-3)} \tag{2}$$

It can be rewritten as:

$$H(z) = \frac{z^{-1}}{(1-2z^{-1})(1-3z^{-1})} = \frac{-1}{1-2z^{-1}} + \frac{1}{1-3z^{-1}} \tag{3}$$

Transfer function has two poles, at $z = 2$ and $z = 3$. This corresponds to the following pole-zero plot is shown in Figure 1.

(b) (4 pts) In order for system to be stable, ROC has to include the unit circle. Thus, ROC is $|z| < 2$. The corresponding impulse response is:

$$h[n] = 2^n u[-n-1] - 3^n u[-n-1] \tag{4}$$

Since ROC is spreading inward, system is anticausal.

(c) (4 pts) In order for system to be causal, ROC has to expand outward. Thus, ROC is $|z| > 3$. The corresponding impulse response is:

$$h[n] = -2^n u[n] + 3^n u[n] \tag{5}$$

Since the ROC doesn't include the unit circle, system is not stable.

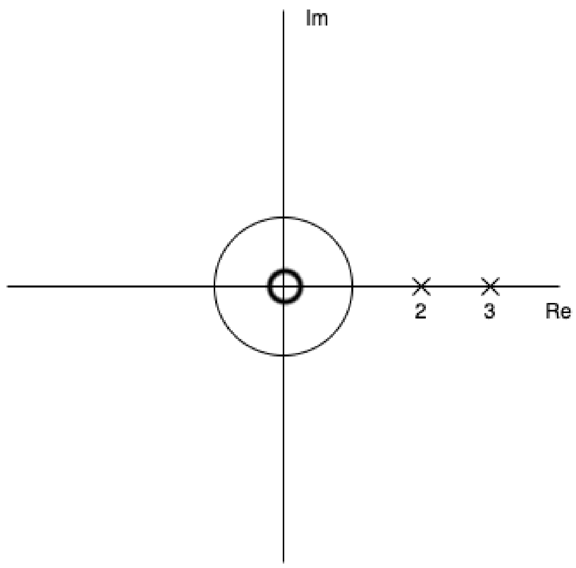


Figure 1: Pole-zero plot for the system

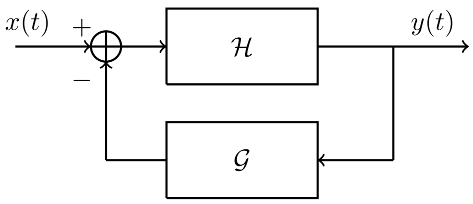
Question 15 *(Feedback Composition)* 12 points

(a) A causal LTI system $\mathcal{H}$ satisfies the following input-output relationship

$$\frac{dy(t)}{dt} = K_0 x(t) + K_1 y(t),$$

where K_0 and K_1 are positive real numbers. Determine if the system $\mathcal{H}$ is stable and find its transfer function.

(b) Next, we are going to analyze the feedback system composition given as follows, where $\mathcal{H}$ is the system from part (a) and $\mathcal{G}$ is another causal, LTI system



Find the transfer function of the overall composite system. Your answer should be in terms of K_0, K_1 and $G(s)$.

(c) Suppose that the system $G(s)$ is constrained to be an amplifier; that is, $G(s) = K_2$ where K_2 is any real constant. Is it possible to design $G(s)$ so that the overall system in (b) is stable? If yes, determine the constraints on K_2 (in terms of K_1 and K_0). If not, explain why.

Solution:

(a) (5 pts)

Taking the Laplace transform of both sides together with the differentiation we obtain

$$sY(s) = K_0 X(s) + K_1 Y(s)$$

and

$$Y(s) = \frac{K_0}{s - K_1} X(s) \implies H(s) = \frac{K_0}{s - K_1}$$

The system has a pole at $s = K_1$ and it is causal so the ROC must be $Re(s) > K_1 > 0$. The ROC does not include the imaginary axis and so the system is not stable.

(b) (3 pts)

$$H_{all}(s) = \frac{H(s)}{1 + H(s)G(s)} = \frac{K_0}{s - K_1} \frac{1}{1 + \frac{K_0}{s - K_1} G(s)} = \frac{K_0}{s - K_1 + K_0 G(s)}$$

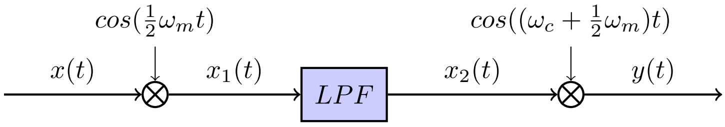
(c) (4 pts) If $G(s) = K_2$ then

$$H_{all}(s) = \frac{K_0}{s - K_1 + K_0 K_2}$$

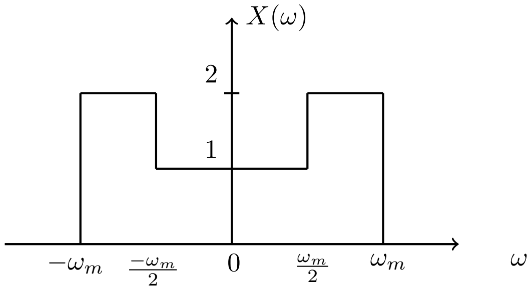
The overall system is still casual, so the poll $s = K_1 - K_0 K_2$ must be in the left half-plane. That is $K_1 - K_0 K_2 < 0$ and so $K_2 > \frac{K_1}{K_0}$.

Question 16 *(Frequency Domain)* 14pts

(a) Consider the following system where $\omega_c >> \omega_m$ and LPF represents an ideal low-pass filter with cutoff frequency $\frac{\omega_m}{2}$



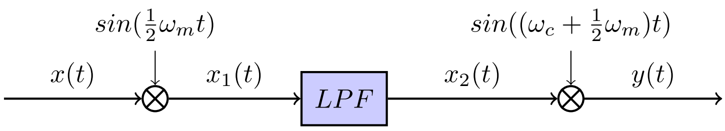
The input signal $x(t)$ has the following Fourier transform,



What is the Fourier transform of $y(t)$? Sketch the Fourier transforms of $x_1(t)$ and $x_2(t)$ first to show your work.

Hint: The notation $\omega_c >> \omega_m$ means that ω_c is much larger than ω_m. If you find that confusing, you may assume, for simplicity, that $\omega_c = 3\omega_m$.

(b) Next, we construct the same system, but with cosine signals replaced by sines. That is

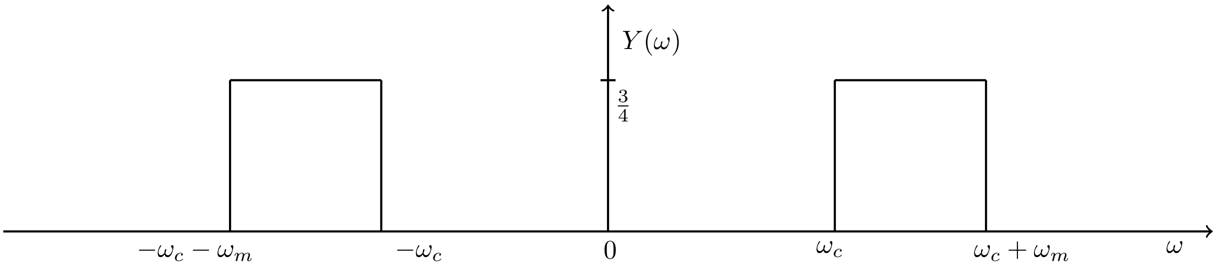


What is the Fourier Transform of $y(t)$ given that the Fourier transform of $x(t)$ is the same as in part (a)?

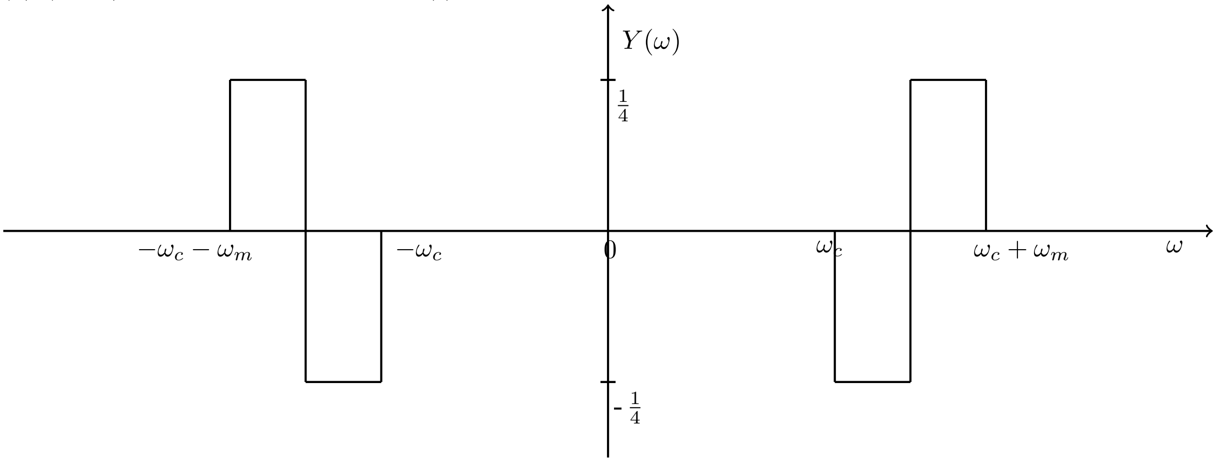
(c) If the systems in parts (a) and (b) are connected in parallel, what is the output given the same input $x(t)$? What do you observe about the output?

SOLUTION

(a) (5 pts) Frequency response of $y(t)$



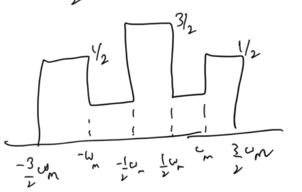
(b) (5 pts) Frequency response of $y(t)$

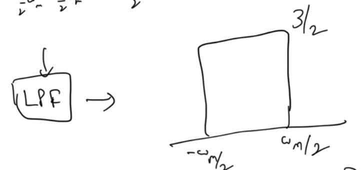


(c) (4 pts) On adding we see that we separate the right and left sides of the original signal into two separate parts but with half of the amplitude.

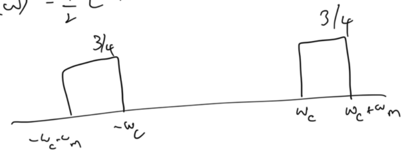
Qn 16.

a) $X_1(\omega) = \frac{1}{2}\left[X\left(\omega - \frac{\omega_m}{2}\right) + X\left(\omega + \frac{\omega_m}{2}\right)\right]$

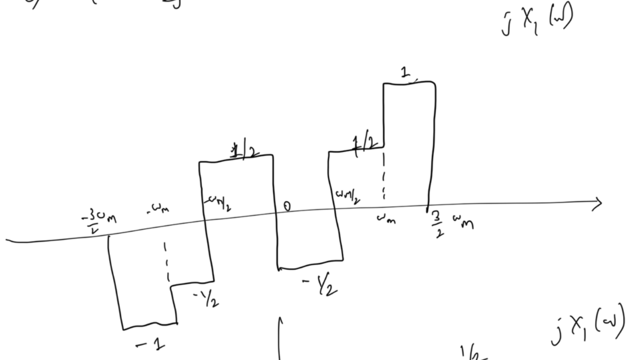


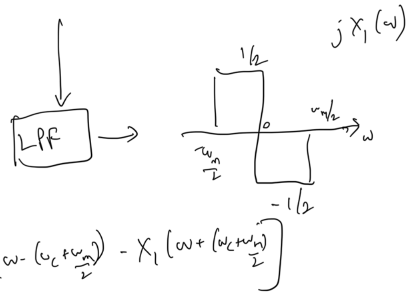


$X_2(\omega) = \frac{1}{2}\left[X_1\left(\omega - (\omega_c + \tfrac{1}{2}\omega_m)\right) + X_1\left(\omega + \omega_c + \tfrac{1}{2}\omega_m\right)\right]$

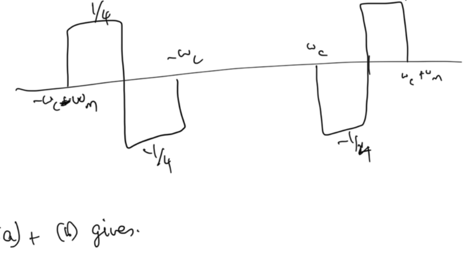


b) $X_1(\omega) = \frac{1}{2j}\left[X\left(\omega - \frac{\omega_m}{2}\right) - X\left(\omega + \frac{\omega_m}{2}\right)\right]$

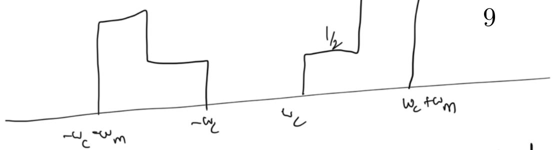




$X_2(\omega) = \frac{1}{2j}\left[X_1\left(\omega - (\omega_c + \frac{\omega_m}{2})\right) - X_1\left(\omega + (\omega_c + \frac{\omega_m}{2})\right)\right]$



c) (a) + (b) gives.



Original signal but scaled by ½ and pushed away from the middle!

Question 17 *(LTI System)* 14pts

We are given a certain LTI system with an impulse response $h_0(t)$. We are told that when the input is $x_0(t)$, the output is

$$y_0(t) = \begin{cases} 0, & t \leq 0 \\ t, & 0 < t \leq 1 \\ 1, & t > 1. \end{cases}$$

(a) Suppose that the input signal $x(t) = x_0(-t)$ is applied to this LTI system. Do we have enough information to determine the output signal $y(t)$? If yes, find and sketch the output signal $y(t)$. If no, prove that this is not possible.

(b) Suppose that the input signal $x(t) = x_0(t-2)$ is applied to an LTI system with the impulse response $h(t) = h_0(t+1)$. Do we have enough information to determine the output signal $y(t)$? If yes, find and sketch the output signal $y(t)$. If no, prove that this is not possible.

(c) Suppose that the input signal $x(t) = x_0(t)$ is applied to an LTI system with the impulse response $h(t) = \int_{t-1}^{t} h_0(\tau)d\tau$. Do we have enough information to determine the output signal $y(t)$? If yes, find and sketch the output signal $y(t)$. If no, prove that this is not possible.

SOLUTION

(a) (4 pts)

This is not possible. One way to see this is to write

$$y(t) = h_0(t) * x(t)$$

and to take CTFT of both sides to obtain

$$Y(\omega) = H_0(\omega)X(\omega) = H_0(\omega)X_0(-\omega).$$

Then we can argue that if we know $Y_0(\omega) = H_0(\omega)X_0(\omega)$ it does not necessarily mean that we know $Y(\omega) = H_0(\omega)X_0(-\omega)$. Indeed, we reverse one signal in time but not the other. Unless one of the signals is symmetric in ω, it is not possible to determine what would happen to their product.

Another way to see this is to construct a counter example. Consider the following signals

$$x_0(t) = y_0(t) \text{ and } h_0(t) = \delta(t).$$

Then

$$y(t) = h_0(t) * x(t) = h_0(t) * x_0(-t) = y(-t).$$

On the other hand, let

$$x_0(t) = \delta(t-1) \text{ and } h_0(t) = y_0(t+1).$$

Then

$$y(t) = h_0(t) * x(t) = h_0(t) * x_0(-t) = y_0(t+1) * \delta(t+1) = y_0(t+2)$$

A very common error was to say that it is possible because of time-invariance. But, this is not a correct application of the time-invariance property. Time invariance applies to shifts in time, not time reversal.

(b) (4 pts)

We have enough information. That is

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= (h_0(t) * \delta(t-2)) * (x_0(t) * \delta(t+1)) \\ &= (h_0(t) * x_0(t)) * (\delta(t-2) * \delta(t+1)) \\ &= y_0(t) * \delta(t-1) = y_0(t-1) \end{aligned}$$

The resulting signal is a shifted version of the original signal.

(c) (6 pts)

This is possible. We can write

$$h(t) = h_0(t) * (u(t) - u(t-1))$$

Then

$$\begin{aligned} y(t) = h(t) * x(t) &= h_0(t) * (u(t) - u(t-1)) * x(t) \\ &= y_0(t) * (u(t) - u(t-1)) = \int_{t-1}^{t} y_0(\tau) d\tau \end{aligned}$$

There are several cases to consider.

(1) If $t < 0$, then $y(t) = 0$.

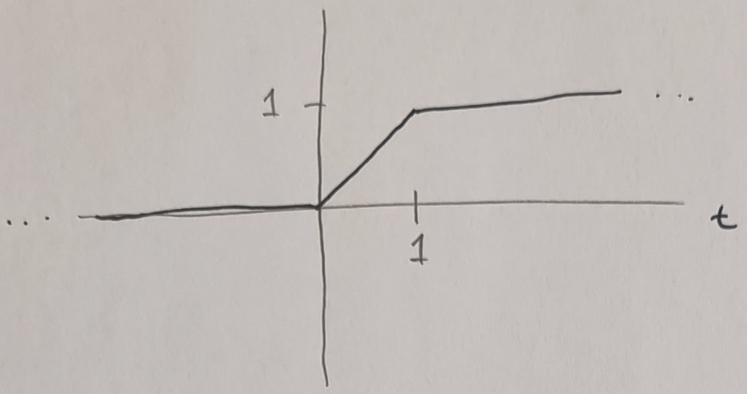
(2) If $0 < t \leq 1$ then $y(t) = \int_0^t \tau d\tau = \frac{1}{2}t^2$.

(3) If $1 < t \leq 2$ then $y(t) = \int_{t-1}^1 \tau d\tau + \int_1^t \tau d\tau = -\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}$.

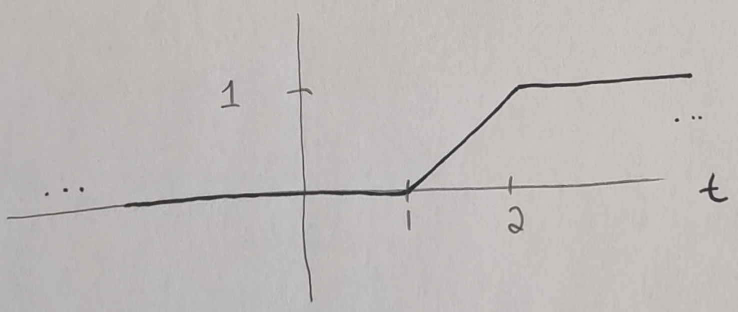
(3) If $t > 2$ then $y(t) = 1$.

Q17 (b)

$y_0(t)$



$y_0(t-1)$



Q 17 (c)

$y(t)$