Signals & Systems
Spring 2020

Ecole Polytechnique Fédérale, Lausanne

Sample Final

Last name	First name	SCIPER Nr	Points

- **Rules:** This exam is closed-book and closed-notes; calculators, computing and communication devices are not permitted. Two handwritten and not photocopied double-sided A4 sheets of notes are allowed. Moreover, copies of the tables in Sections 4.A, 4.B, 4.C, 4.D, 6.A, 6.B, 7.A, and 7.B in the lecture notes will be attached to the exam sheet.
- You have 180 minutes to complete this exam.
- Unless explicitly stated otherwise, detailed derivations of the results are required for full credit on all **open problems**.
- For the **multiple choice with unique answer** questions, we give
  - +3 points if your answer is correct,
  - 0 points if your answer is incorrect.
- For the **multiple choice with multiple answers** questions, we give
  - +4 points for all correct answers,
  - +2 points for one incorrect answer and three correct answers,
  - 0 points for other possibilities of answers.

1.*(3 Pts)* The integral $\int_{-\infty}^{\infty} \delta(t-6) \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau) d\tau dt$ evaluates to **(check the unique answer!)**

1. $f(6t)$,
2. 0,
3. $f(6)$,
4. $f(0)$.

2. *(4 Pts)* Which of the following claims about properties of signals and systems are true? **(check all that apply!)**

1. The discrete-time signal $x[n] = \frac{1}{\sqrt{n}} u[n]$ is not an energy signal.
2. The discrete-time signal $z[n] = \cos(2n)$ has fundamental period π.
3. The continuous-time system $\mathcal{H}\{x(t)\} = \frac{d}{dt}x(t)$ is linear and time-invariant.
4. The continuous-time system $\mathcal{H}\{x(t)\} = (x(t) - \mu)^3$ is memoryless but not causal.

3. *(4 Pts)* Which of the following systems are linear? **(check all that apply!)**

1. $y(t) = x(t)$.
2. $y(t) = (x(t))^n$ for any $n \in \mathbb{Z}^+$.
3. $y(t) = x(t^2)$.
4. $y[n] = \sum_{k=-\infty}^{n} (-1)^k x[k]$.

4. *(4 Pts)* Consider a discrete-time LTI system with frequency response $H(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$. Which of the following claims are true? **(check all that apply!)**

1. The input $x[n]$ and output $y[n]$ satisfy $y[n] - \frac{1}{3}y[n-2] = x[n]$
2. If the input is $x[n] = e^{-j\frac{\pi}{2}n}$, then the output is $y[n] = \frac{e^{-j\frac{\pi}{2}n}}{1-\frac{j}{3}}$.
3. The system is causal and memoryless.
4. The input $x[n]$ and output $y[n]$ satisfy $y[n] - \frac{1}{3}y[n-1] = x[n]$.

5. *(3 Pts)* In the time domain, every LTI system can be characterized by its impulse response, and the input-output relationship can be written as $y(t) = (h * x)(t)$. Let $x(t) = \frac{2}{\pi}\left(\text{sinc}(\frac{2}{\pi}t)\right)^{100}$, and $h(t) = \frac{4}{\pi}\text{sinc}(\frac{4}{\pi}(t-1))$. Find $Y(\omega = 6)$, where $Y(\omega)$ is the Fourier Transform of $y(t)$. **(check the unique answer!)**

1. 22.125,
2. 0,
3. $\frac{\pi}{6}$,
4. None of the other options

6. *(3 Pts)* Let $x_\tau(t) = x(t-\tau)$ and $y_\mu(t) = y(t-\mu)$. Let $z(t) = (x * y)(t)$. Then, we have **(check the unique answer!)**

1. $(x_\tau * y_\mu)(t) = z(t - \tau - \mu)$,
2. $(x_\tau * y_\mu)(t) = z(t - \tau + \mu)$,
3. $(x_\tau * y_\mu)(t) = z(t + \tau - \mu)$
4. $(x_\tau * y_\mu)(t) = z(t + \tau + \mu)$

7. *(4 Pts)* A sampling system that samples continuous-time signals with a sampling frequency $\omega_s = 1000\pi$ is applied to the signal

$$x(t) = \sin(200\pi t).$$

The result is the following discrete time signal: **(check all that apply!)**

1. $x[n] = \sin(200\pi n)$
2. $x[n] = \sin(\frac{2}{5}\pi n)$
3. $x[n] = \sin(5\pi n)$
4. $x[n] = \sin(5n)$

8. *(4 Pts)* A signal $x(t)$ is sampled with frequency $\omega_s = 1000\pi$ using the impulse-train sampling procedure covered in lecture, and then reconstructed with a low-pass filter with cut-off frequency $\omega_c = 500\pi$. The reconstructed signal is

$$x_r(t) = \cos(300\pi t).$$

We do not know anything else about $x(t)$. Which of the following signals could be $x(t)$? **(check all that apply!)**

1. $x(t) = \cos(200\pi t)$
2. $x(t) = \cos(300\pi t)$
3. $x(t) = \cos(1300\pi t)$
4. $x(t) = \cos(700\pi t)$

9.*(3 Pts)* The autocorrelation sequence of a sequence $x[n]$ is defined as

$$r[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k]$$

Let $X(z)$ denote the Z-transform of $x[n]$. Which one of the following expressions is the Z-transform of $r[n]$? **(check the unique answer!)**

1. $R(z) = X(z)X(-z)$.
2. $R(z) = X(z)X(z^{-1})$.

3. $R(z) = X(-z)X(z^{-1})$.

4. $R(z) = X(z^{-1})X(-z^{-1})$.

10. *(3 Pts)* If two unstable causal LTI systems are placed in a parallel connection, then the overall (end-to-end) system must be **(check the unique answer!)**

1. LTI, causal, and unstable
2. LTI, causal, but may be stable
3. LTI, but may be anti-causal,
4. Causal, but may not be LTI.

11. *(3 Pts)* Which of the following is the impulse response of the causal LTI system described by the differential equation,

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + 3x(t).$$

(check the unique answer!)

1. $h(t) = e^{-4t}u(t) + 2e^{-2t}u(t)$.
2. $h(t) = e^{-t}u(t) + e^{-4t}u(t)$.
3. $h(t) = 2e^{t}u(t) + e^{2t}u(t)$.
4. $h(t) = e^{-t}u(t) + e^{-2t}u(t)$.

12. *(3 Pts)* Consider a discrete-time LTI system such that if we feed it with an input $x[n] = \left(-\frac{1}{3}\right)^n u[n]$, the corresponding output would be $y[n] = \delta[n] + \left(-\frac{1}{3}\right)^n u[n]$. Suppose now we feed the same system with an input signal $x[n] = \left(\frac{1}{9}\right)^n$, $\forall n$, that is, for $-\infty < n < \infty$. Then, the corresponding output signal $y[n]$ would be, **(check the unique answer!)**

1. $y[n] = 4\left(\frac{1}{9}\right)^n \;, \forall n$.
2. $y[n] = \left(\frac{1}{3}\right)^{2n} \;, \forall n$.
3. $y[n] = 5\left(\frac{1}{9}\right)^n \;, \forall n$.
4. $y[n] = \left(\frac{1}{9}\right)^{n-1} \;, \forall n$.

Problem 1 *(LTI System)* 12 points

Consider an LTI system with an input and output related through the equation

$$y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k-n} x[k-1]$$

(a) (4 Pts) What is the impulse response $h[n]$ of the system?

(b) (4 Pts) Is the system stable? Briefly justify your answer.

(c) (4 Pts)

What is the output $y_1[n]$ of the system when the input

$$x_1[n] = \left(\frac{1}{2}\right)^{n+1} u[n]$$

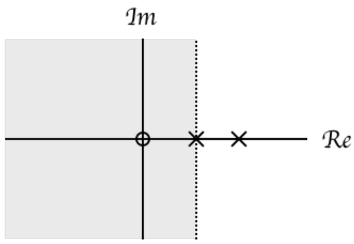
is applied?

Problem 2 *(System Properties)* 12 points

Determine if each system being described is causal or not. Briefly justify your answer.

(a) (4 Pts) An LTI system with impulse response $h(t) = (1 + e^{-t+1})u(t-1)$

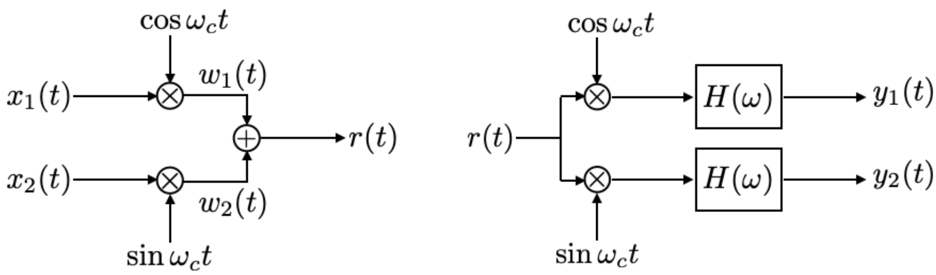
(b) (4 Pts) A continuous-time LTI system with system function $H(s)$ that has the following pole-zero plot and ROC



(c) (4 Pts) A discrete-time LTI system with system function $H(z) = 1$ for all z

Problem 3 *(Communication System)* 10 points

In this problem we will perform a simple analysis of an LTI communication system depicted in the figure below.



(a) (5 Pts) What are the Fourier transforms $W_1(\omega)$ and $W_2(\omega)$ of the signals

$$w_1(t) = x_1(t)\cos\omega_c t \quad \text{and} \quad w_2(t) = x_2(t)\sin\omega_c t\ ?$$

You answers should be in terms of $X_1(\omega)$ and $X_2(\omega)$.

(b) (5 Pts) Suppose $x_1(t)$ and $x_2(t)$ are both assumed to be band limited continuous-time signals with Fourier transforms that satisfy

$$X_1(\omega) = 0, \quad |\omega| \geq \omega_M$$

and

$$X_2(\omega) = 0, \quad |\omega| \geq \omega_M.$$

Moreover, $\omega_M < \omega_c$. Let $r(t) = w_1(t) + w_2(t)$. The same low-pass filter $H(\omega)$ is applied to the signals

$$r(t)\cos\omega_c t \quad \text{and} \quad r(t)\sin\omega_c t$$

in order to obtain y_1 and y_2 (see figure above). Is it possible to design $H(\omega)$ so that $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$? If yes, determine the gain and the cut-off frequency of the desired low-pass filter. If no, briefly explain why.

Problem 4 *(System Composition)* 10 points

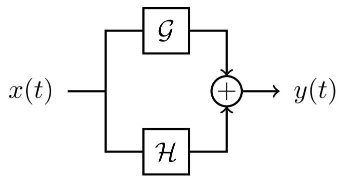
This is a design problem where subparts **do not have a unique answer**.

(a) (5 Pts) Assume that the overall system has the following frequency response

$$\frac{8+3j\omega}{(2+j\omega)(3+j\omega)}. \tag{1}$$

With respect to the figure below, find stable LTI systems $\mathcal{G}$ and $\mathcal{H}$ such that the overall system in the figure has exactly the frequency response given above.

- Provide the impulse response $g(t)$ and $h(t)$ of the systems $\mathcal{G}$ and $\mathcal{H}$, respectively.
- For full credit, the impulse responses $g(t)$ and $h(t)$ cannot be scaled and/or shifted Dirac delta functions (i.e. $g(t), h(t) \neq \alpha\delta(t-\beta)$ for constants α, β).

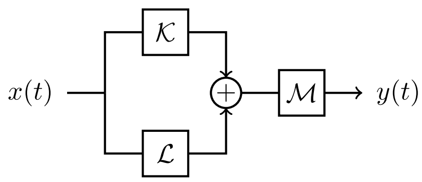


(b) (5 Pts) Assume that the overall system has the following frequency response same as in part (a)

$$\frac{8+3j\omega}{(2+j\omega)(3+j\omega)}. \tag{2}$$

With respect to the figure below, find stable LTI systems $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ such that the overall system in the figure has exactly the frequency response given above.

- Provide the impulse response $k(t)$, $\ell(t)$ and $m(t)$ of the systems $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$, respectively.
- For full credit, the impulse responses $k(t)$ and $m(t)$ cannot be scaled and/or shifted Dirac delta functions (i.e. $k(t), m(t) \neq \alpha\delta(t-\beta)$ for constants α, β).



Problem 5 *(Sampling)* 15 points

The signal

$$x(t) = \frac{15}{2\pi}\,\mathrm{sinc}\left(\frac{15}{2\pi}t\right) - \frac{5}{\pi}\,\mathrm{sinc}\left(\frac{5}{\pi}t\right) + \frac{5}{2\pi}\,\mathrm{sinc}\left(\frac{5}{2\pi}t\right)$$

is sampled with a sampling frequency $\omega_s = 10$.

(a) (5 Pts) Find the Fourier Transform $X(\omega)$. Find ω_M such that $X(\omega) = 0$ whenever $|\omega| > \omega_M$.

(b) (5 Pts) In lecture, we modeled the sampling operation with multiplication by a periodic impulse train. Mathematically this could be written as

$$x_p(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

The reconstruction $x_r(t)$ was obtained from $x_p(t)$ by applying a low-pass filter $h(t)$ with gain T and cutoff frequency $\omega_c = \frac{\omega_s}{2}$, where $\omega_s = \frac{2\pi}{T}$.

In this case, does $x_r(t) = x(t)$ hold? If yes, explain why. If not, find $x_r(t)$.

(c) (5 Pts) Let $x(t)$ be as given above and $y(t)$ be some arbitrary signal. The signal

$$z(t) = y(t) * x(t)$$

is sampled with a sampling frequency $\omega_s = 10$.

We again model the sampling operation with multiplication by a periodic impulse train to obtain

$$z_p(t) = z(t)p(t)$$

and apply some filter $g(t)$ to obtain a reconstructed signal $z_r(t)$.

Design the reconstruction filter $g(t)$ such that $z_r(t) = z(t)$ regardless of the value of $y(t)$.

Hint: $g(t)$ will no longer be a simple low-pass filter.