EPFL

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Final Exam Signals and Systems - (n/a)
20/08/2020
Duration : 3h

n/a

n/a

SCIPER : **999999**

Do not turn the page before the start of the exam. This document is double-sided, has 24 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- For the **multiple choice with unique answer** questions, we give
  +3 points if your answer is correct,
  0 points if your answer is incorrect.
- For the **multiple choice with multiple answers** questions, we give
  +4 points for all correct answers,
  +2 points for one incorrect answer and three correct answers,
  0 points for other possibilities of answers.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Read these guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First Part: Multiple Choice Questions

For each question, mark all boxes corresponding to the correct answers. Note that some questions are marked as having multiple correct answers – **check all that apply!** – while others have **a unique correct answer**.

Question 1 : Let the Laplace transform of the signal $x(t)$ be

$$X(s) = \frac{1}{s^2 + s}, \tag{1}$$

with the region of convergence $Re(s) > 0$. Then, the corresponding signal $x(t)$ would be, **(check the unique answer!)**

- ☐ $x(t) = (1 - e^{-t})u(t)$
- ☐ $x(t) = (t + e^{t})u(t)$
- ☐ $x(t) = (t - 1 + e^{-t})u(t)$
- ☐ $x(t) = (t + e^{-t})u(t)$

Question 2 : Suppose that a causal and stable LTI system has a rational transfer function $H(s)$ and that $H(s)$ has exactly two poles and four zeros. Which of the following statements are true? **(check all that apply!)**

- ☐ If $G(s)$ is the transfer function of an inverse of $\mathcal{H}$, $G(s) = \frac{1}{H(s)}$
- ☐ $\lim_{s\to\infty} H(s) = 0$
- ☐ If $G(s)$ is the transfer function of an inverse of $\mathcal{H}$, it has exactly 4 poles
- ☐ $H(s)$ must always have a stable inverse system

Question 3 : Which of the following claims about properties of signals and systems are true? **(check all that apply!)**

- ☐ The continuous-time system $\mathcal{H}\{x(t)\} = \sqrt{|(x(t) - \mu)|}$ is memoryless and causal
- ☐ The discrete-time signal $z[n] = \cos(\frac{n}{2})$ has fundamental period 4π
- ☐ The continuous-time system $\mathcal{H}\{x(t)\} = x(t^2 - 1)$ is linear and time-invariant
- ☐ The discrete-time signal $x[n] = u[n] - u[n-2]$ is an energy signal

Question 4 : A signal $x(t)$ is sampled with frequency $\omega_s = 1000\pi$ using the impulse-train sampling procedure covered in lecture, and then reconstructed with a low-pass filter with cut-off frequency $\omega_c = 500\pi$. The reconstructed signal is

$$x_r(t) = \sin(300\pi t).$$

We do not know anything else about $x(t)$. Which of the following signals could be $x(t)$? **(check all that apply!)**

- ☐ $x(t) = \sin(1300\pi t)$
- ☐ $x(t) = \sin(200\pi t)$
- ☐ $x(t) = \sin(700\pi t)$
- ☐ $x(t) = \sin(300\pi t)$

Question 5 : A sampling system that samples continuous-time signals with a sampling frequency $\omega_s = 800\pi$ is applied to the signal

$$x(t) = e^{j\pi t}.$$

The result is the following discrete time signal: **(check all that apply!)**

- ☐ $x[n] = e^{j\frac{800\pi}{400}n}$
- ☐ $x[n] = 1$
- ☐ $x[n] = e^{j\frac{\pi}{400}n}$
- ☐ $x[n] = e^{j\frac{801\pi}{400}n}$

Question 6 : The Basel problem asks for the sum of reciprocals of the squares of the nature numbers, i.e., $\sum_{n=1}^{\infty}\frac{1}{n^2}$. The sum can be considered as half of the *energy* of a discrete time signal $x[n]$ such that

$$x[n] = \begin{cases} \dfrac{\cos(\pi n)}{jn}, & n \neq 0, \\ 0, & n = 0. \end{cases}$$

Which one of the following values is equal to the sum, $\sum_{n=1}^{\infty}\frac{1}{n^2}$? **(check the unique answer!)**

- ☐ $\frac{\pi^2}{6}$
- ☐ $\frac{\pi^2}{4}$
- ☐ $\frac{\pi^2}{2}$
- ☐ $\frac{\pi^2}{3}$

Question 7 : Let $X(\omega)$ denote the Fourier transform of $x(t)$. Then, the Fourier transform of $x_1(t-t_0) + x_2(-t)$ is **(check the unique answer!)**

- ☐ $e^{-jt_0\omega}X_1(\omega) + X_2(-\omega)$
- ☐ $e^{jt_0\omega}X_1(\omega) - X_2(\omega)$
- ☐ $e^{-jt_0\omega}X_1(\omega) - X_2(-\omega)$
- ☐ $e^{jt_0\omega}X_1(\omega) + X_2(\omega)$

Question 8 : Consider a discrete-time LTI system with frequency response $H(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$. Which of the following claims are true? **(check all that apply!)**

- ☐ The system is stable
- ☐ The input $x[n]$ and output $y[n]$ satisfy $y[n] + \frac{1}{2}y[n-1] = x[n]$
- ☐ The system is causal and memoryless
- ☐ If the input is $x[n] = e^{-j\frac{\pi}{2}n}$, then the output is $y[n] = \frac{e^{-j\frac{\pi}{2}n}}{1-\frac{j}{2}}$

Question 9 : Let $x_n(t) = \frac{d^n}{dt^n}x(t)$ and $y_k(t) = \frac{d^k}{dt^k}y(t)$. Let $z(t) = (x * y)(t)$. Then, we have **(check the unique answer!)**

- ☐ $(x_n * y_k)(t) = \frac{d^{\max(n,k)}}{dt^{\max(n,k)}}z(t)$
- ☐ $(x_n * y_k)(t) = \frac{d^{n+k}}{dt^{n+k}}z(t)$
- ☐ $(x_n * y_k)(t) = \frac{d^{\min(n,k)}}{dt^{\min(n,k)}}z(t)$
- ☐ $(x_n * y_k)(t) = \frac{d^{nk}}{dt^{nk}}z(t)$

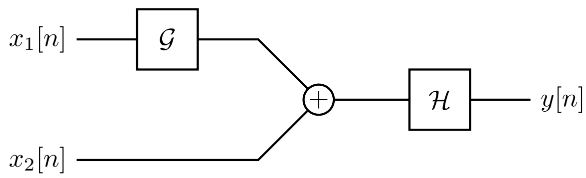
Question 10 : Consider the continuous signals $x(t) = e^{-2t}u(t) - e^{-t}u(-t)$ and $y(t) = e^{t}u(t)$. Which of the following claims are true? **(check all that apply!)**

- ☐ The signal $y(t)$ has a finite energy
- ☐ Laplace transform of $x(t)y(t)$ exists
- ☐ Laplace transform of $x(t) + y(t)$ exists
- ☐ The signal $x(t)$ is not absolutely integrable

Question 11 : Let the signal $y(t) = \int_{-\infty}^{t}(\delta(\tau+1) - \delta(\tau-1))d\tau$. Then **(check the unique answer!)**

- ☐ $y(t) = \delta(t+1) - \delta(t-1)$
- ☐ $y(t) = u(t+1) - u(t-1)$
- ☐ $y(t) = 2$
- ☐ $y(t) = 2u(t)$

Question 12 : Consider the following system composite with two input signal $x_1[n]$, $x_2[n]$ and output signal $y[n]$.



Let the transfer functions of LTI systems $\mathcal{G}$ and $\mathcal{H}$ be $G(z) = \frac{4}{1+2z^{-1}}$ and $H(z) = \frac{1}{1+z^{-1}}$, respectively. Find the output $y[n]$ when the inputs $x_1[n] = 2^n, \forall n$ and $x_2[n] = 3^n, \forall n$ are applied to the system. **(check the unique answer!)**

- ☐ $y[n] = \frac{1}{5}6^n, \forall n$
- ☐ $y[n] = \frac{4}{3}2^n + \frac{3}{4}3^n, \forall n$
- ☐ $y[n] = \frac{5}{6}6^n, \forall n$
- ☐ $y[n] = \frac{3}{4}2^n + \frac{4}{3}3^n, \forall n$

Second Part: Open Problems

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in detail. Leave the check-boxes empty, they are used for the grading.

Question 13: *(12 Points)*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 ☐11 ☐12 *Do not write here.*

Consider an LTI system with an input and output related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 5) d\tau.$$

(a) What is the impulse response $h(t)$ of the system?

(b) Is the system causal? Justify your answer.

(c) What is the output $y(t)$ of the system when the input

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

is applied?

Question 14: *(12 Points)*

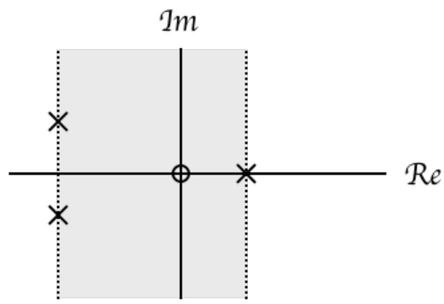
☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 10 ☐ 11 ☐ 12 *Do not write here.*

Determine if each system being described is stable or not. Justify your answer.

(a) An LTI system with impulse response $h(t) = (1 + e^{-t+1})u(t-1)$

(b) A system that given an input $x(t) = \cos(2t)$ produces an output $y(t) = e^{(2j-1)t}$

(c) A continuous-time LTI system with transfer function $H(s)$ that has the following pole-zero plot and ROC

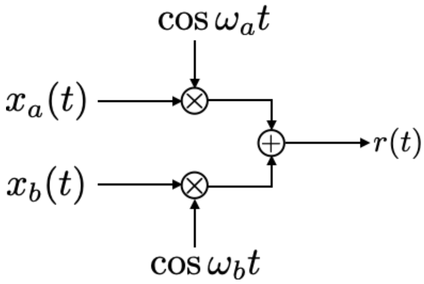


(d) Two stable LTI systems (continuous or discrete-time) $\mathcal{G}$ and $\mathcal{H}$ connected in series

Question 15: *(12 Points)*

0 1 2 3 4 5 6 7 8 9 10 11 12 *Do not write here.*

In this problem we will perform a simple analysis of an LTI communication system depicted in the figure below.



(a) What is the Fourier transforms $R(\omega)$ of the carrier signal $r(t) = x_a(t) \cos \omega_a t + x_b(t) \cos \omega_b t$? You answers should be in terms of $X_a(\omega)$ and $X_b(\omega)$.

(b) Suppose that $x_a(t)$ and $x_b(t)$ are both assumed to be band limited continuous-time signals with Fourier transforms that satisfy

$$X_a(\omega) = 0, \quad |\omega| \geq \omega_M \quad \text{and} \quad X_b(\omega) = 0, \quad |\omega| \geq \omega_M.$$

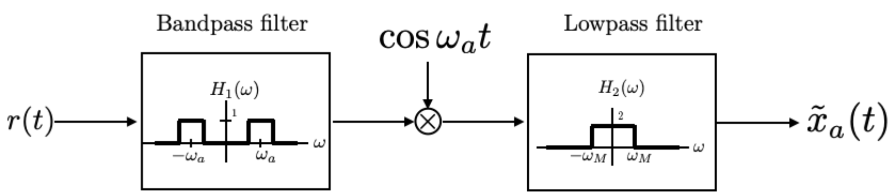
A reconstruction system for $x_a(t)$ is proposed below where

$$H_1(\omega) = \begin{cases} 1, & \omega_a - \omega_M < |\omega| < \omega_a + \omega_M \\ 0, & \text{otherwise,} \end{cases}$$

and

$$H_2(\omega) = \begin{cases} 2, & |\omega| < \omega_M \\ 0, & \text{otherwise.} \end{cases}$$

(A similar system could also be designed to reconstruct $x_b(t)$, by replacing ω_a with ω_b.)



What conditions must ω_a and ω_b satisfy in order to guarantee that $\tilde{x}_a(t) = x_a(t)$? Assume that $\omega_a < \omega_b$.

(c) Finally, suppose that $x_a(t)$ and $x_b(t)$ are both assumed to be band limited continuous-time signals with Fourier transforms that satisfy

$$X_a(\omega) = 0, \quad |\omega| \geq \omega_M \quad \text{and} \quad X_b(\omega) = 0, \quad |\omega| \geq 2\omega_M.$$

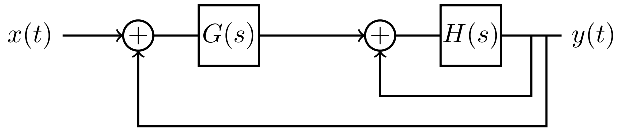
That is, $x_b(t)$ has twice as much bandwidth as $x_a(t)$.

What conditions must ω_a and ω_b satisfy now in order to guarantee that $\tilde{x}_a(t) = x_a(t)$?

Question 16: *(10 Points)*

0 1 2 3 4 5 6 7 8 9 10 *Do not write here.*

Consider the interconnection of continuous-time casual LTI systems shown in figure below.



(a) Express the overall system function for this interconnection in terms of $G(s)$ and $H(s)$.

(b) Let $H(s) = \frac{1}{s+2}$ and $G(s) = 1$. Given the input $x(t) = e^{-2t}u(t)$, find the output $y(t)$.

Question 17: *(12 Points)*

0 1 2 3 4 5 6 7 8 9 10 11 12 *Do not write here.*

Recall that in lecture, we derived the sampling theorem by assuming that $x(t)$ was *band limited.* That is, its Fourier transform satisfies

$$X(\omega) = 0, \quad |\omega| \geq \omega_h.$$

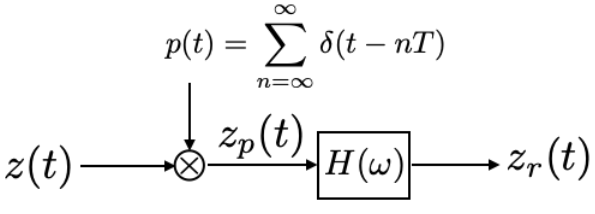
In this problem we study the sampling of a continuous time signal $z(t)$ such that its Fourier transform satisfies

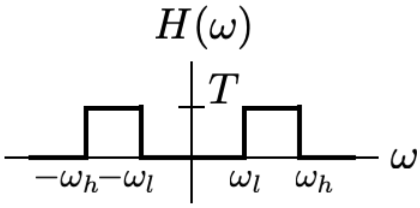
$$Z(\omega) = 0, \quad |\omega| \geq \omega_h \text{ or } |\omega| \leq \omega_l.$$

A signal like this whose energy is concentrated in an energy band is often referred to as a *bandpass signal.*

(a) Using the sampling theorem, as derived in class, find the Nyquist rate of the signal $z(t)$. In other words, find smallest sampling frequency ω_s needed to reconstruct $z(t)$ exactly from its samples. What is the corresponding sampling interval T?

After graduating from EPFL you get a job at an exciting new startup that promises to design and build the first ever artificially intelligent android (Congratulations!). Your colleague notices that all of the signals sampled by a certain audio circuit are bandpass signals, just like $z(t)$. She proposes a new sampling system (depicted below) which she claims should work better for bandpass signals.





That is, the sampling operation is modeled by an impulse train and is the same as before. However, the reconstruction is now performed by a bandpass filter, rather than a low-pass filter.

(b) If you use the same sampling frequency as in part (a) with this new reconstruction procedure, does $z_r(t) = z(t)$? Explain why or why not.

Hint: You may find it helpful to recall that in lecture we derived $Z_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - \omega_s)$ *where* ω_s *is the sampling frequency.*

(c) Assuming that $\omega_h = 2\omega_l$, find the smallest sampling frequency ω_s and the largest sampling interval T such that $z_r(t) = z(t)$ for the new proposed sampling system. How does this compare to part (a)?