Lecture 6

Area law in 1D

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To complete a proof of the area law for a gapped 1D local Hamiltonian, it is sufficient to find a $(D,\Delta)AGSP$ that satisfies the hypotheses of Theorem 5.10, i.e. has $D\Delta \leq \frac{1}{2}$. The gap is denoted as $\delta := \lambda_1(H) - \lambda_0(H) > 0$.

Theorem 6.1. *$\exists (D,\Delta)AGSP$ with $D\Delta \leq \frac{1}{2}$ and $D \sim \left(\frac{1}{\delta}\right)^{\tilde{O}(\frac{1}{\delta})}$ ($\tilde{O}(\frac{1}{\delta})$ does not depend on n).*

The AGSP $K = P_1 P_2 \dots P_n$ from the previous lecture, which is an intuitive choice, has $D = 4$, $\Delta = 1 - \Omega(\delta)$ given by the Detectability Lemma (Th. 5.1). This is not good enough to prove Theorem 6.1, but it can be progressively refined until the hypotheses are met. A first idea to improve it is using

$$K = \left(\mathrm{Id} - \frac{H_1 + H_2 + \ldots + H_n}{n}\right)^l \qquad \begin{array}{l} \text{for which:} \\ |\Gamma\rangle \to |\Gamma\rangle \\ \forall |\psi\rangle \perp |\Gamma\rangle, \quad \||K|\psi\rangle\|^2 \leq \left(1 - \frac{\delta}{n}\right)^{2l} \||\psi\rangle\|^2 \end{array} \tag{6.1}$$

The connection with the previous AGSP becomes clearer when the H_i are expressed in terms of the P_i:

$$K = \left[\frac{1}{n}(P_1 + \ldots + P_n)\right]^l = \frac{1}{n^l} \sum_{i_1,\ldots,i_l} P_{i_1} P_{i_2} \dots P_{i_l} \tag{6.2}$$

which is the average of all products of l P operators. Setting $l = n$ one obtains $\Delta = 1 - \Omega(\delta)$. Each term in the sum can increase the bond dimension of an MPS by up to a factor $4^{l/2}$ (see following comments on bond dimension) and one can show that the average makes the bond dimension bounded by $n^l 4^{l/2}$, which is far too large.

In order to come up with a better AGSP, it is useful to picture the map it induces on the energies. As shown in Figure 6.1, the condition that we want is that it should map 0 to 1 to leave the ground state unchanged and it should map the rest of the eigenvalues, which range from δ to n, to values smaller than Δ.

The operator defined in Equation 6.2 can be expressed as $K = f_K(H)$, with $f_K(x) = (1 - \frac{x}{n})^l \sim e^{-\frac{xl}{n}}$, which is shown in Figure 6.2a. In this plot, a valid AGSP is obtained if for values of E larger than δ, the function $f_K(E)$ is bounded between Δ and $-\Delta$. This constraint can be satisfied by other functions that may nevertheless oscillate and appear less simple, such as in Figure 6.2b. The ideal function would be a low degree polynomial, and an optimal choice for $f_K(x)$ is a Chebyshev polynomial. This leads to the following result:

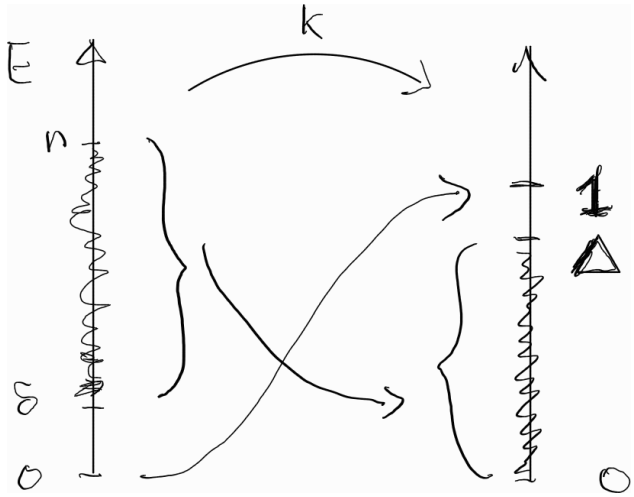
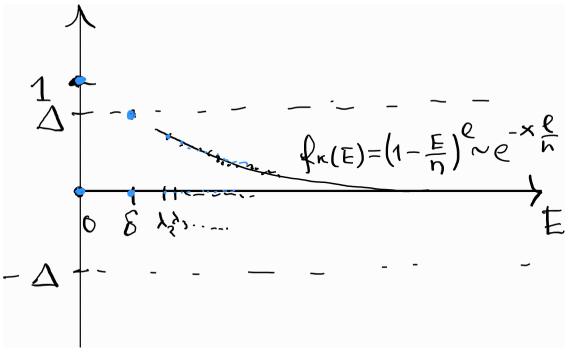
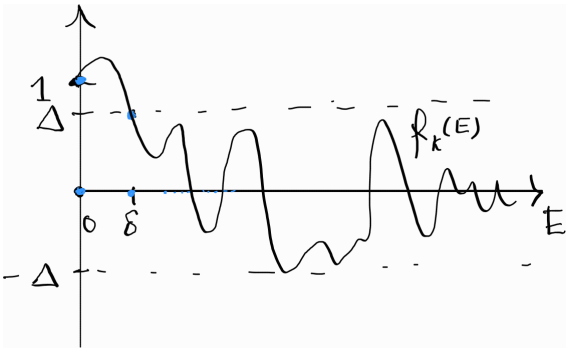


Figure 6.1: The AGSP should keep the ground state unchanged ($|\Gamma\rangle$, which has energy 0, is an eigenvector of K of eigenvalue 1). The rest of the eigenvalues, which range from δ to n, should be shrunk to at most Δ.



(a) The AGSP defined in Equation 6.2, which smoothly suppresses eigenvalues larger than δ with an exponential-like decay.



(b) A generic AGSP defined by $K = f_K(H)$.

Theorem 6.2. *$\forall l \geq 1$ $\exists P_l$ polynomial of degree l s.t.*

$$P_l(0) = 1$$
$$\forall x \in [\delta, t],\ |P_l(x)| \leq 2e^{-2l\sqrt{\frac{\delta}{t}}}$$

It follows that $K_l = P_l(H)$ is an AGSP with $\Delta \leq 4e^{-4l\sqrt{\frac{\delta}{n}}}$. It can be shown that $D \sim 4^{\frac{l}{n}}2^n$, up to lower-order terms (see the following box for a sketch of the argument).

Comments on bond dimension

To identify the value of D, first note that

$$P_l(H) = \sum_{i_1,\dots,i_l \in \{1,\dots,n\}} \alpha_{i_1,\dots,i_l} P_{i_1} \dots P_{i_l} \tag{6.3}$$

where $\alpha_{i_1,\dots,i_l}$ are some coefficients determined by P_l.
The typical number of times that the operator P_i appears in one term of the sum $P_{i_1} \dots P_{i_l}$ is $\frac{l}{n}$. Since P_i is the only operator that increases the bond dimension between sites i and $i+1$, we have that each product of l projections P_{i_j} increases the bond dimension between sites i and $i+1$ by, on average, $D \sim 4^{\left(\frac{l}{n}\right)}$.
However, since D is a bound on the growth of the bond dimension due to the application of the AGSP, what matters is the worst case, which is $D \sim 4^{l/2}$. Since $P_iP_i \dots P_i = P_i$, one of the worst products is actually $P_iP_{i-1}P_iP_{i-1} \dots$, where P_i appears $l/2$ times.
This estimate is too pessimistic, as it is not possible to increase entanglement indefinitely by only acting on qubits $i, i+1$, which can only encode a limited amount of information. If the bond dimension between $i, i+1$ is large, it must be due to entanglement with other qubits, which indicates that other terms than P_i and P_{i+1} must be present in the sequence. As the bond dimension cannot increase more than 2^m times by moving along the MPS for m steps, the presence of a low-entangled pair of adjacent qubits restricts the bond dimension around it, putting a stricter bound on D, which is found to be $D \sim 4^{\frac{l}{n}}2^n$. This is shown by the pigeonhole principle: for any product of l projections P_{i_j}, there is an index $j \in \{1, \dots n\}$ such that P_j appears at most l/n times in the product. Therefore, the bond dimension across qubits j and $j+1$ increases by at most $4^{\frac{l}{n}}$. Since qubits i and j can be at distance at most n, it follows that the bond dimension across qubits i and $i+1$ increases by at most $4^{\frac{l}{n}}2^n$.
It remains to sum over all the terms. This is non-trivial to handle and we skip the argument. Overall, the final scaling is similar up to lower-order terms.

As $l = n^2$, the proposed K has $D \sim 4^n$, $\Delta \sim e^{-n^{\frac{3}{2}}\sqrt{\delta}}$ and for n large enough $D\Delta < 1$.

Unfortunately we are not quite done yet. The greatest limitation of the current construction is that the parameter t from Theorem 6.2 has to be set to the value of the largest eigenvalue of H, which is n. The presence of large eigenvalues makes l and Δ unnecessarily large. Furthermore, the corresponding bound on the entropy is $S \leq O(1) \log D = n$ which is trivial. Intuition suggests that large eigenvalues should not be relevant for properties of low-energy states. Following this intuition, and focusing on the bond dimension between i and $i+1$ the Hamiltonian is rewritten as

$$H = H_1 + \ldots + H_{i-\frac{u}{2}} + \ldots + H_i + H_{i+1} + \ldots + H_{i+\frac{u}{2}} + \ldots + H_n \tag{6.4}$$
$$= H_L + H_{i-\frac{u}{2}} + \ldots + H_i + H_{i+1} + \ldots + H_{i+\frac{u}{2}} + H_R \tag{6.5}$$

Since the operators $H_{L,R}$ act far from i, it is possible to replace them with simpler operators $\tilde{H}_{L,R}$ which have smaller eigenvalues, but without significantly changing the action of the operator in the low-energy subspace. To this end, define

$$\tilde{H} = \tilde{H}_L + H_{i-\frac{u}{2}} + \ldots + H_i + H_{i+1} + \ldots + H_{i+\frac{u}{2}} + \tilde{H}_R \tag{6.6}$$

The operators $\tilde{H}_{L,R}$ are obtained by putting a threshold t on the eigenvalues of $H_{L,R}$, such that $\lambda_i \to \min(\lambda_i, t)$

$$\tilde{H}_{L,R} = trunc_{\leq t}(H_{L,R}) \tag{6.7}$$

It can be shown that, as far as the low-energy subspace (defined as the span of eigenvectors whose associated eigenvalue is $\ll 1/t$) is concerned, the errors introduced by this operation are exponentially small:

$$|\lambda_0(\tilde{H}) - \lambda_0(H)| \leq 2^{-t} \tag{6.8}$$
$$\left\| |\psi_0(\tilde{H})\rangle - |\psi_0(H)\rangle \right\| \leq 2^{-t} \tag{6.9}$$

This operation puts a bound on the norm of the Hamiltonian $\left\|\tilde{H}\right\| \leq 2t + u$, improving the previous AGSP to

$$\tilde{\Delta} \leq 4e^{-4l\sqrt{\frac{\delta}{2t+u}}} \qquad \tilde{D} \leq 4^{\frac{l}{u}+u} \tag{6.10}$$

Setting $l = u^2,\ u = t = \frac{C}{\delta}$ leads to

$$\tilde{\Delta} \sim e^{-\frac{C^{3/2}}{\delta}} \qquad \tilde{D} \sim 4^{\frac{2C}{\delta}}. \tag{6.11}$$

Note that by scaling u by some parameter C, the exponent of $\tilde{D}$ in Equation 6.10 grows linearly with the parameter, while the exponent $\tilde{\Delta}$ scales with a power of $3/2$. Therefore, by scaling u by a sufficiently large amount, the product $\tilde{D}\tilde{\Delta}$ can be made smaller than $\frac{1}{2}$, concluding the proof. The corresponding bound on the entropy is $S \leq O(1)\log D = O(\frac{1}{\delta})$.

While the previous is a statement on the entanglement of $|\Gamma\rangle$, it is also useful to prove the existence of an efficient MPS approximation of the ground state.

Lemma 6.3.

$$\forall \epsilon > 0\ \exists |\psi\rangle \quad s.t. \quad |\langle\psi|\Gamma\rangle| \geq 1 - \epsilon \tag{6.12}$$
$$\text{and } |\psi\rangle \text{ has MPS representation with } B \sim 2^{o(\log^{3/4} n)} \tag{6.13}$$

Proof. In the previous construction set $t \sim \log n,\ u \sim \log^{3/4} n$ and $l = \log^{3/2} n$ to obtain $\Delta \sim e^{-\Omega(\log n)}$ and $D \sim e^{-\Omega(\log^{3/4} n)}$. Then it follows that $\forall i,\ \exists |\psi_i\rangle$ s.t. $|\langle\psi_i|\Gamma\rangle \geq 1 - \frac{1}{poly(n)}$ and $B_i \sim 2^{\log^{3/4} n}$. Using an iterative argument based on the triangle inequality (for the vector norm $\|\cdot\|$) it is then possible to obtain the lemma, which guarantees the existence of a single MPS approximation to the ground state whose bond dimension across all bonds is bounded. □

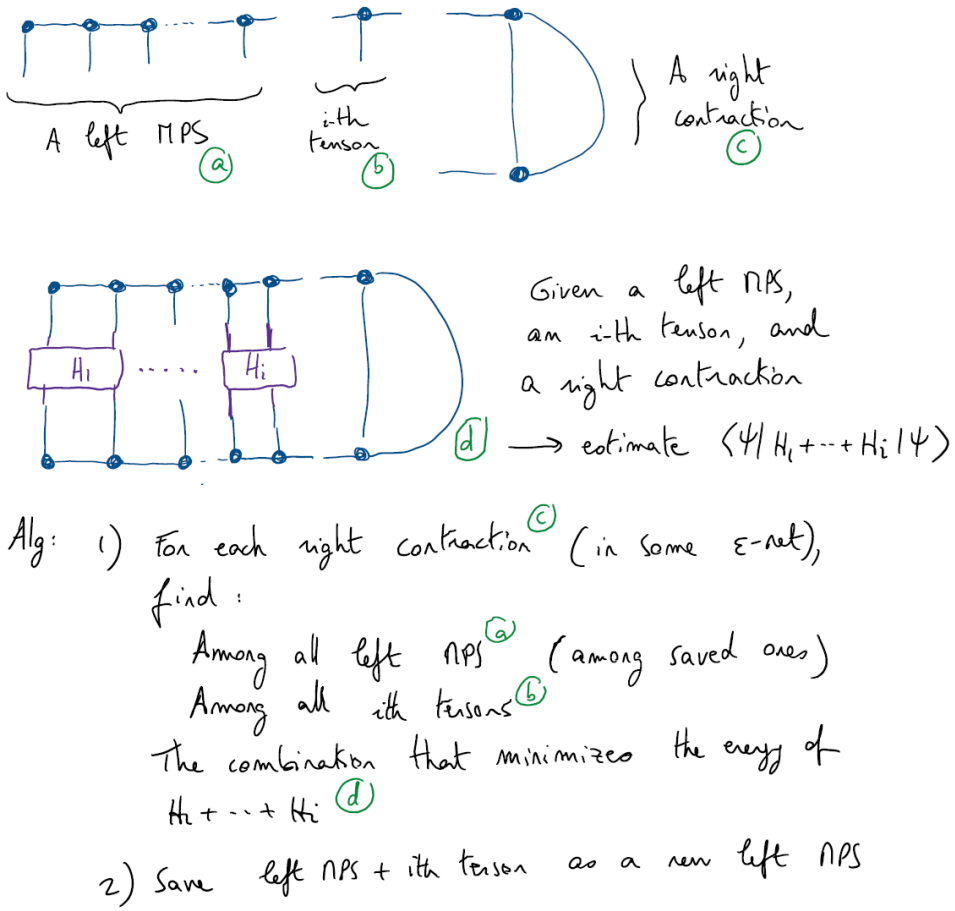


Figure 6.3: Main steps of the algorithm.

An algorithm for the ground state in subexponential time

An algorithm to find an MPS representation of the ground state can be constructed using dynamic programming. The algorithm treats each local term H_i one at a time, in a single sweep from H_1 to H_n. For each step i, and for each possible right contraction of an n-qubit MPS with bond dimension B (see Figure 6.3), the left MPS such that, together with the right contraction the resulting MPS minimizes the energy of $H_1 + \ldots + H_{i-1}$ is saved to be extended at the next iteration.

This means that for each element of an ε-net over right contractions, we need to iterate over all saved left-half MPS and tensors for the current location, i. Taking into account that there are n steps, the running time scales as $n \left(\frac{1}{\epsilon}\right)^{O(B^2)}$. For B as in Lemma 6.3 this is sub-exponential.

Note that there exists a polynomial time algorithm which will not be described in the lecture.