# CS-631 Quantum Hamiltonian Complexity

Exercise sheet 2

24/03/2025

**Some Matrix Product States.** Recall than an MPS on $n$ qubits is specified by $n$ tensots $A_1 \in \mathbb{C}^{2\times B}, A_2 \in \mathbb{C}^{B\times 2\times B}, \dots, A_{n-1} \in \mathbb{C}^{B\times 2\times B}, A_n \in \mathbb{C}^{B\times 2}$, where $B$ is an integer parameter (the "bond dimension").

1. Give two tensors $A_1, A_2$ that represent an EPR pair.
2. Give $n$ tensors $A_1, \dots, A_n$ that represent a CAT state $\frac{1}{\sqrt{2}}(|0\cdots 0\rangle + |1\cdots 1\rangle)$.
3. Can you give an MPS representation of the Motzkin state from lecture 1? Here, the qubits are qutrits so the dimension '2' is replaced by a '3'. What $B$ do you use?

**Some facts about bond dimension of MPS.**

1. Show that $|\psi\rangle$ has an MPS whose bond dimension between qubits $i$ and $i+1$ is $B_i$ if and only if $|\psi\rangle$ has a Schmidt decomposition across the cut between qubits $\{1,\dots,i\}$ and $\{i+1,\dots,n\}$ with at most $B_i$ nonzero coefficients. (Note: in general it may not be possible to achieve this bound across all cuts simultaneously.)
2. Show that if $|\psi\rangle$ has an MPS whose bond dimension between qubits $i$ and $i+1$ is $B_i$, then $|\psi\rangle$ has an MPS whose bond dimension between qubits $i+j$ and $i+j+1$ is at most $2^j B_i$, for any $j \geq 0$ (simultaneously, meaning that there is a single MPS representation that satisfies all conditions).
3. Let $A$ be an operator acting on two consecutive qudits. Show how to compute an MPS representation of $A|\psi\rangle$ from the MPS representation of $|\psi\rangle$ and an explicit matrix representation of $A$. Give an upper bound on the bond dimension of the resulting MPS (as a function of the bond dimension of the initial MPS).

**Quantum union bound.** Recall the definition of the Schatten 1-norm $\|M\|_1 = \mathrm{Tr}\sqrt{MM^\dagger} = \mathrm{Tr}\sqrt{M^\dagger M}$ and the fidelity between two density matrices $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

Let $\rho$ be a density matrix and $A_1, \dots, A_m$ orthogonal projectors. Define $\mathrm{E}_\rho[A_i] = \mathrm{Tr}(A_i\rho) = 1 - \varepsilon_i$, and $L = \sum \varepsilon_i$. We think of $L$ as a "loss", i.e. the chance that, on average, a measurement of the PVM $\{A_i, \mathrm{Id} - A_i\}$ would return the outcome "fail" $(\mathrm{Id} - A_i)$. Now consider a process by which all PVM $\{A_i, \mathrm{Id} - A_i\}$ are performed sequentially, for $i = 1, 2, \dots, m$. Call $S$ the chance that all "good" outcomes occurr, i.e. $S = \mathrm{Tr}(A_m \cdots A_1 \rho A_1 \cdots A_m)$. Let $F = 1 - S$. The *Quantum Union Bound* states that $F \leq 4L$. The goal of this exercise is to prove this bound.

1. Comment on the choice of terminology "Quantum Union Bound". Can you express the "classical" union bound using similar terms of the "quantum" one, and compare the two bounds?
2. Verify the following equality, valid for any density matrices $\rho, \sigma$ and $A$: $\|\sqrt{\rho}A\sqrt{\sigma}\|_1 = \sqrt{F(\rho, A\sigma A^\dagger)}$.
3. Show that under the same conditions and furthermore assuming that $0 \le \overline{A} \le \mathrm{Id}$, we have the inequality $\|\sqrt{\rho}\overline{A}\sqrt{\sigma}\|_1 \le \sqrt{\mathrm{E}_\sigma[\overline{A}]}\sqrt{\mathrm{E}_\rho[\overline{A}]}$. *[Hint: use the Cauchy-Schwary inequality $\|XY\|_1 \le \|X\|_2\|Y\|_2$]*

Introduce new notation $\sigma_{|A} = A\sigma A^\dagger/\mathrm{E}_\rho[A^\dagger A]$ for the renormalized post-measurement state. Combining the previous two questions, we deduce the following inequality

$$\sqrt{F(\rho,\sigma)} \le \sqrt{\mathrm{E}_\sigma[A^\dagger A]}\sqrt{F(\rho,\sigma_{|A})} + \sqrt{\mathrm{E}_\sigma[\overline{A}]}\sqrt{\mathrm{E}_\rho[\overline{A}]}\ .$$

4. For $0 \le t \le m$ let $p_t$ be the probability that $A_1, \dots, A_t$ occur, i.e. $p_t = \mathrm{Tr}(A_t \cdots A_1 \rho A_1 \cdots A_t)$, and $\rho_t$ the associated (re-normalized) post-measurement state. Let $r_t = \sqrt{p_t}\sqrt{F(\rho,\rho_t)}$. Show that $r_{t-1} - r_t \le \sqrt{q_t}\sqrt{\varepsilon_t}$.
5. Deduce the inequality $1 \le \sqrt{S}\sqrt{F(\rho,\rho_m)} + \sqrt{F}\sqrt{L}$.
6. Conclude the proof of the quantum union bound.
7. From the same inequality (item 5.), deduce a "gentle sequential measurement" bound $\|\rho - \rho_m\|_1 \le \sqrt{L}$.