CS-631 Quantum Hamiltonian Complexity

Exercise sheet 1

25/02/2025

Error reduction for QMA. Let V be a QMA verifier whose maximum acceptance probability on n-qubit witnesses $|\xi\rangle$ is p. For any integer k and $1 \leq t \leq k$ let $V_{t,k}$ be the verifier which expects a (nk)-qubit witness $|\xi'\rangle$, processes each chunk of n qubits of $|\xi'\rangle$ independently according to V, and accepts if and only if at least t of the k chunks resulted in the "accept" outcome.

1. Show that the maximum acceptance probability of $V_{k,k}$ is exactly p^k.
2. What is the maximum acceptance probability of $V_{t,k}$, for $1 \leq t \leq k$?
3. Deduce $\mathrm{QMA}_{c,s} \subseteq \mathrm{QMA}_{2/3,1/3}$ for any $c - s = \Omega(1/\operatorname{poly})$. In terms of c and s, how much larger a proof does the modified verifier require?

5-local Hamiltonian is QMA**-complete.** In class we saw that the the local Hamiltonian problem with Hamiltonians acting on $O(\log n)$ qubits is QMA-hard. In this problem we improve the construction to constant locality, by showing that 5-local Hamiltonian is also QMA-hard. In order to achieve this the main idea is to represent the clock in *unary*: time $|t\rangle$ is represented as $|0\cdots 01\cdots 1\rangle$, with $(T-t)$ zeroes and t ones.

1. Show how to modify H_{in}, H_{out} and H_{prop} so that each term is 5-local and is designed to act on the new clock.
2. The new construction is insufficient: check that invalid clock states, such as $|101\rangle$, have eigenvalue 0 with respect to $H_{in} + H_{out} + H_{prop}$. Design a "penalty" Hamiltonian H_{stab} such that any state of the form $|\psi\rangle|\text{valid clock state}\rangle$ has eigenvalue 0 with respect to H_{stab}, but any state of the form $|\psi\rangle|\text{invalid clock state}\rangle$ has eigenvalue at least 1. What is the locality of H_{stab}?
3. Use the projection lemma to analyze the new Hamiltonian $H = J_{in}H_{in} + H_{out} + J_{prop}H_{prop} + J_{stab}H_{stab}$ and show that it has a small eigenvalue if and only if the original circuit had an accepting state (provided the weights J_{in}, J_{prop} and J_{stab} are chosen appropriately). Conclude.

The Motzkin Hamiltonian. Let $|\Phi\rangle = \frac{1}{\sqrt{2}}(|LR\rangle - |00\rangle)$, $|\Psi_l\rangle = \frac{1}{\sqrt{2}}(|L0\rangle - |0L\rangle)$ and $|\Psi_r\rangle = \frac{1}{\sqrt{2}}(|0R\rangle - |R0\rangle)$. The Motzkin Hamiltonian H is

$$H = |)\rangle\langle)|_1 + |(\rangle\langle(|_n + \sum_{j=1}^{n-1}\big(|\Phi\rangle\langle\Phi|_{j,j+1} + |\Psi_l\rangle\langle\Psi_l|_{j,j+1} + |\Psi_r\rangle\langle\Psi_r|_{j,j+1}\big).$$

Let the Motzkin state $|\mathcal{M}_n\rangle$ be the uniform superposition over all well-parenthesized states $|s\rangle$ for $s \in \{L, R, 0\}$.

1. Let $S_{p,q} \subseteq \{L, R, 0\}^n$ be the set of all strings s that are equivalent to a string $s_{p,q} = L \cdots L0 \cdots 0R \cdots R$, where there are p left brackets, $(n - (p + q))$ zeros and q right brackets, under local transformations $00 \leftrightarrow LR$, $L0 \leftrightarrow 0L$ and $0R \leftrightarrow R0$. Prove that the sets $S_{p,q}$ form a partition of $\{(,), 0\}^n$. Show that $S_{0,0}$ is precisely the set of Motzkin paths.

2. Let $\tilde{H} = \sum_{j=1}^{n-1} (|\Phi\rangle\langle\Phi|_{j,j+1} + |\Psi_l\rangle\langle\Psi_l|_{j,j+1} + |\Psi_r\rangle\langle\Psi_r|_{j,j+1})$. Prove that the eigenspace of $\tilde{H}$ associated with eigenvalue 0 (i.e. the ground space) is spanned by the states $|S_{p,q}\rangle$ that are the uniform superposition over all strings in $S_{p,q}$.

3. Conclude that $|\mathcal{M}_n\rangle = |S_{0,0}\rangle$ is the unique ground state of H.