Exercise XI, Computational Complexity 2024

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

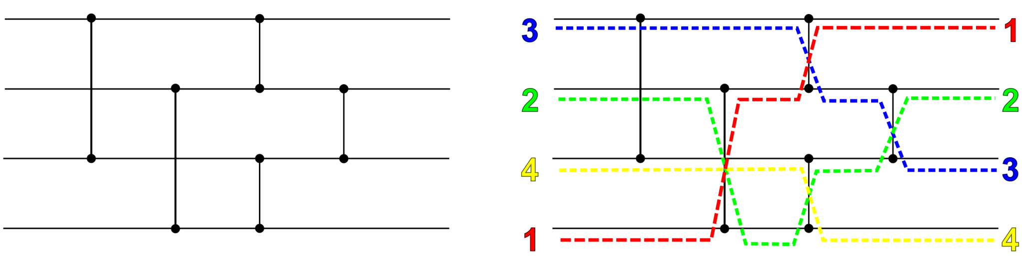
Set-Intersection, and Applications

1 Consider the directed s–t connectivity function $\textsc{Conn}\colon \{0,1\}^{n\times n} \to \{0,1\}$ where the input $x \in \{0,1\}^{n\times n}$ is the adjacency matrix of a graph $G = ([n], E)$ (that is, $(i,j) \in E$ iff $x_{ij} = 1$) and $\textsc{Conn}(x) = 1$ iff this graph has a directed path from vertex $s = 1$ to vertex $t = n$. Note that $\textsc{Conn}$ is monotone. Show that the monotone KW game for $\textsc{Conn}$ can be solved with a $O(\log^2 n)$-bit deterministic protocol. (*Hint: What would Savitch do?*)

2 Recall Problem X-1: Alice has $A \subseteq [n]$, $|A| \geq n/2$, Bob has $B \subseteq [n]$, $|B| < n/2$, and they want to output $i \in A \smallsetminus B$.

(a) Observe that this problem is essentially the monotone Karchmer–Wigderson game for the n-bit majority function $\textsc{Maj}_n\colon \{0,1\}^n \to \{0,1\}$

(b) A *sorting network*[1] consists of n wires together with *comparator gates* that given two input numbers (a,b) output the pair $(\min(a,b), \max(a,b))$. On input a list of n integers, the network outputs the integers in sorted order. Explain how a sorting network of *depth* d (maximum number of comparisons in any input-to-output path) can be used to construct a monotone circuit of depth d for $\textsc{Maj}_n$.



(picture credit: Wikipedia)

(c) The famous AKS network has depth $O(\log n)$. Assuming this, conclude that there is a $O(\log n)$-bit deterministic protocol for Problem X-1.

[1] `https://en.wikipedia.org/wiki/Sorting_network`

3 Consider the following problem: Alice holds a graph $G_A = ([n], E_A)$, Bob holds a graph $G_B = ([n], E_B)$, and their goal is to decide whether $G_A \cup G_B = ([n], E_A \cup E_B)$ is connected.

(a) Find a deterministic protocol of cost $O(n \log n)$ for this problem.

(b) Prove an $\Omega(n)$ lower bound by a reduction from set-intersection SI.
(*Formally, a reduction from* SI_n *to* $f\colon \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ *consists of a pair of functions* $A\colon \{0,1\}^n \to \mathcal{X}$ *and* $B\colon \{0,1\}^n \to \mathcal{Y}$ *such that* $\mathrm{SI}_n(x,y) = f(A(x), B(y))$ *for all* x, y.)

4 *Set-intersection is "*NP*-complete."* Let $f\colon \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be any two-party function with nondeterministic communication complexity $\mathrm{N}_1(f) = k$. Show that f reduces to the set-intersection function SI_n with input length $n = 2^k$.