Computational Motor Control

Lecture 2:

Intro to dynamical systems

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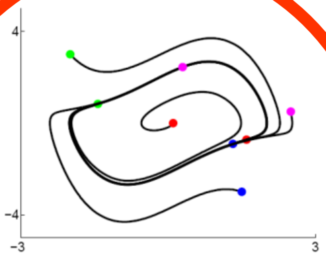
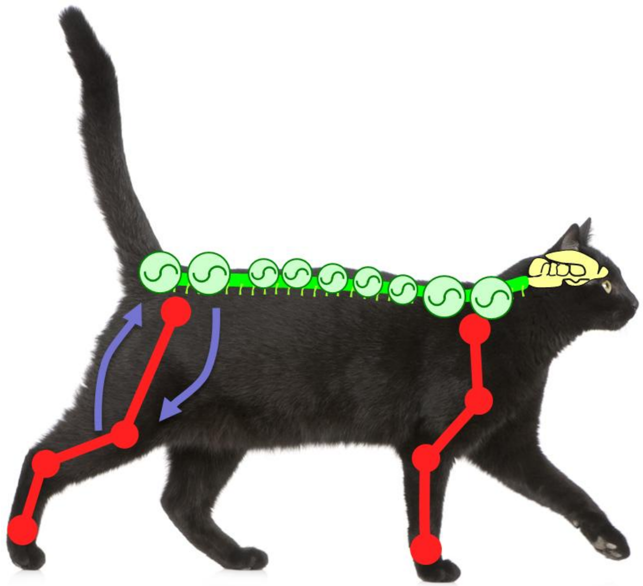
Recap of goals:

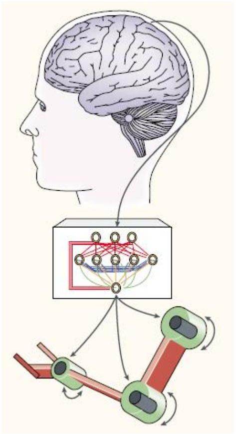
- To explore how **numerical simulations can be used to explain *motor control* in biology**
- To learn **how to design good numerical models**, and **how to evaluate them**
- To present how inspiration from biology can bring useful contributions to the **new design and control principles for robotics**
- To apply concepts from the lectures to (1) **design and test simple models in Python**, and (2) **develop sensory-motor models applied to a simulated zebrafish**

Contents of lectures

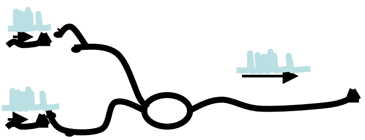


Dynamical systems

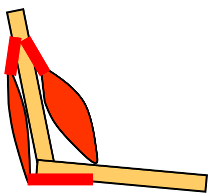




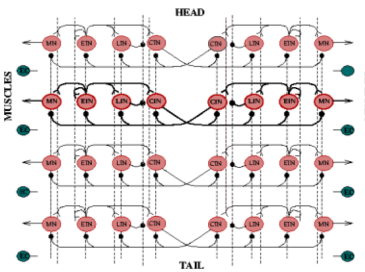
Neuroprosthetics



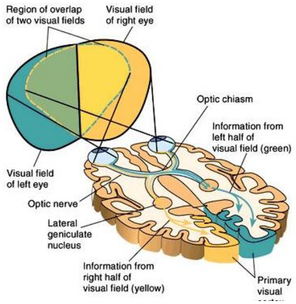
Neuron models



Muscle and Biomech. models



Motor system models



Visual system models

Lecture 2:
Methodology of modeling and intro to dynamical systems

Topics:

- Dynamical systems, ODEs
- Solving ODEs
- Definition of stability
- Linear dynamical systems

Dynamics is Everywhere

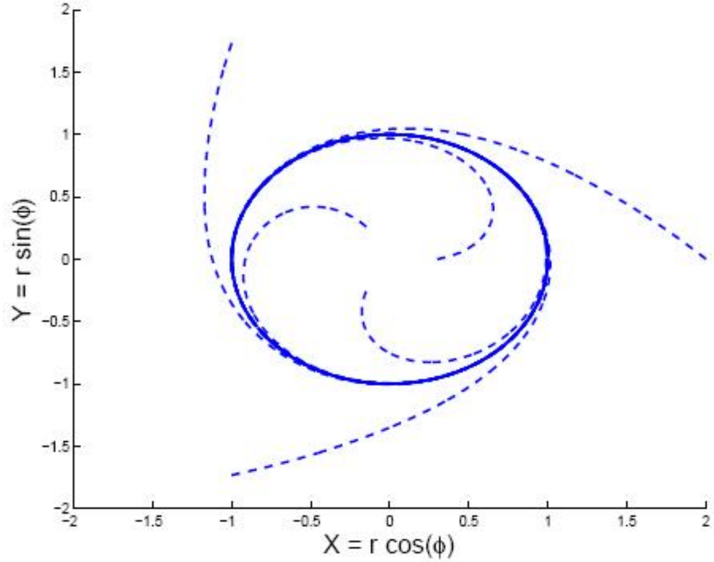
Physics
Chemistry
Weather Prediction, …

Neurons in the brain
Models of single neurons
Oscillations, synchronization
Neural network behaviors

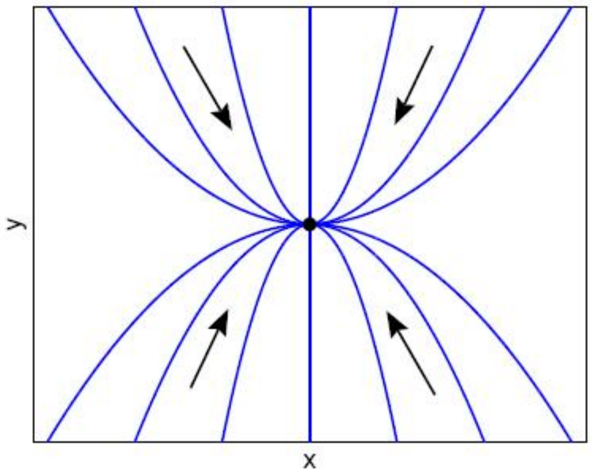
Central pattern generators,
neuromechanical models

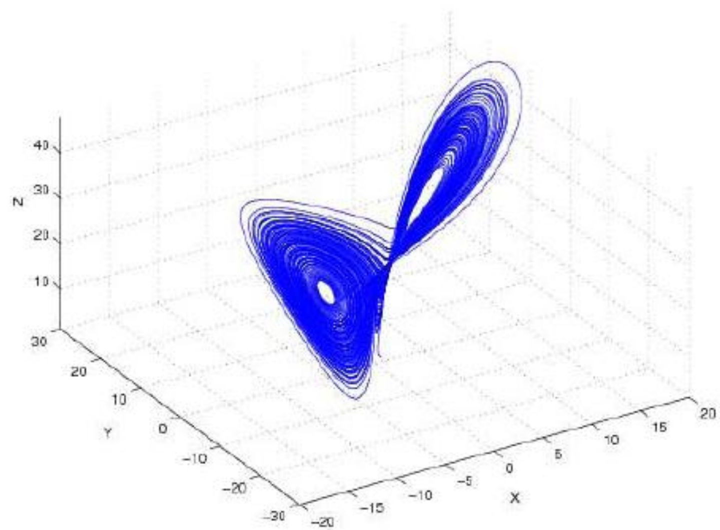
Dynamical systems features

Limit cycle attractors



Fixed point attractors





Chaos

Dynamical Systems Theory is...

- ... a powerful mathematical framework for modeling and understanding dynamic processes

- ... Highly relevant for many domains in science and engineering

- ... NOT a new topic, the first dynamical system appeared with the invention of calculus (Newton in 1666!)

- Development of the modern theory since Poincaré (~1890)

Dynamical Systems

- Two kinds of dynamical systems

1. **Differential equations** (continuous in time):

   - ***Ordinary Differential Equations*** (ODE) e.g.

$$\frac{dx}{dt} = \dot{x} = f(x)$$

   - ***Partial Differential Equations*** (PDE) e.g.

$$\frac{du}{dt} = \frac{\partial^2 u}{\partial x^2}$$

2. **Iterated maps** (discrete in time):

$$x_{n+1} = f(x_n)$$

Ordinary differential equations

Most models presented in the course will be based on **ordinary differential equations (ODEs)**:

$$\frac{d}{dt}\vec{x} = f(\vec{x}, \vec{\alpha}, t)$$

These types of equations are used in many types of numerical models. They determine how the *state variables* $\vec{x}$ vary over time. The time derivative of the state variables $\vec{x}$ are described as a (usually nonlinear) function of the state variables, some *parameters* $\vec{\alpha}$, and (possibly) the time *t*.

***Autonomous* ODEs**: $\frac{d}{dt}\vec{x} = f(\vec{x}, \vec{\alpha})$ no explicit dependence on time

***Non-autonomous* ODEs**: $\frac{d}{dt}\vec{x} = f(\vec{x}, \vec{\alpha}, t)$ explicit time dependence, much harder to deal with

Ordinary differential equations

Note, any **higher order differential equation**:

$$\frac{d^2}{dt^2} x_1 = -\omega x_1$$

Can be transformed into a **set of first order equations** by the introduction of additional variables:

$$\frac{d}{dt} x_2 = -\omega x_1$$

$$\frac{d}{dt} x_1 = x_2$$

➔ we restrict our analysis to first order systems: $\frac{d}{dt}\vec{x} = f(\vec{x}, \vec{\alpha}, t)$

Introduction to nonlinear dynamical systems, part I

Topics:

- Dynamical systems, ODEs
- **Solving ODEs**
- Definition of stability
- Linear dynamical systems

First Example

- Consider the following nonlinear system

$$\dot{x} = \frac{dx}{dt} = \sin x$$

- Such a system can be analyzed in different ways:

    - Analytical integration
    - Numerical integration
    - Geometrical analysis

First Example: analytical integration

- This particular nonlinear system

$$\dot{x} = \frac{dx}{dt} = \sin x$$

- can be solved analytically by separating the variables:

$$dt = \frac{dx}{\sin x}$$

- which implies

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| - \ln\left|\tan\frac{x_0}{2}\right|$$

First Example: analytical integration

- We have the exact solution:

$$t - t_0 = \ln\left|\tan\frac{x}{2}\right| - \ln\left|\tan\frac{x_0}{2}\right|$$

- This is very valuable, but it is difficult to interpret because in this case, it does not provide $x(t)$

- Given some initial conditions x_0, what are the qualitative features of the solution $x(t)$?
- What happens when $t \rightarrow \infty$?
- Is this behavior the same for any initial condition?
- We cannot so easily answer these questions with the exact analytic solution of the equation

➔ it is often useful to also use numerical integration and geometrical interpretation

First Example

- Consider the following nonlinear system

$$\dot{x} = \frac{dx}{dt} = \sin x$$

- Such a system can be analyzed in different ways:

  - Analytical integration
  - **Numerical integration**
  - Geometrical analysis

Numerical integration

Almost all interesting problems in biology are **nonlinear and multidimensional**

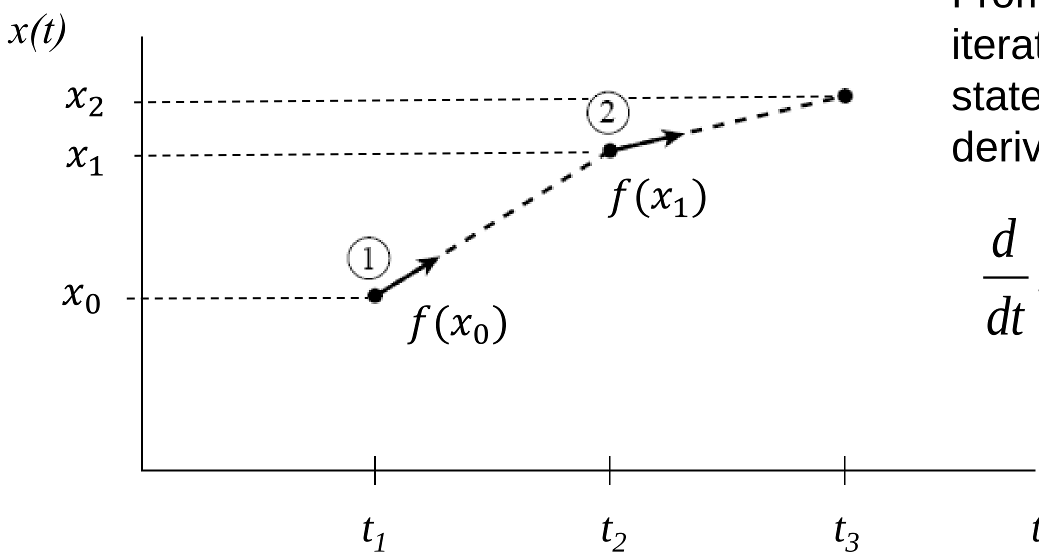
Analytical (closed form) solutions very rarely exist → therefore **use of numerical tools to solve the equations**

Useful integration methods for solving differential equations:

- **Euler Method**
- **Runge Kutta method**

Note: numerical integration typically corresponds to transform a differential equation into an **iterative map** (discretization of time)

Solving differential equations: Euler method



From an initial condition x_0 iteratively update your state using the local derivative value:

$$\frac{d}{dt}\vec{x} = f(\vec{x},\vec{\alpha},t)$$

A simple iterative method, with small Δt steps:

$$\vec{x}(t+\Delta t) = \vec{x}(t) + \frac{d}{dt}\vec{x}(t)\cdot\Delta t = \vec{x}(t) + f(\vec{x}(t),\vec{\alpha},t)\cdot\Delta t$$

That is directly derived from the definition of **the time derivative**:

$$\frac{d}{dt}\vec{x}(t) = \lim_{\Delta t\to 0}\frac{\vec{x}(t+\Delta t)-\vec{x}(t)}{\Delta t}$$

Solving differential equations: Euler method

Advantage of Euler method:
- Very simple and very quickly implemented

Disadvantages:
- Less accurate than other methods, e.g. Runge-Kutta
- Not always stable

It is better not to use it in practice.

Solving differential equations: 4th order Runge-Kutta method

= A more sophisticated method, that updates the states based on **four estimations** of the derivatives:

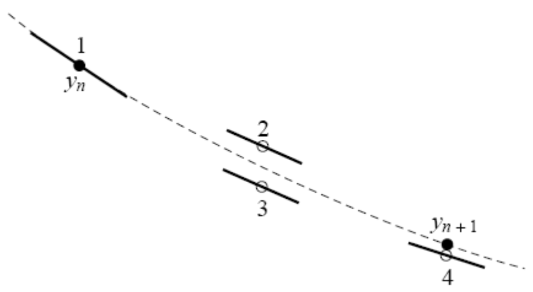


Figure 16.1.3. Fourth-order Runge-Kutta method. In each step the derivative is evaluated four times: once at the initial point, twice at trial midpoints, and once at a trial endpoint. From these derivatives the final function value (shown as a filled dot) is calculated. (See text for details.)

Runge Kutta:

$$k_1 = f(x(t), t) \cdot \Delta t$$

$$k_2 = f\left(x(t) + \frac{k_1}{2}, t + \tfrac{\Delta t}{2}\right) \cdot \Delta t$$

$$k_3 = f\left(x(t) + \frac{k_2}{2}, t + \tfrac{\Delta t}{2}\right) \cdot \Delta t$$

$$k_4 = f(x(t) + k_3, t + \Delta t) \cdot \Delta t$$

$$x(t + \Delta t) = x(t) + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\Delta t^5)$$

Corresponds to the first terms of the Taylor expansion

Solving differential equations: 4th order Runge-Kutta method

Fourth order

Runge Kutta:

$$k_1 = f(x(t),t)\cdot\Delta t$$

$$k_2 = f(x(t)+\frac{k_1}{2}, t+\tfrac{\Delta t}{2})\cdot\Delta t$$

$$k_3 = f(x(t)+\frac{k_2}{2}, t+\tfrac{\Delta t}{2})\cdot\Delta t$$

$$k_4 = f(x(t)+k_3, t+\Delta t)\cdot\Delta t$$

First order

Euler:

$$x(t+\Delta t) = x(t) + f(x(t),t)\cdot\Delta t + O(\Delta t^2)$$

$$x(t+\Delta t) = x(t) + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\Delta t^5)$$

Solving differential equations: 4th order Runge-Kutta method

Advantage of the Runge-Kutta method:
- Always more accurate than Euler for the same step size
- Normally one step of Runge-Kutta is more accurate than 4 smaller steps with Euler (see Lab 1)

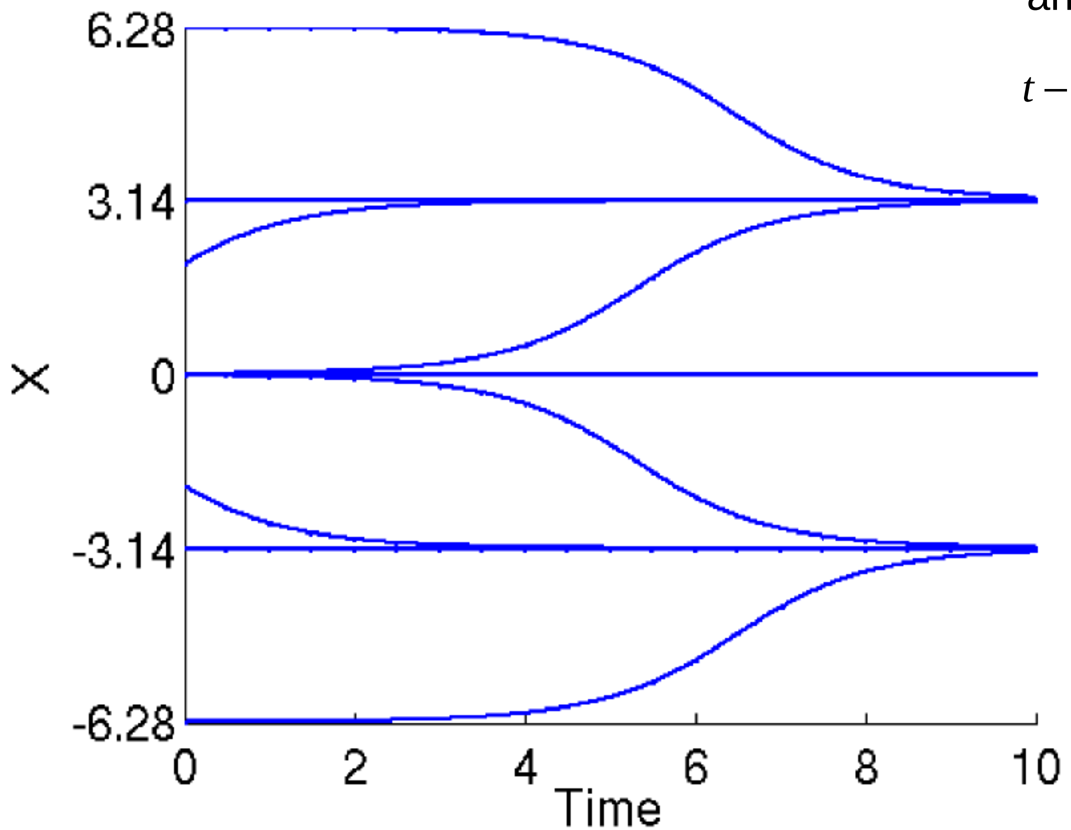
Disadvantages:
- Can have difficulties with *stiff* problems (problems with different times scales in which derivatives sometimes change abruptly). E.g. contacts in a mechanical simulation.
- Can be slower than other, more sophisticated, methods.

First Example: numerical integration

- We can understand the behavior of the system by numerically integrating the equations for several initial conditions

Exact match with the analytical solution:

$$t - t_0 = \ln\left|\tan\frac{x}{2}\right| - \ln\left|\tan\frac{x_0}{2}\right|$$



Matlab example: sinx_interface

Solving differential equations: use small integration steps!

With any numerical integration method, the **integration step size must be sufficiently small**, otherwise it leads to numerical errors.

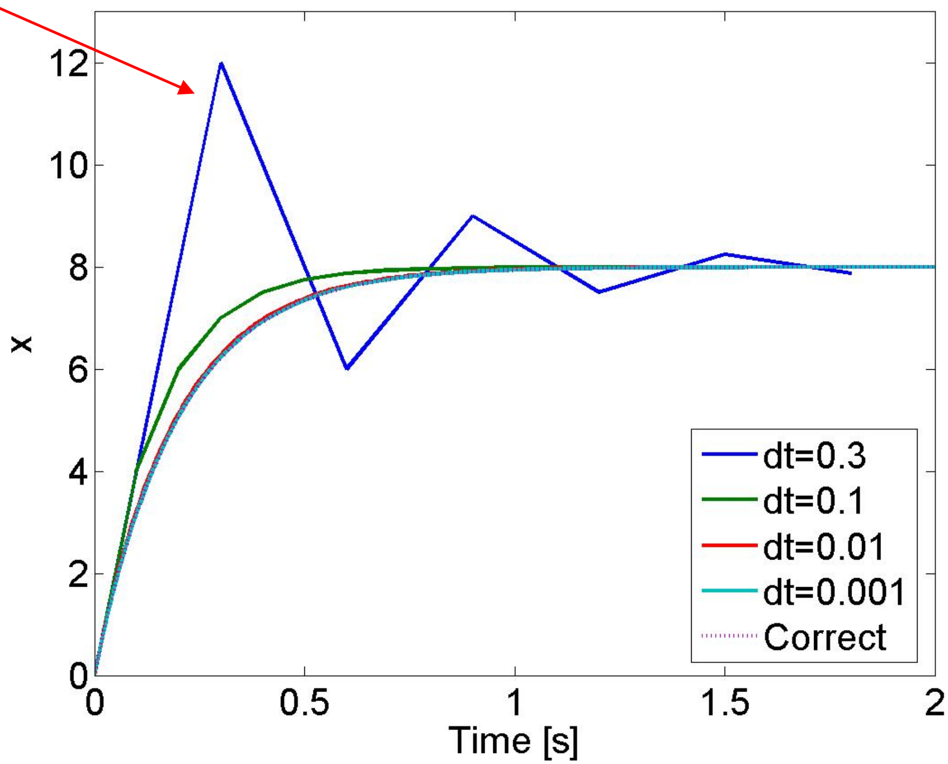
One possibility to choose a step size: make several integration trials: start relatively large, and **gradually decrease until the results do not differ significantly**

See next.

Role of time step size

Numerical instability: the solution is wrong not only quantitatively but also qualitatively

Euler Integration:



Simple example:

$$\frac{d}{dt}x = \alpha(g - x)$$

$$x(t=0) = 0.0$$

Analytical solution:

$$x(t) = (x_0 - g)e^{-\alpha t} + g$$

A too large time step will lead to **numerical instability**. In the practicals: if you see strange behavior (e.g. an exploding body), the first thing to try is to reduce the integration time step.

Role of time step size

Simple example:

$$\frac{d}{dt}x = \alpha(g - x)$$

$$x(t=0) = 0.0$$

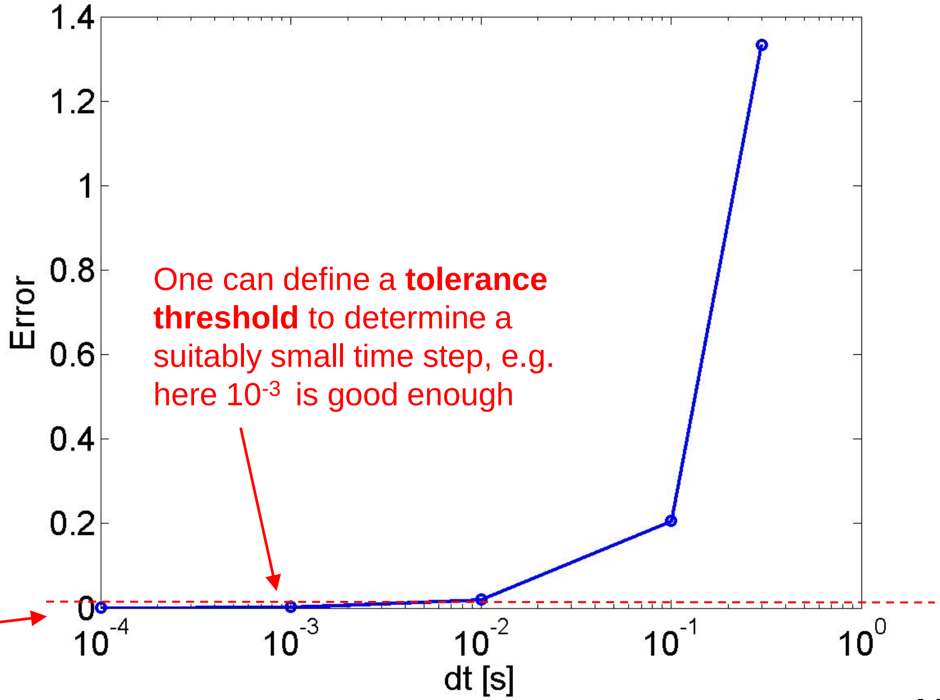
Analytical solution:

$$x(t) = (x_0 - g)e^{-\alpha t} + g$$

Analytical solution — Numerical integration

$$E = \sum_{t=1}^{N} \left| \frac{x(t) - \tilde{x}(t)}{N} \right|$$

The error decreases with the step size:



But what can happen with **too small** steps?

More sophisticated methods

It is better to choose **integration methods with adaptive time steps**: the size of the integration time step is automatically adjusted based on an estimation of the error and a chosen **tolerance value**. Typically: smaller integration time steps when derivatives are large (i.e. when state variables change rapidly)

See for instance ode45 in Matlab (4^{th} order Runge Kutta with variable time step).

Some algorithms even switch between integration methods, e.g. the **LSODA algorithm** (see lab 1), which adaptively switches between methods for stiff and nonstiff ODEs,

Petzold, L. Automatic selection of methods for solving stiff and nonstiff systems of ordinary differential equations. SIAM. *J. Sci. Stat. Comput. Stat. Comput.* **4**, 136–148 (1983).

Numerical integration

Note: how to properly do numerical integration is a big topic by itself. We only scratched the surface here.

In the lab 1 practical, you will explore this a bit further, including other methods such as LSODA and DOPRI.

For an interesting comparison of several methods and the influence of hyper parameters, see:

Städter, P., Schälte, Y., Schmiester, L., Hasenauer, J., & Stapor, P. L. (2021). Benchmarking of numerical integration methods for ODE models of biological systems. *Scientific Reports*, *11*(1), 2696. https://doi.org/10.1038/s41598-021-82196-2

First Example

- Consider the following nonlinear system

$$\dot{x} = \frac{dx}{dt} = \sin x$$

- Such a system can be analyzed in different ways:

    - Analytical integration
    - Numerical integration
    - **Geometrical analysis**

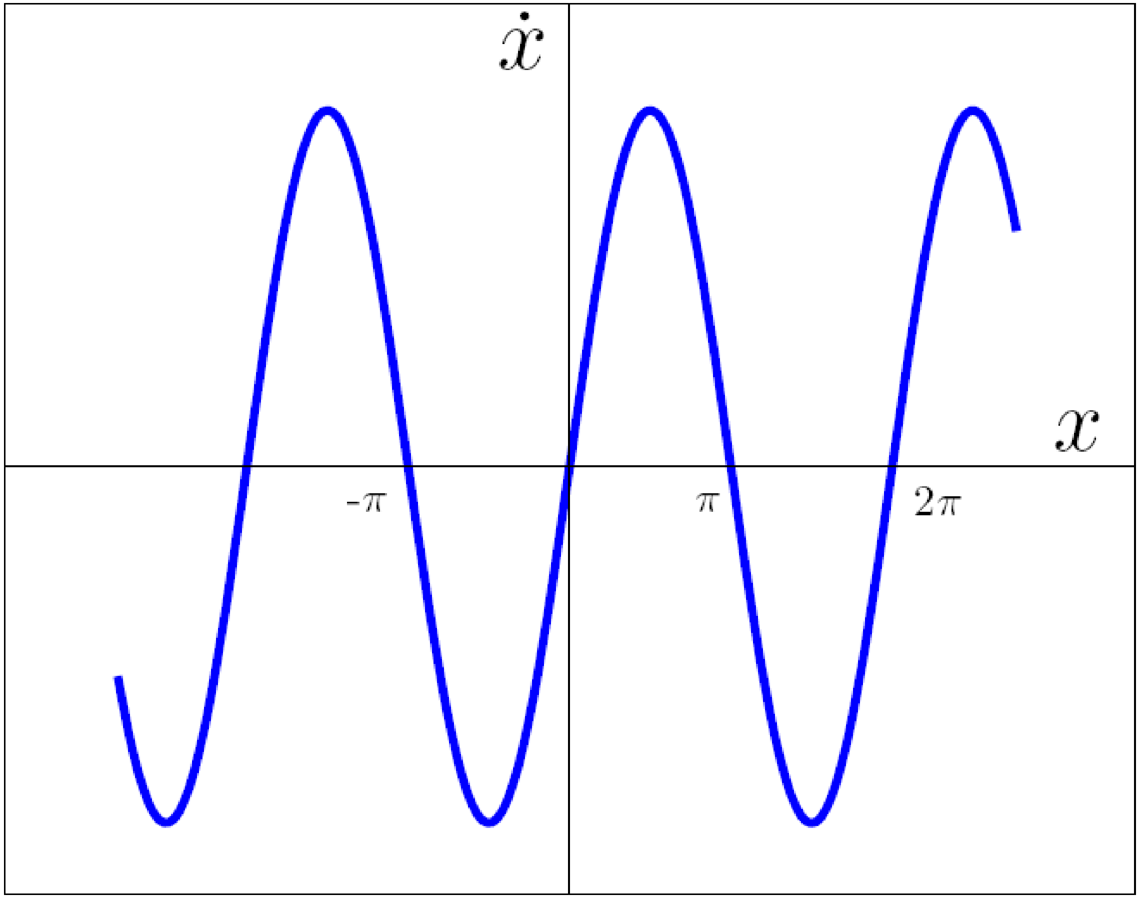
First Example: the geometric way

- We can extract qualitative information from the differential equation without solving it explicitly and without numerical integration
- From a physical point of view, $\dot{x}$ is the rate of change of x (its "velocity")

- It is therefore very useful to make the ***derivative vs state* plot**

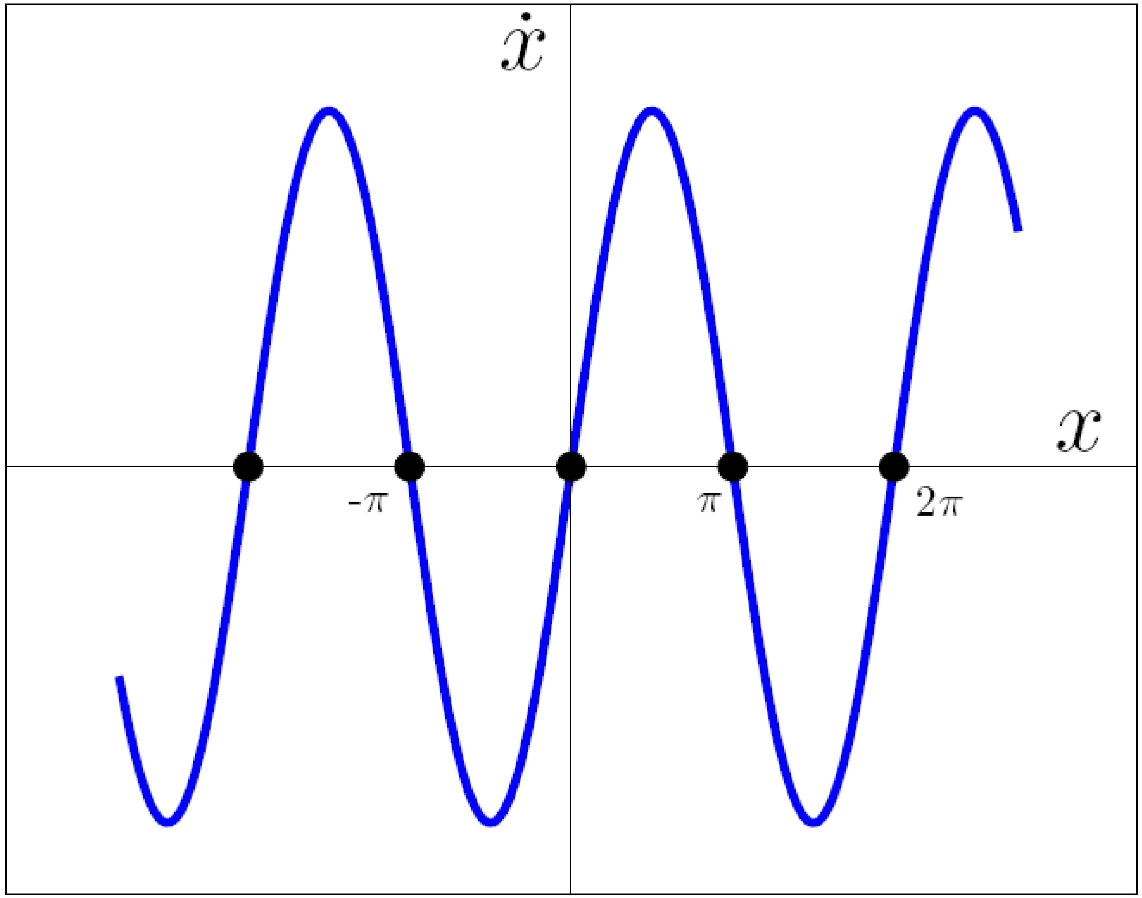
- For each point x *in* R, we know its rate of change $\dot{x} = \sin x$
- If $\dot{x}>0$, x increases
- If $\dot{x}=0$, x stays fixed
- If $\dot{x}<0$, x decreases

Derivative vs state plot



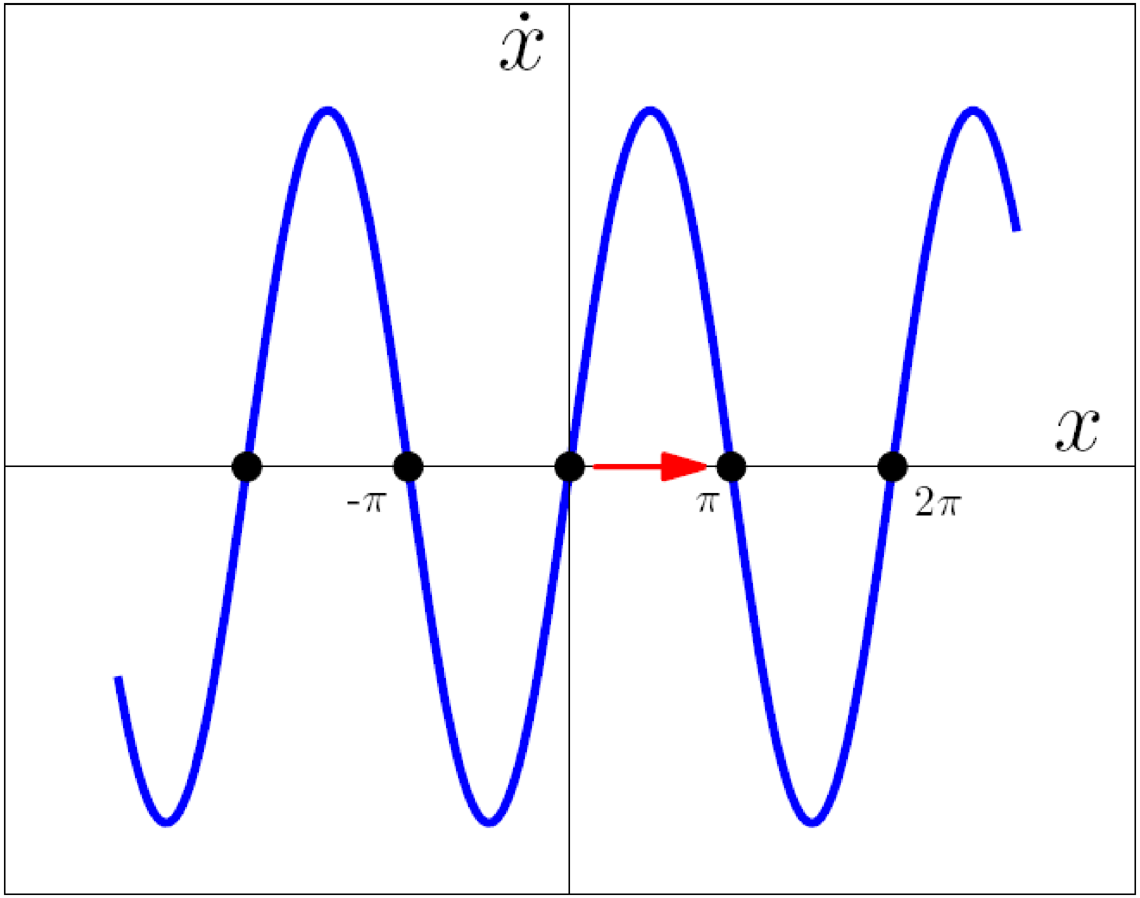
- We plot $\dot{x} = \sin x$ as a function of x to better understand the flow of x

Derivative vs state plot



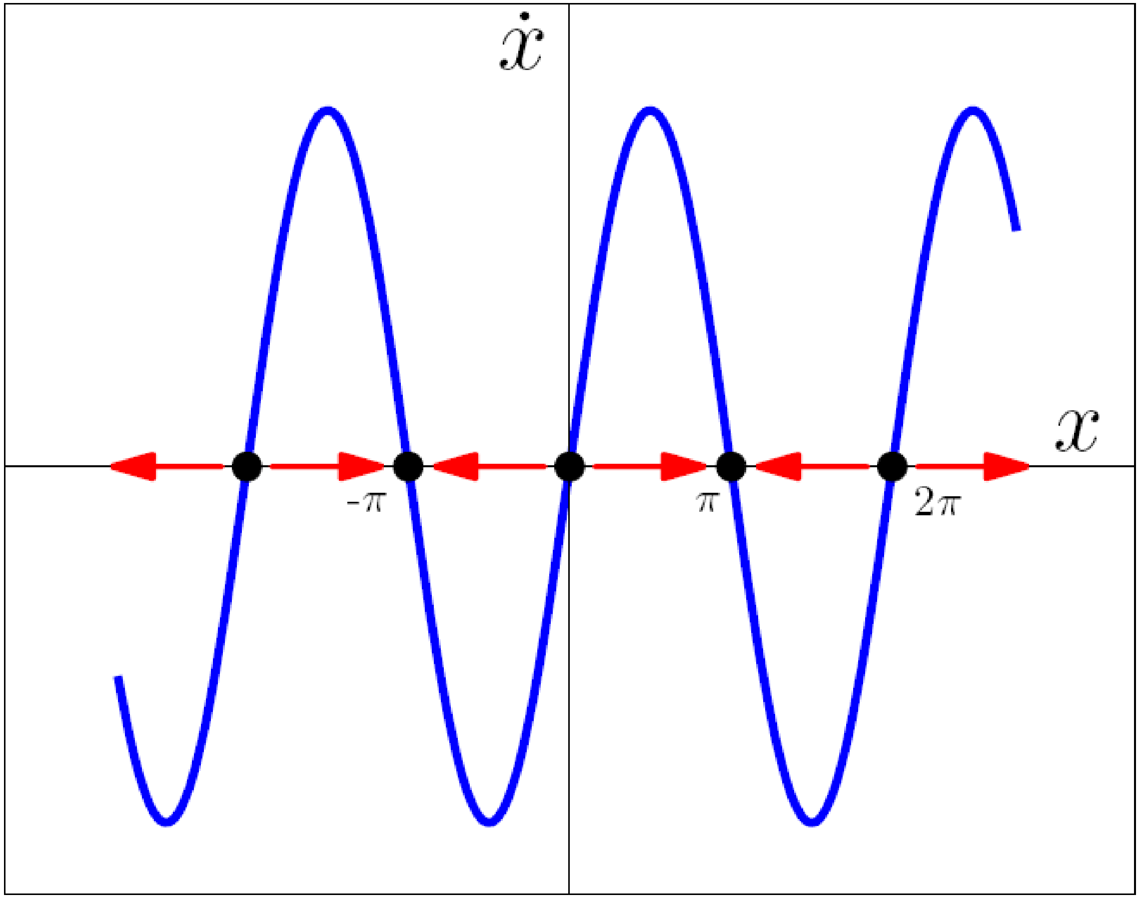
- If $x = n\,\pi$ then $\dot{x} = 0$
- And $x(t)$ does not change anymore
- These points are called ***Fixed Points*** (or *equilibrium points* or *rest points)*

Derivative vs state plot



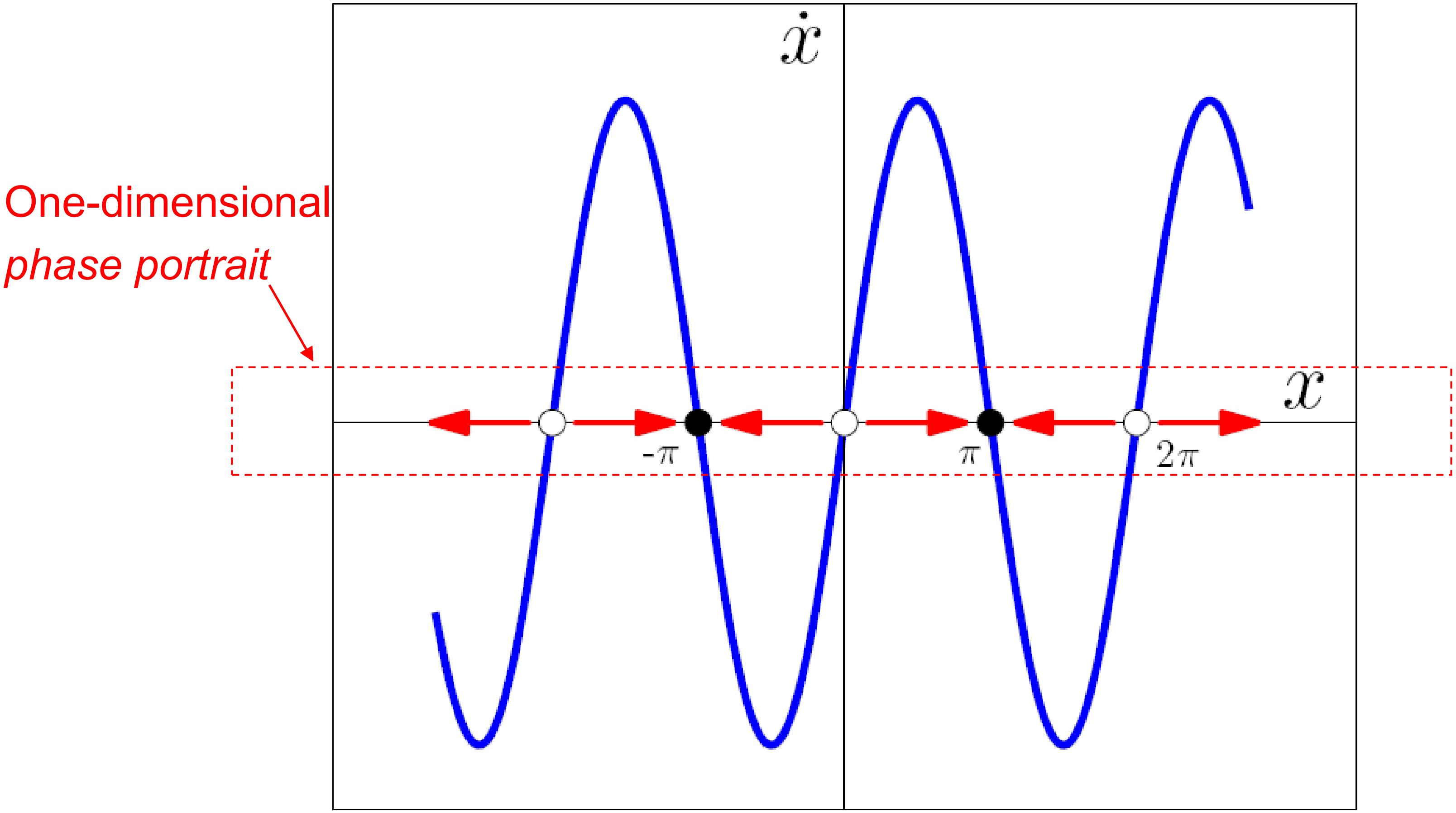
- If $0 < x < \pi$ then $\dot{x} > 0$ and x converges to π

Derivative vs state plot



- For each region between fixed points we find the
- convergence properties

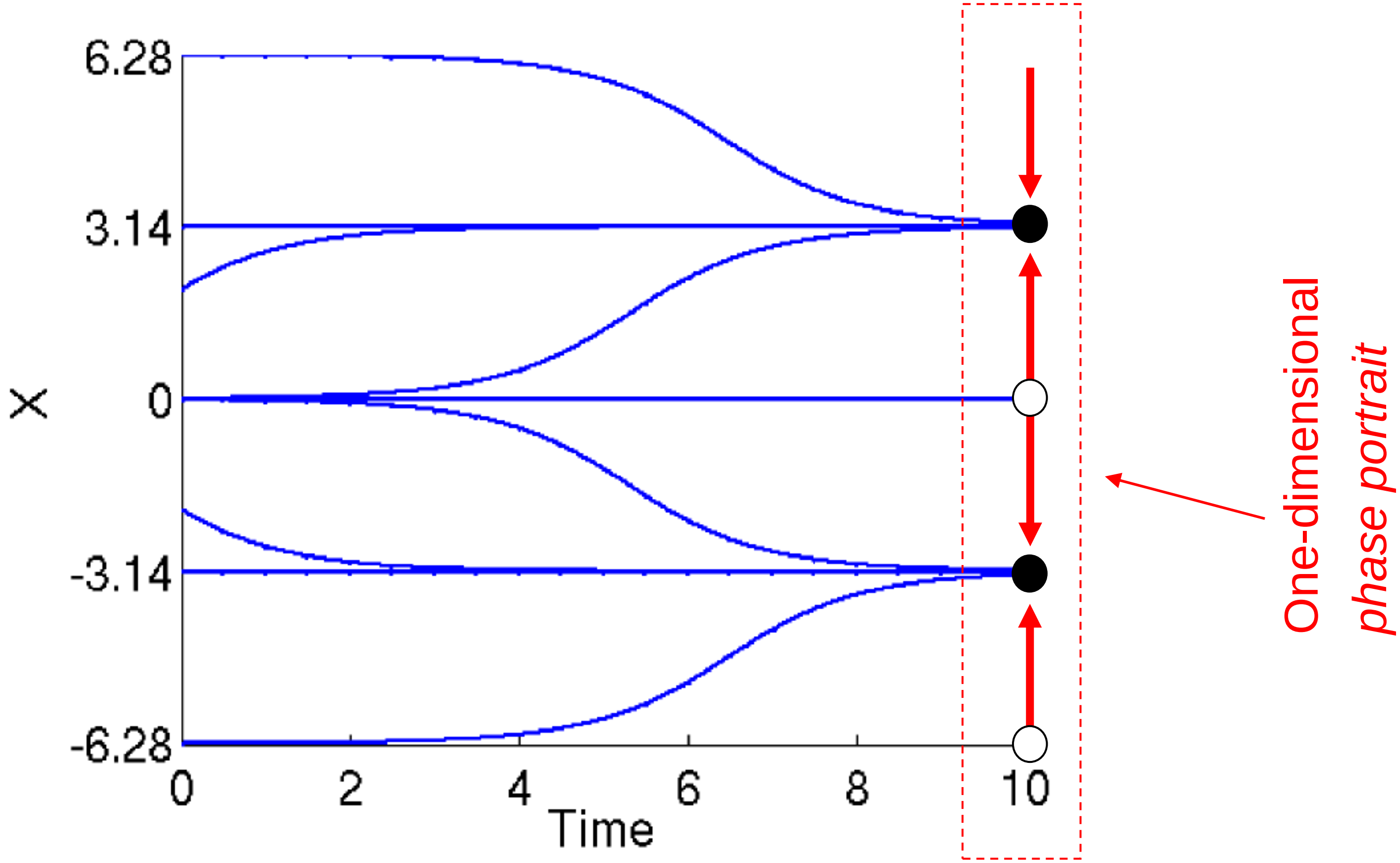
Derivative vs state plot



- We see 2 kinds of fixed points
- ***Stable*** *Fixed Points* are at $(2n+1)\pi \quad n \in Z$
- ***Unstable*** *Fixed Points* are $2n\pi \quad n \in Z$

First Example: the geometric way

The *phase portrait* correctly predicts the qualitative behavior of the system:



Note: this geometric approach works well for 1D dynamical systems. For 2D systems, one can sketch **2D direction fields**, i.e. arrows showing the local flow, see next week.

First Example: Principles

- For a 1-D system, plotting $\dot{x}$ as a function of x allows us to obtain the ***Phase Portrait***

- Phase portraits show all the qualitatively different trajectories of the system

- ***Fixed points*** represent **equilibrium solutions** (or steady state solutions).

- They are such that $\dot{x} = f(x) = 0$

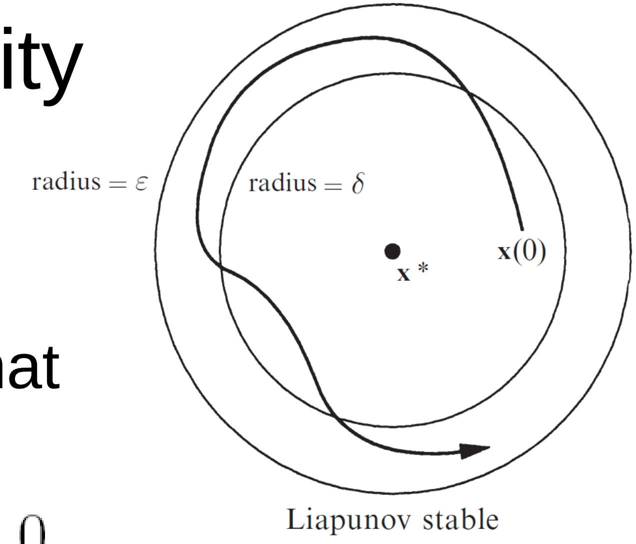
- Fixed points can be *stable* or *unstable*

Introduction to nonlinear dynamical systems, part I

Topics:

- Dynamical systems, ODEs
- Solving ODEs
- **Definition of stability**
- Linear dynamical systems

Definition of Stability



A fixed point x^* is

- ***(Lyapunov) stable*** if,
  for <u>each</u> $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|x(0) - x^*\| < \delta \Rightarrow \|x(t) - x^*\| < \epsilon, \qquad \forall t \geq 0$$

In words, (Lyapunov) stable means that trajectories that start within δ of x^* remain within ε of x^* for all positive time.
~ **It stays in the neighborhood**.

We talk about ***asymptotic stability*** if δ can be chosen such that

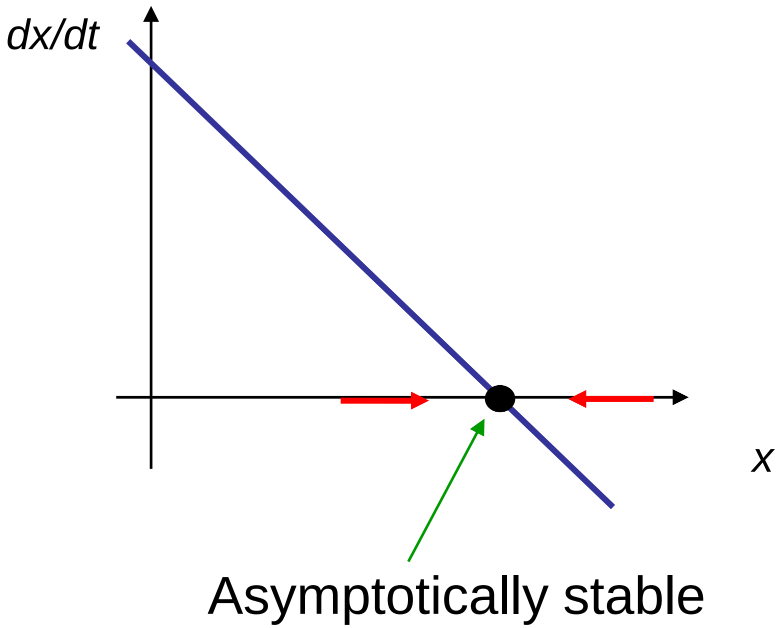
$$\|x(0) - x^*\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = x^*$$

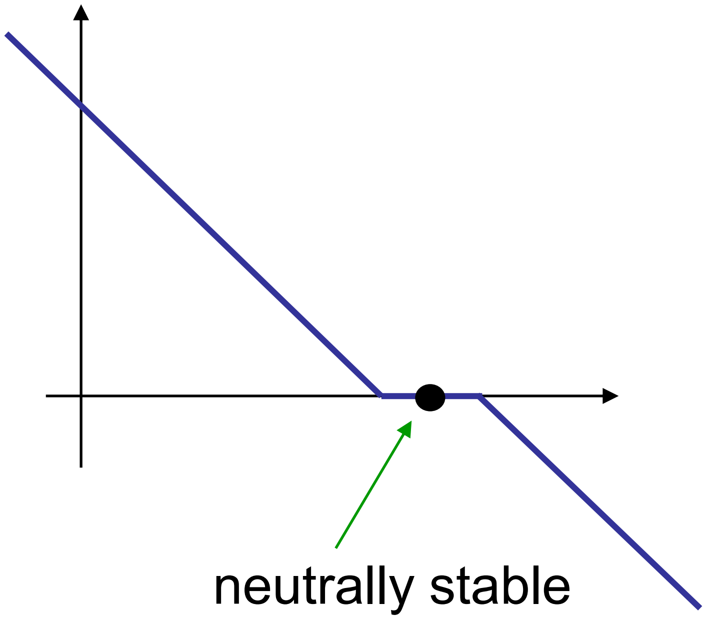
Definition of Stability

A fixed point x^* is

- ***neutrally stable*** or ***marginally stable*** if it is (Lyapunov) stable but not asymptotically stable

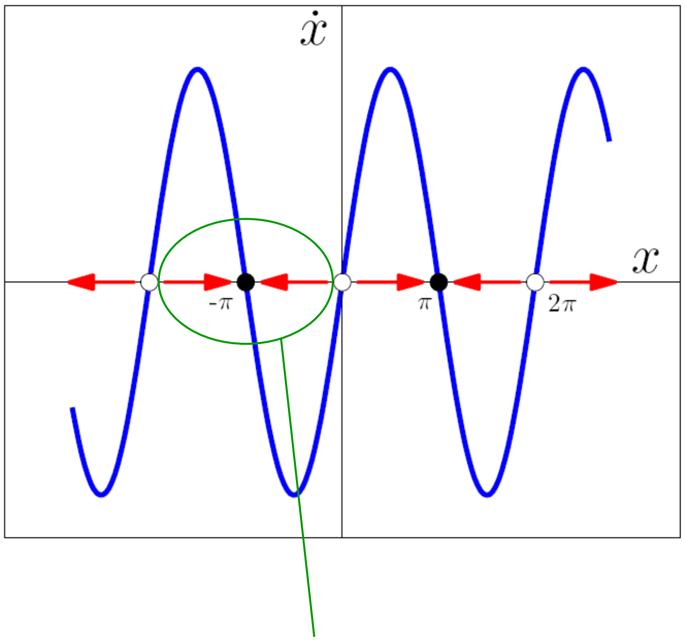
- Examples (derivative vs state plots):

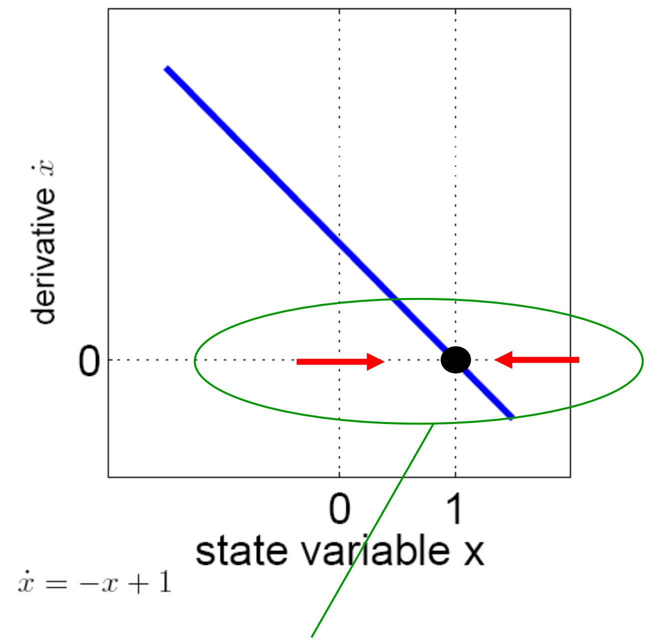




Local versus global stability

The stability of a fixed point can be *local* or *global*





Locally stable fixed point

Limited *basin of attraction*

Globally stable fixed point

Infinite *basin of attraction*

Introduction to nonlinear dynamical systems, part I

Topics:

- Dynamical systems, ODEs
- Solving ODEs
- Definition of stability
- **Linear dynamical systems**

Linear systems

- Linear systems that can be written on the form

$$\dot{x} = Ax$$

Where $x \in \mathbb{R}^n$ and A is a $n \times n$ matrix

- These systems are very well known and can often be solved analytically (except for some special cases, e.g. when *A* can not be diagonalized, because it does not have *n* linearly independent eigenvectors).

- We often linearize nonlinear systems to study their local behavior around some point

Linear systems

- Assuming that *A* is diagonalizable (i.e. it has *n* linearly independent eigenvectors), the generic solution to equation $\dot{x} = Ax$ where A is an n by n matrix is:

$$x(t) = \sum_{i=1}^{n} c_i \, e^{\lambda_i t} v_i \quad + \quad x_0$$

- Where λ_i is the i[th] **eigenvalue** of *A* and $v_i \in \mathbb{R}^n$ the associated **eigenvector**.

- Thus linear systems are limited to (fairly) simple behaviors.

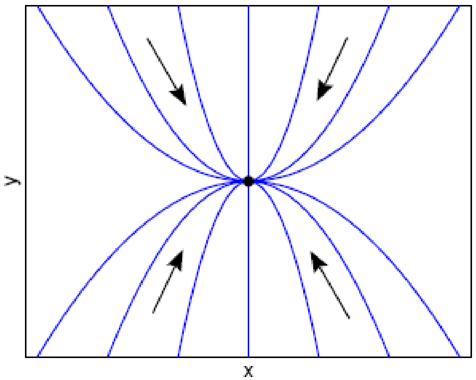
Reminder, the *n* eigenvalues λ and eigenvectors $\vec{v}$ of matrix A (*n* x *n*) are found by solving the following equations:

$$\det(A - \lambda I) = 0 \qquad\qquad A\vec{v} = \lambda\vec{v}$$

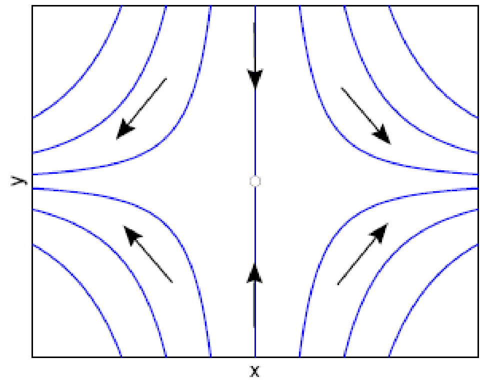
Where *det* is the determinant and *I* is the identity matrix

Linear systems (2D)

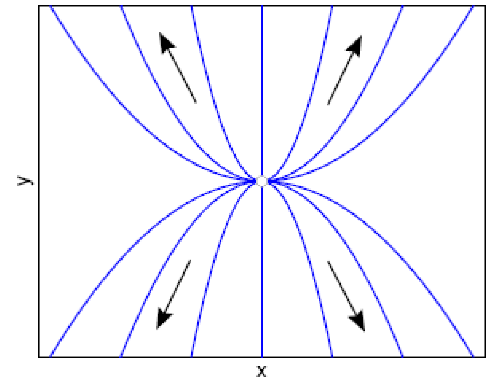
$\lambda_1 < 0$ and $\lambda_2 < 0$
Stable Fixed Point



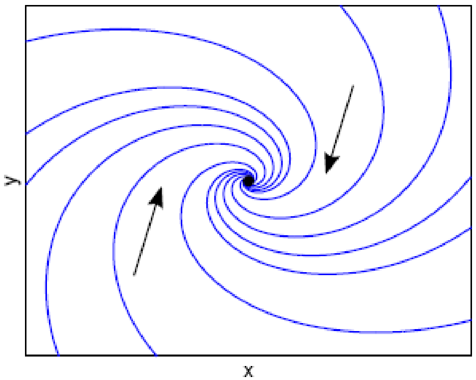
$\lambda_1 > 0$ and $\lambda_2 < 0$
Saddle



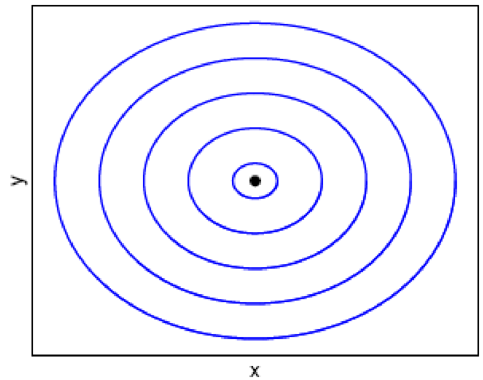
$\lambda_1 > 0$ and $\lambda_2 > 0$
Unstable Fixed Point



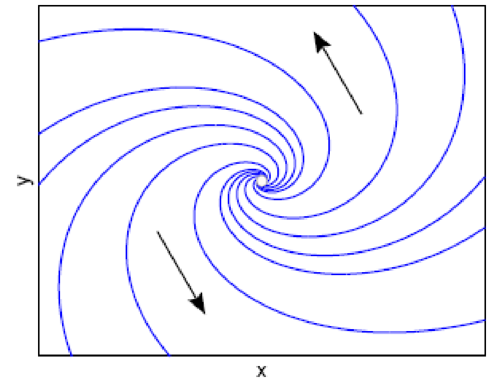
$\lambda_i \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i < 0)$
Stable Spiral



$\lambda_i \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i = 0)$
Neutrally Stable Center



$\lambda_i \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i > 0)$
Unstable Spiral



Note: here the eigenvectors are orthogonal. In general, they might not be.

Eigenvectors and eigenvalues

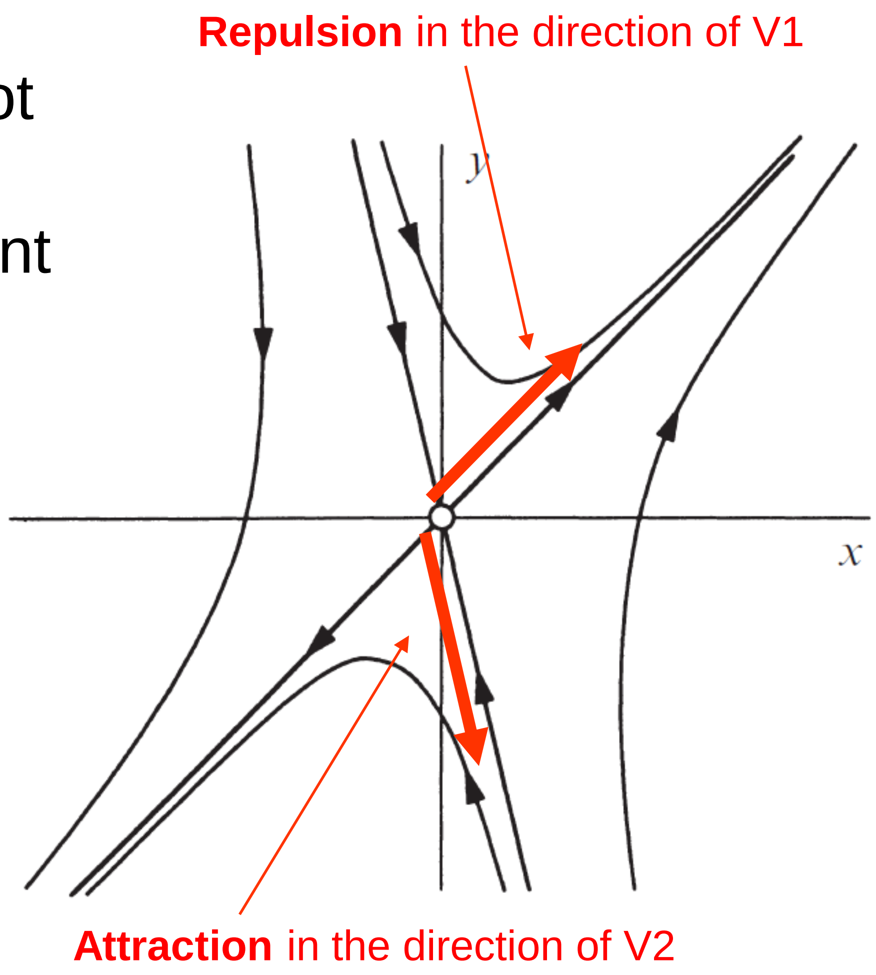
- Eigenvectors are in general not orthogonal.
- Eigenvectors are very important as they define **important directions in phase space**.

- Example

$$\dot{x} = x + y$$
$$\dot{y} = 4x - 2y$$

$$\lambda_1 = 2,\ \lambda_2 = -3.$$

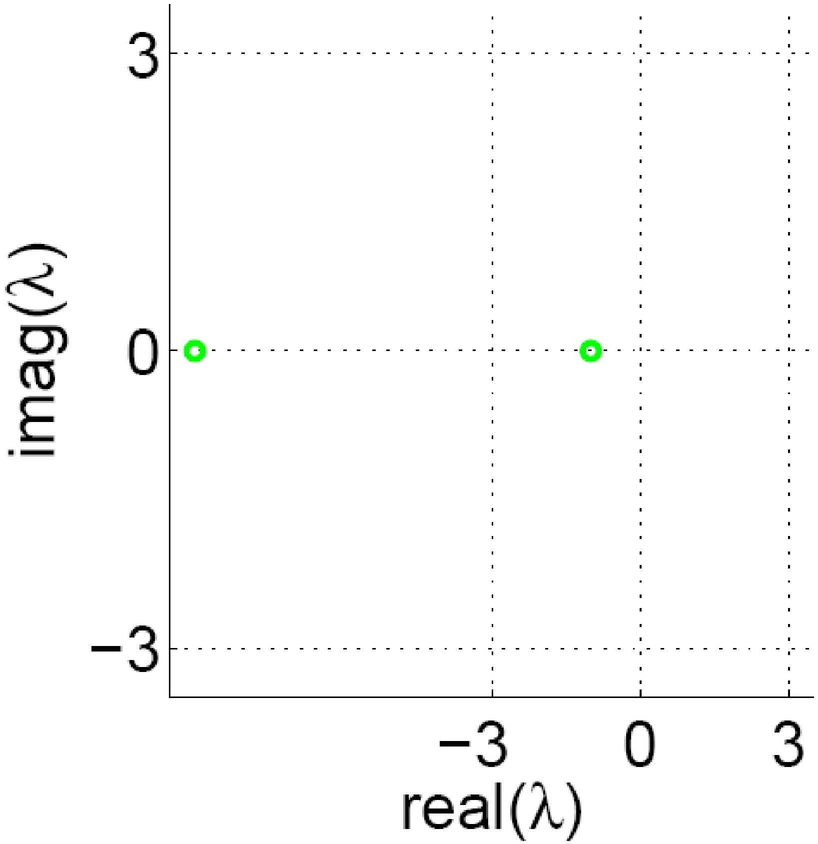
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$



Linear systems

- If $\lambda_i > 0$ the trajectory diverges exponentially fast to ∞
- If $\lambda_i < 0$ the trajectory converges exponentially fast to 0
- If $\lambda_i \in \mathbb{C}$ the trajectory has a periodic component since $e^{\mathrm{i}\lambda_i t} = e^{\mathrm{Re}(\lambda_i)t}\big(\cos(\mathrm{Im}(\lambda_i)t) + \mathrm{i}\sin(\mathrm{Im}(\lambda_i)t)\big)$
- The amplitude of the oscillations
  - increases if $\mathrm{Re}(\lambda_i) > 0$
  - decreases if $\mathrm{Re}(\lambda_i) < 0$
  - is constant if $\mathrm{Re}(\lambda_i) = 0$
- No stable oscillations exists, a perturbation pushes the system in a new oscillatory mode

Example of a 2D linear system, **root locus diagram**



$d = -10$

$\lambda_1 = -1$
$\lambda_2 = -9$

Let's analyze the following 2D linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ -3 & d \end{pmatrix}$$

The eigenvalues of this 2D system can be plotted in the space of imaginary numbers. This is known as a ***root locus diagram***.

Here we will plot the eigenvalues for different values of the critical parameter d.

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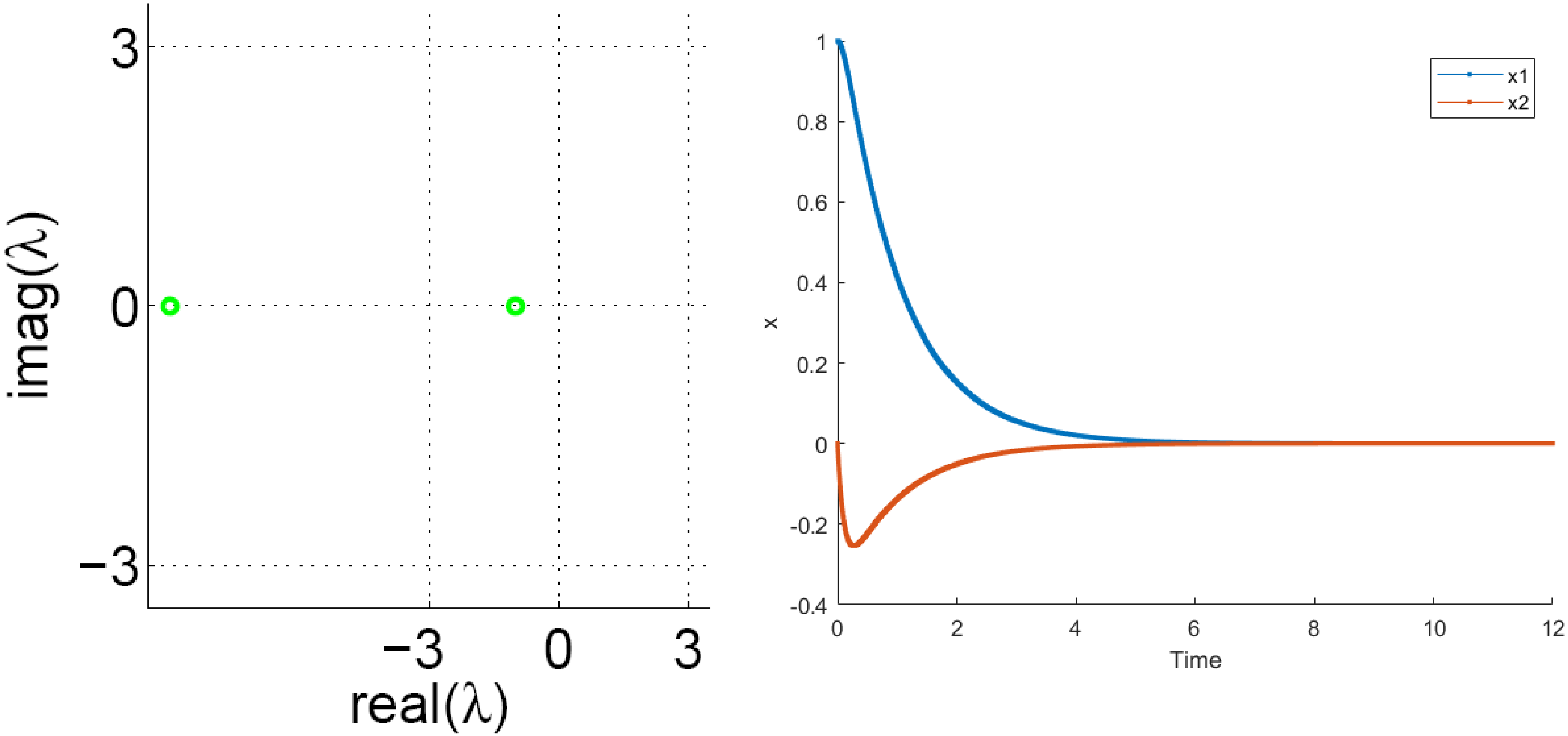
Example of a 2D linear system, **root locus diagram**



$$d = -10$$

$\lambda_1 = -1$
$\lambda_2 = -9$

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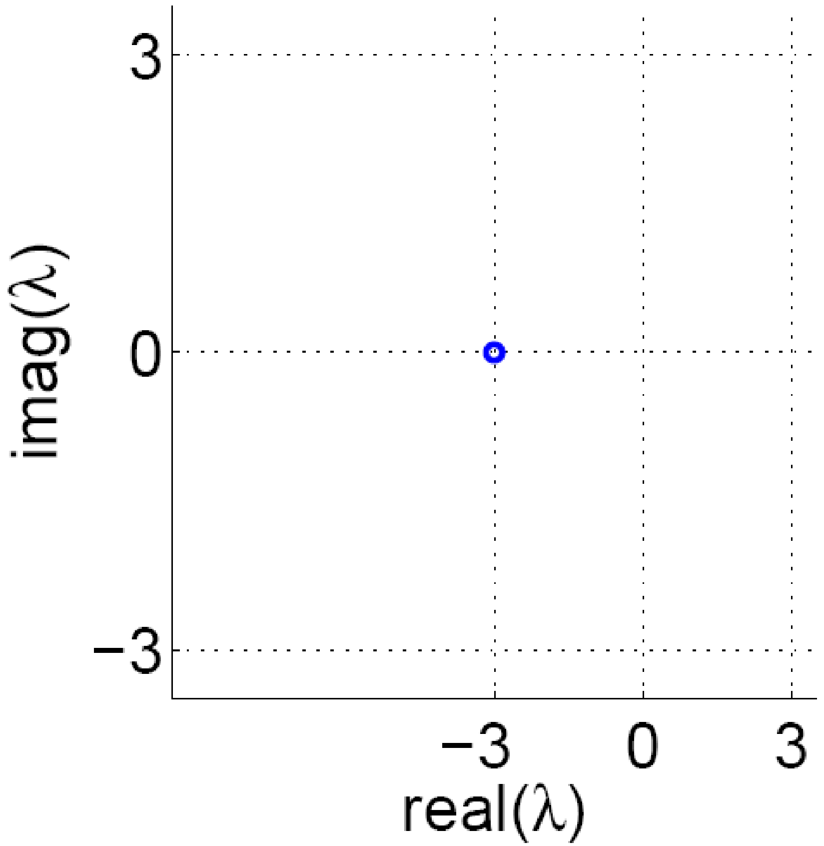
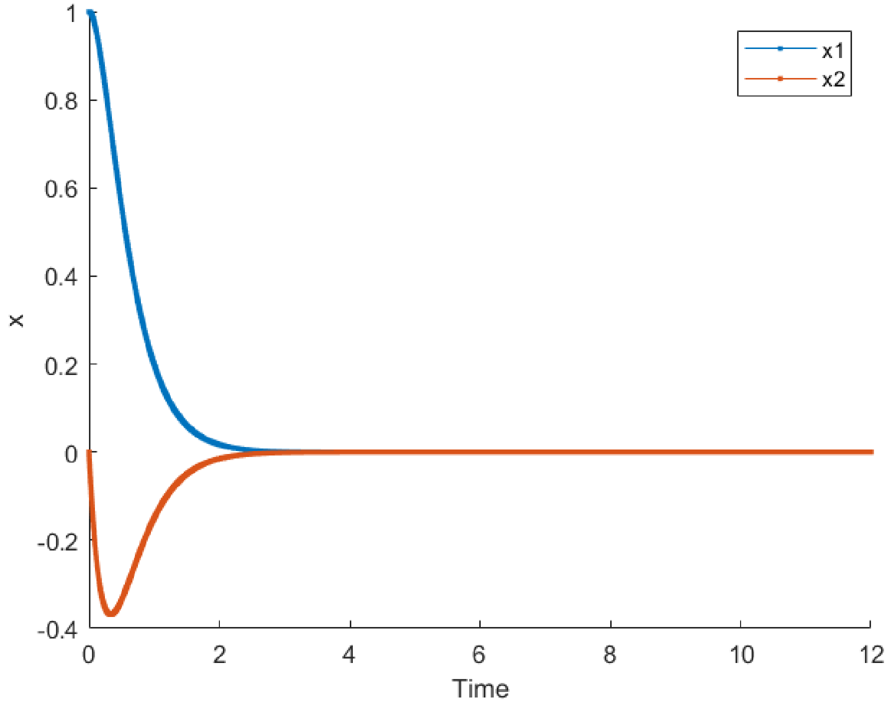
Example of a 2D linear system, **root locus diagram**

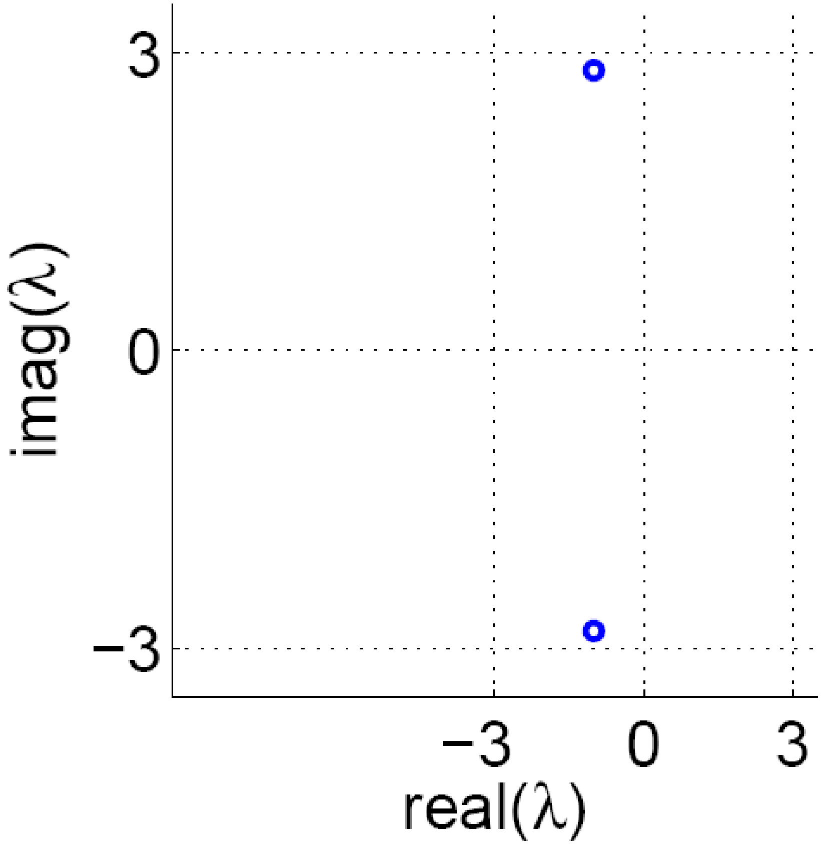


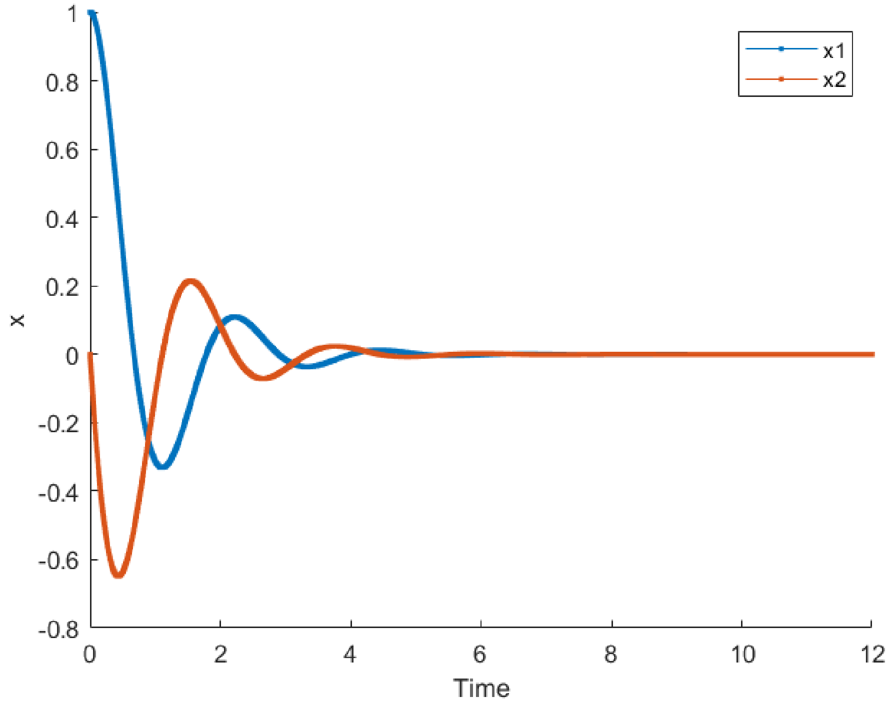


$$d = -6$$

$$\lambda_1 = -3$$
$$\lambda_2 = -3$$

Example of a 2D linear system, **root locus diagram**

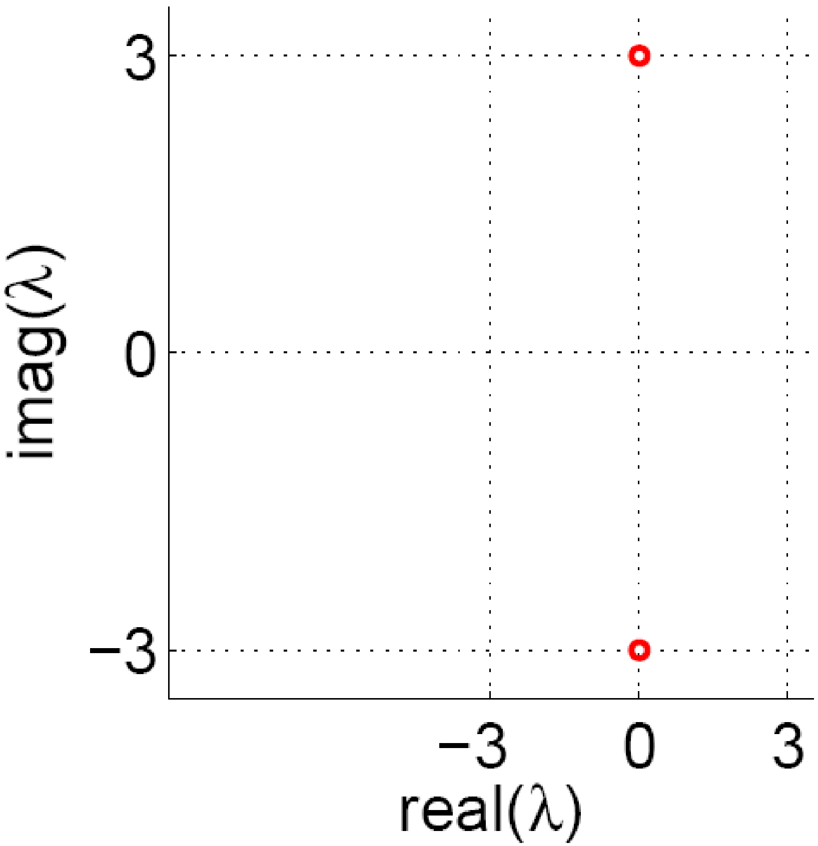


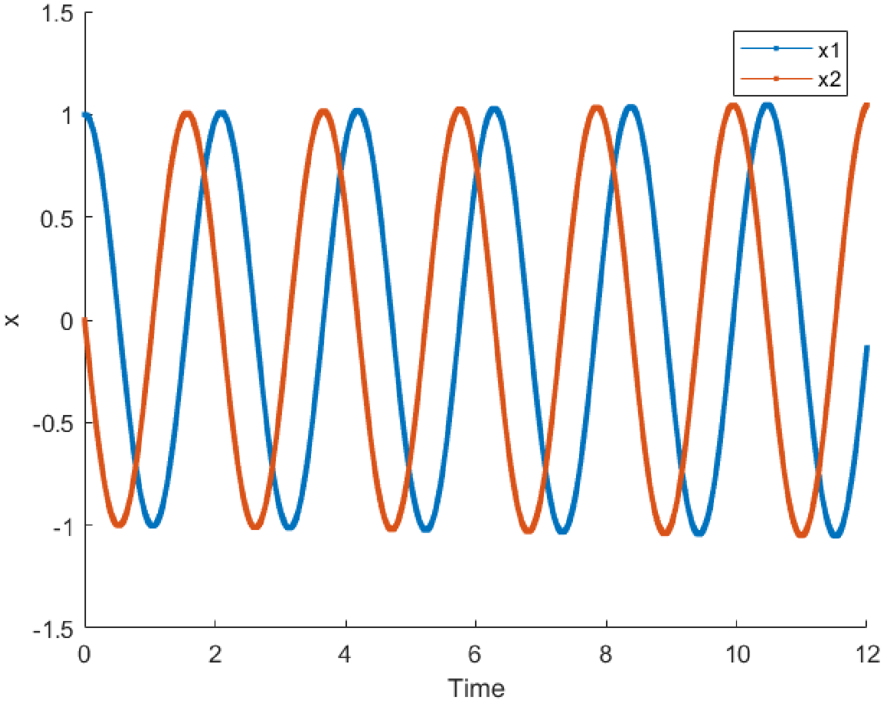


$$d = -2$$

$$\lambda_1 = -1 + 2.83i$$
$$\lambda_2 = -1 - 2.83i$$

Example of a 2D linear system, **root locus diagram**

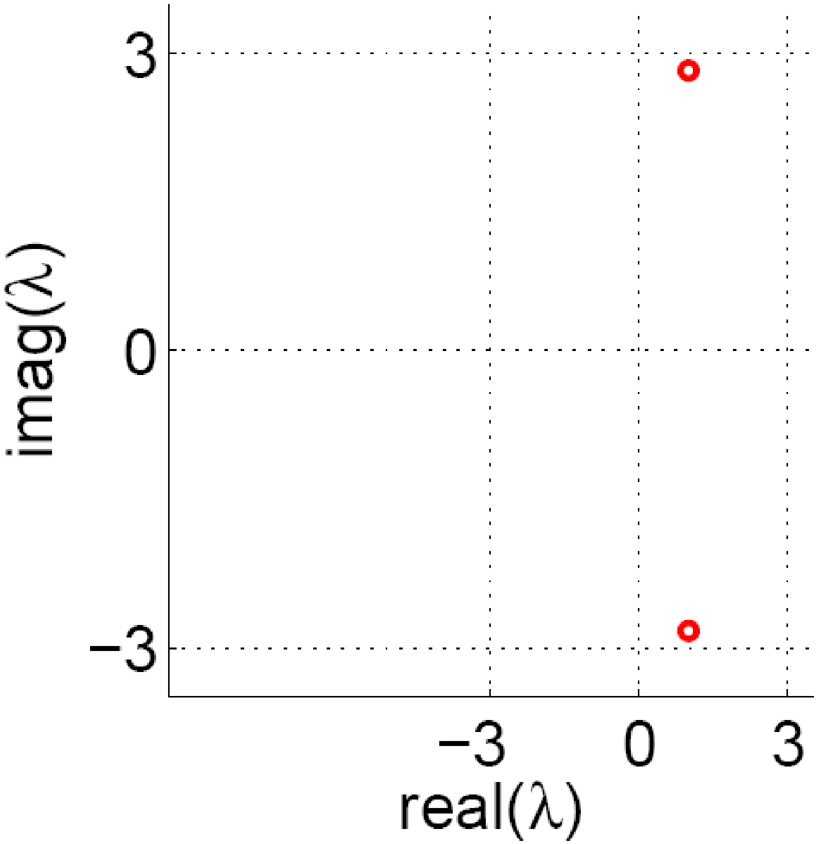


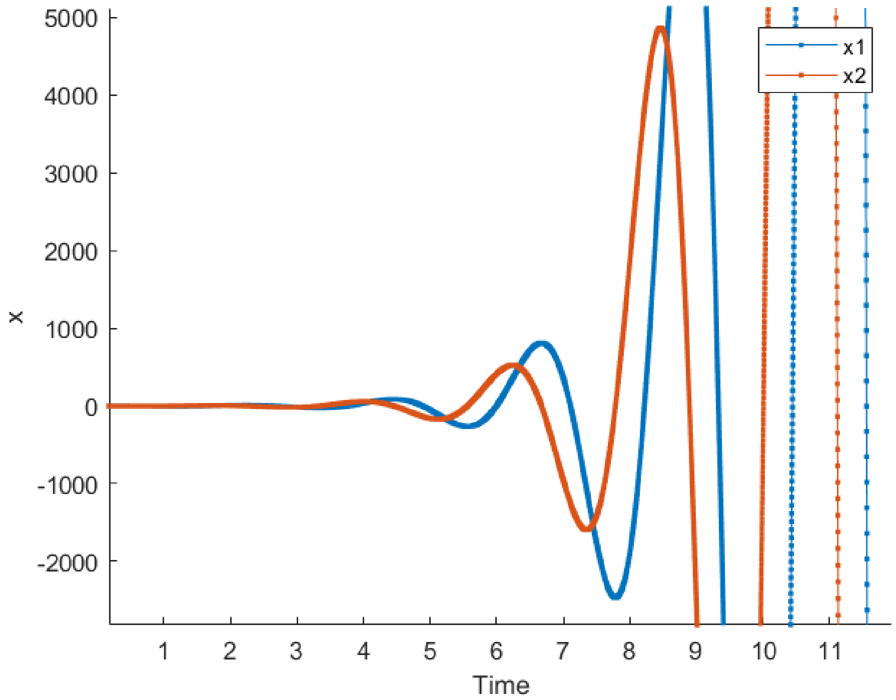


$$d = 0$$

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

Example of a 2D linear system, **root locus diagram**

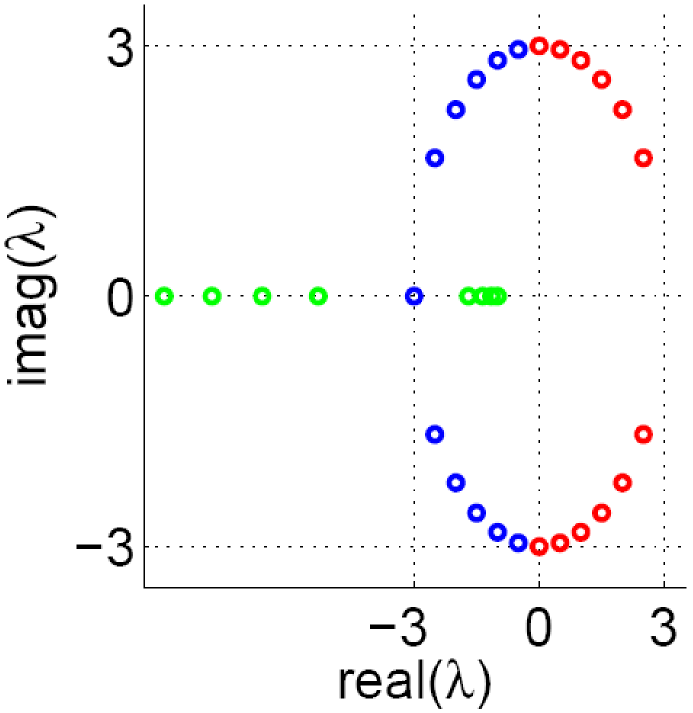




$$d = 2$$

$$\lambda_1 = 1 + 2.83i$$
$$\lambda_2 = 1 - 2.83i$$

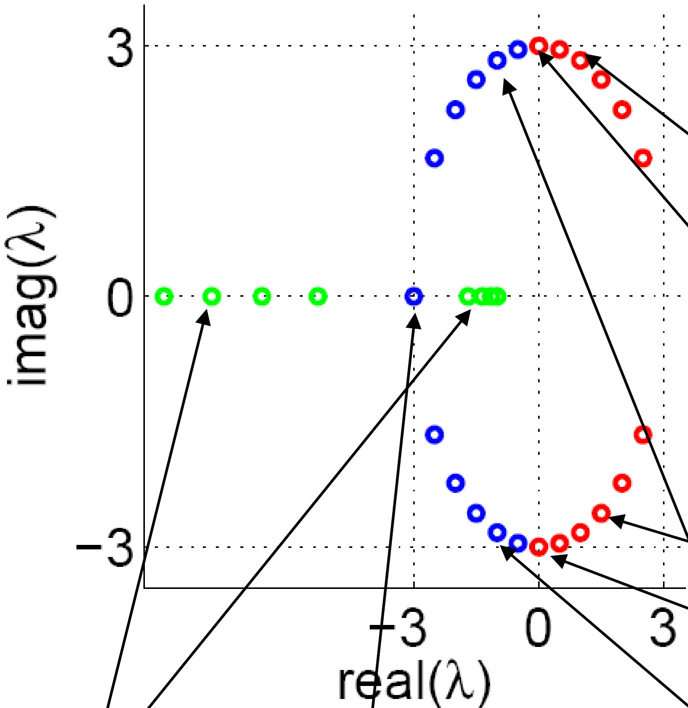
Example of a 2D linear system, **root locus diagram**



In a root locus diagram, the **stability and behavior of the system** is immediately visible from the **position of the eigenvalues.**

As we will see in the next lecture, linearizing around the fixed points of a nonlinear system always leads to a linear system of which the eigen values can be investigated

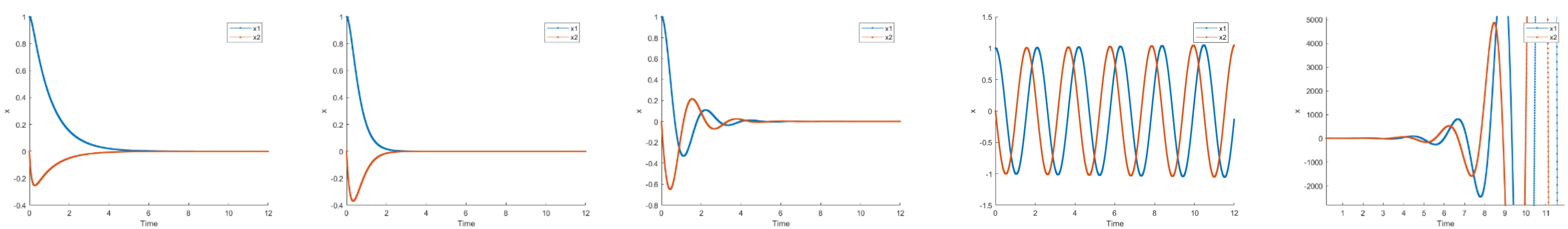
Example of a 2D linear system, **root locus diagram**



In a root locus diagram, the **stability and behavior of the system** is immediately visible from the **position of the eigenvalues.**

As we will see in the next lecture, linearizing around the fixed points of a nonlinear system always leads to a linear system of which the eigen values can be investigated.

$d = -10$ $d = -6$ $d = -2$ $d = 0$ $d = 2$



Example of a 2D linear system, **root locus diagram**

Note this 2D linear system represent **dynamic regimes** that are quite representative of many systems:

Overdamped system, slow convergence

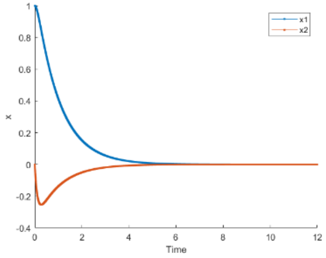
Critically damped system, fastest possible convergence without overshoot

Under damped system, overshoot and oscillations that gradually decrease

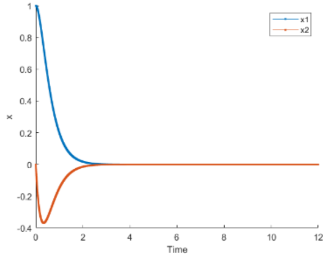
Marginally stable system, oscillations but no limit cycle (cf next lecture)

Unstable system, amplitudes increase exponentially

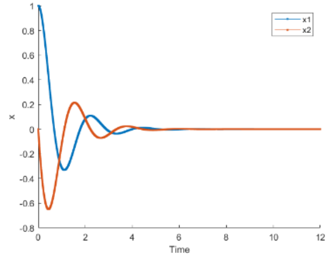
$d = -10$



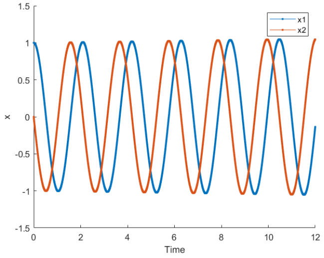
$d = -6$



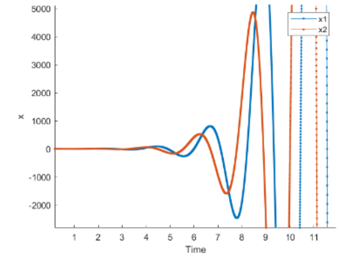
$d = -2$



$d = 0$



$d = 2$



Excellent book on dynamical systems:

Strogatz, S. H. (2019). *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (2nd ed.). CRC Press. https://doi.org/10.1201/9780429492563

PDF provided on Moodle

Possible exam questions

- What should you be carefully with when performing **numerical integration**? What happens if the integration step is too large or too small?

- **Geometrically analyze** the following dynamical system: $\dot{x} = \frac{dx}{dt} = \sin x$

Show the fixed points, determine their stability properties, draw examples of x(t) as accurately as possible (e.g. with inflection points at the right place).

- Discuss different **types of stability**, Lyapunov stable, asymptotically stable, neutrally stable, global versus local stability.

- Discuss the type of **dynamical regimes** a linear dynamical system can have.

- Compute the **fixed points** of the following 2D linear dynamical system, $\dot{x} = Ax$ and analytically **determine their stability** (compute eigenvalues, etc.). Discuss the type of behavior of such a system.

$$A = \begin{pmatrix} 0 & 3 \\ -3 & -1 \end{pmatrix}$$

- **Analyze** the following system: $\dot{x} = x + y \qquad \dot{y} = 4x - 2y$

Compute the stability of its fixed points by computing the eigenvalues. Compute also the **eigenvectors** and **draw typical trajectories in phase space**.

End of Lecture 2

Practical at 13:15

Start thinking about **teams**, but not needed for first labs that are not graded.