Computer Graphics

Rasterization Pipeline

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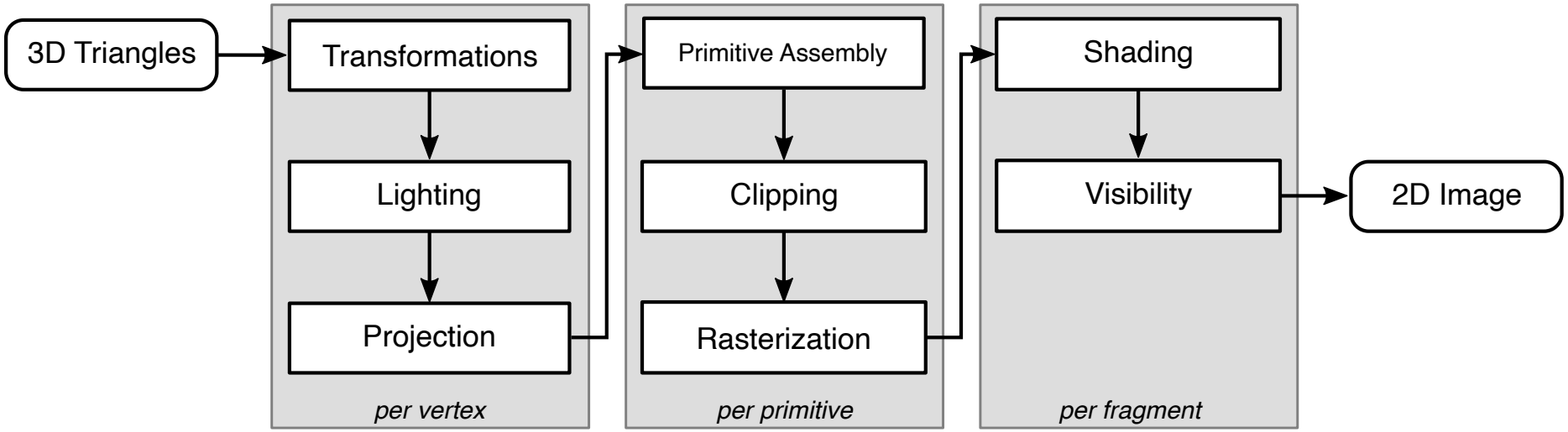
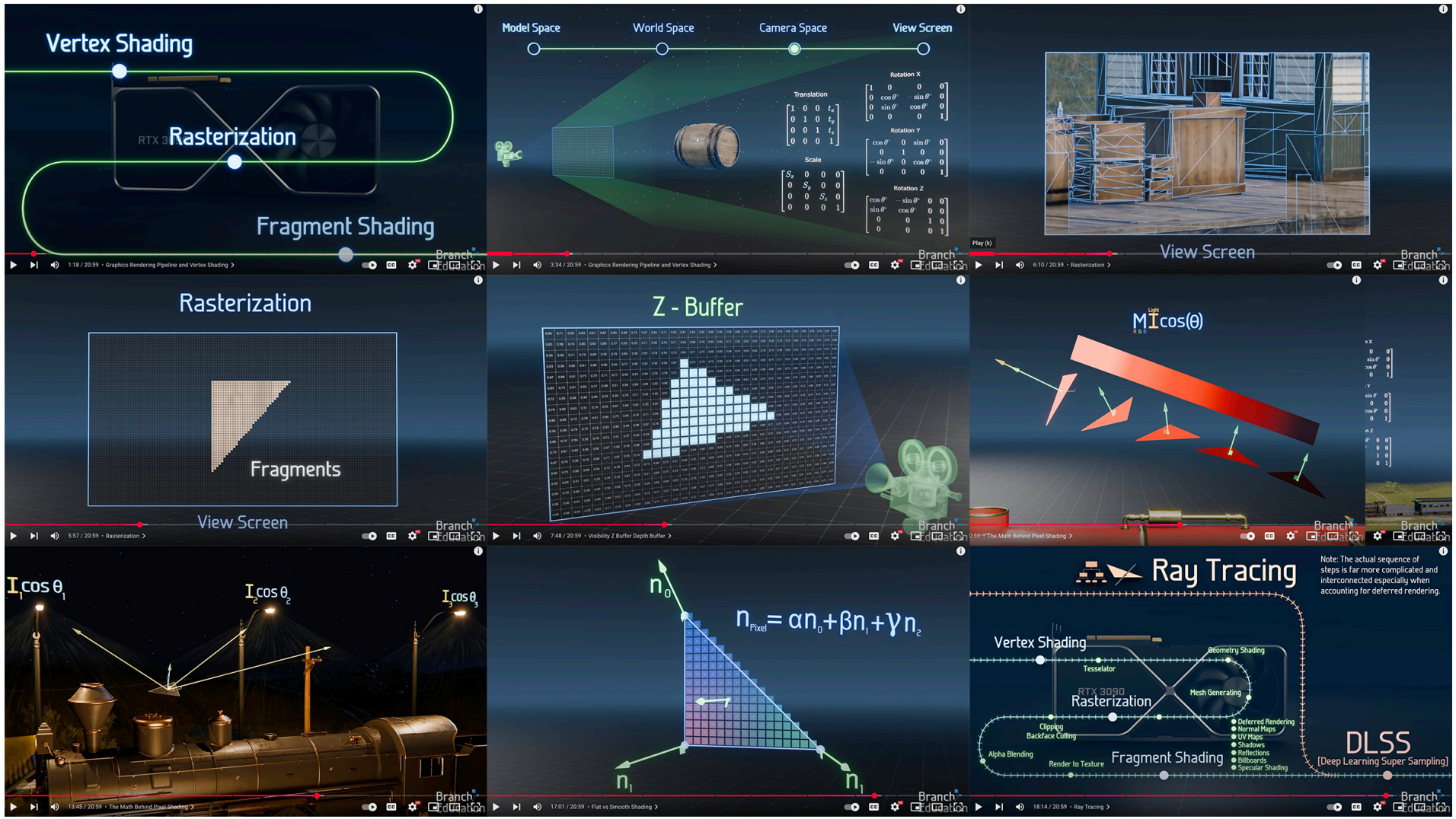
Geometric Computing Laboratory

Rasterization Pipeline

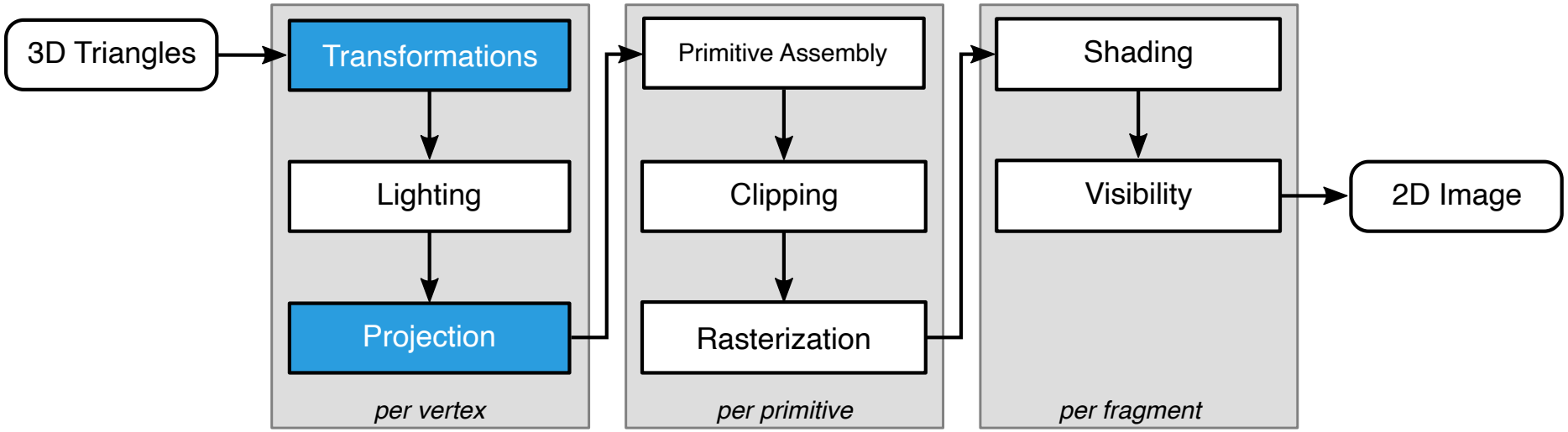


Rasterization Pipeline



Thanks to Amine and Haythem! https://www.youtube.com/watch?v=C8YtdC8mxTU

Transformations & Projections



Transformations & Projections

- Project edge or triangle from 3D to 2D
  - Simply transform/project its endpoints/vertices?
- Let us first analyze this for a line between $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{X}(\alpha) = (1-\alpha)\mathbf{A} + \alpha\mathbf{B}\,, \quad \alpha \in [0,1]$$

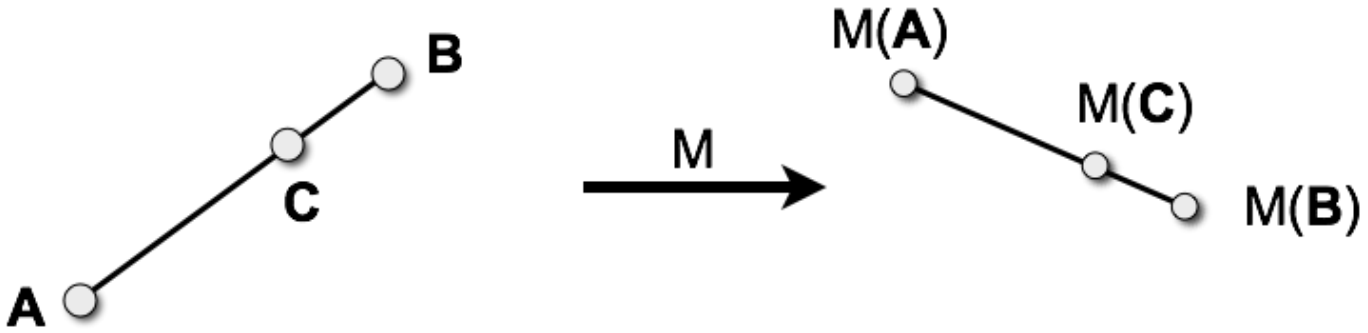
- For an affine/projective transformation $\mathbf{M}$ it has to hold

$$\forall \alpha \in \mathbb{R} \;\; \exists \beta \in \mathbb{R} \;:\; \mathbf{M}(\mathbf{X}(\alpha)) \;=\; (1-\beta)\mathbf{M}(\mathbf{A}) + \beta\mathbf{M}(\mathbf{B})$$

Affine Transformations of Lines

- Affine transformation $\mathbf{M}$ preserves affine combinations

$$\mathbf{M}((1-\alpha)\mathbf{A} + \alpha\mathbf{B}) \;=\; (1-\alpha)\mathbf{M}(\mathbf{A}) + \alpha\mathbf{M}(\mathbf{B})$$



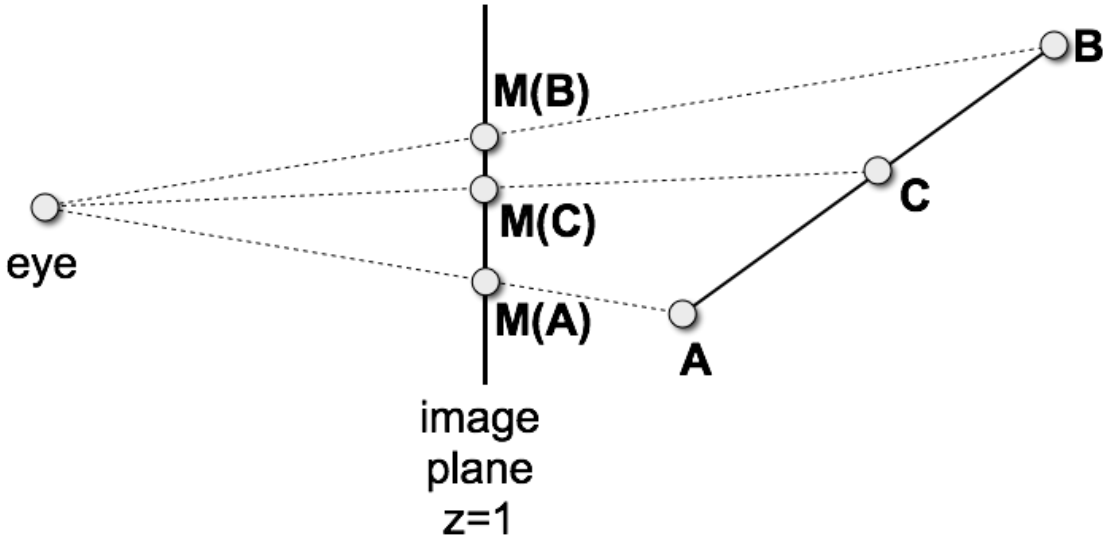
Projective Transformations of Lines

- For a central projection $\mathbf{P}(\mathbf{X}) = \frac{\mathbf{X}}{\mathbf{X}_z}$ we can show that

$$\mathbf{P}(\mathbf{X}(\alpha)) = \frac{(1-\alpha)\mathbf{A} + \alpha\mathbf{B}}{(1-\alpha)\mathbf{A}_z + \alpha\mathbf{B}_z} = (1-\beta)\frac{\mathbf{A}}{\mathbf{A}_z} + \beta\frac{\mathbf{B}}{\mathbf{B}_z}$$

$$\text{if we chose } \beta = \frac{\alpha\mathbf{B}_z}{(1-\alpha)\mathbf{A}_z + \alpha\mathbf{B}_z}$$

- Same argument holds for general projective transformations.



Transformations of Triangles

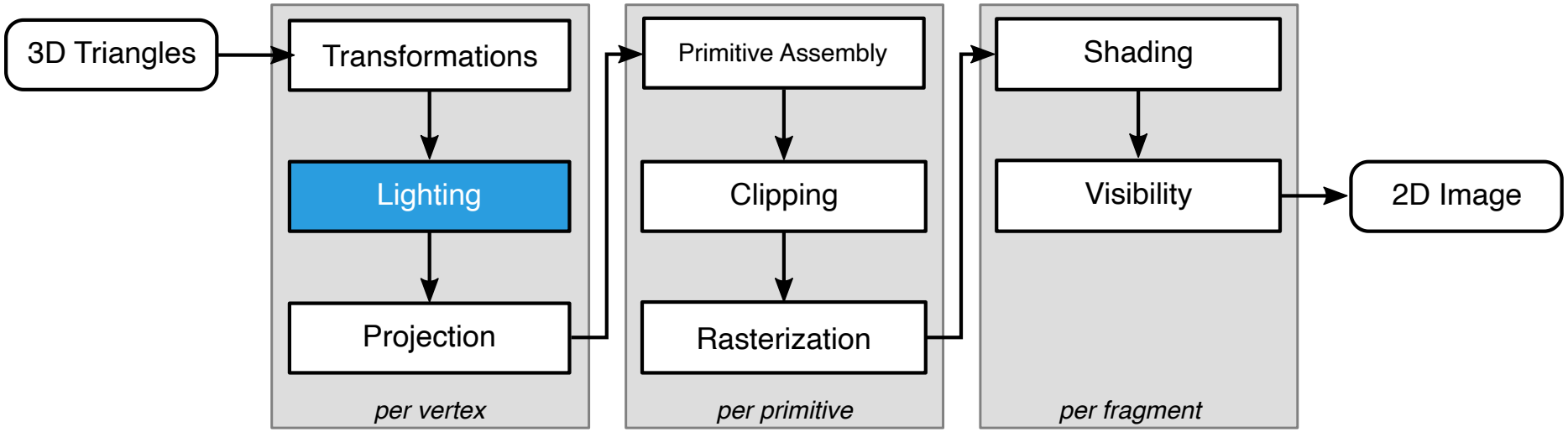
- A triangle is an affine combination of three points

$$\mathbf{X}(\alpha, \beta, \gamma) \;=\; \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

  with $\alpha + \beta + \gamma = 1$.

- Affine transformations preserve affine combinations
  - Triangles are transformed to triangles
- Planar projections map triangles to triangles
  - Similar derivation as for lines

Lighting


3D Triangles
Transformations
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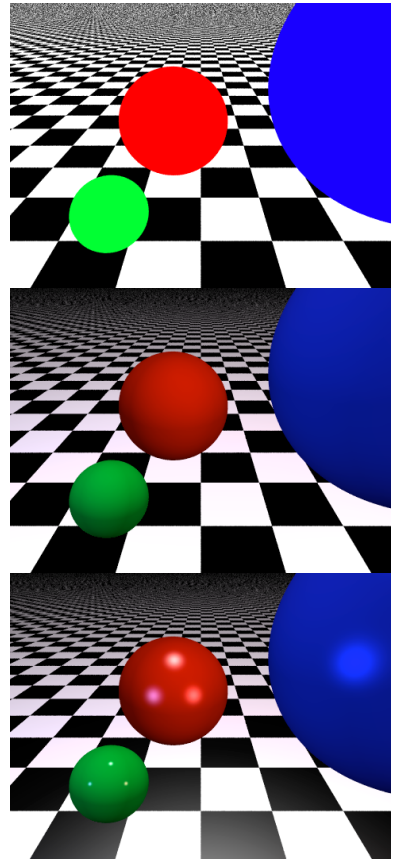
per vertex
Primitive Assembly
Clipping
Rasterization
per primitive
Shading
Visibility
per fragment
2D Image

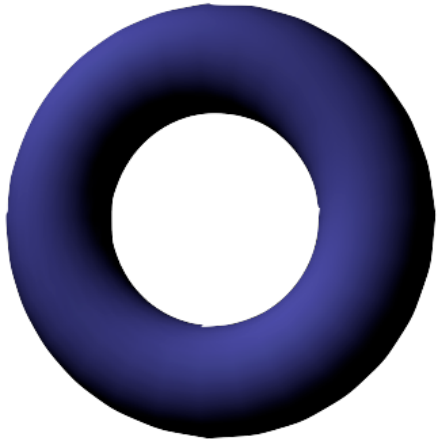
Phong Lighting Model

- **Ambient lighting**
  - approximate global light transport / exchange
  - uniform in, uniform out
- **Diffuse lighting**
  - dull / mat surfaces
  - directed in, uniform out
- **Specular lighting**
  - shiny surfaces
  - directed in, directed out

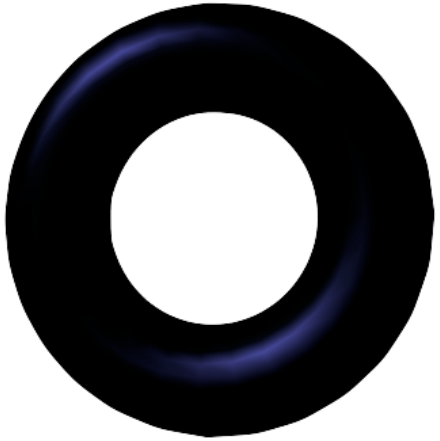


Phong Lighting Model

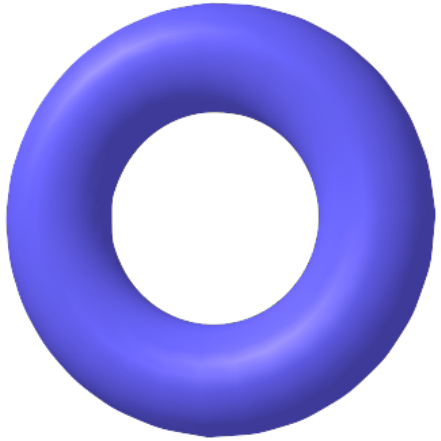
ambient: $I_a m_a$



diffuse: $I_l m_d \, (\mathbf{n} \cdot \mathbf{l})$



specular: $I_l m_s (\mathbf{r} \cdot \mathbf{v})^s$

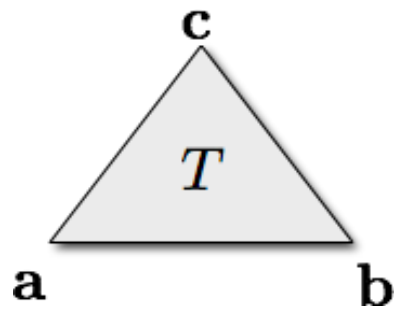


ambient+diffuse+specular

Normal Vectors

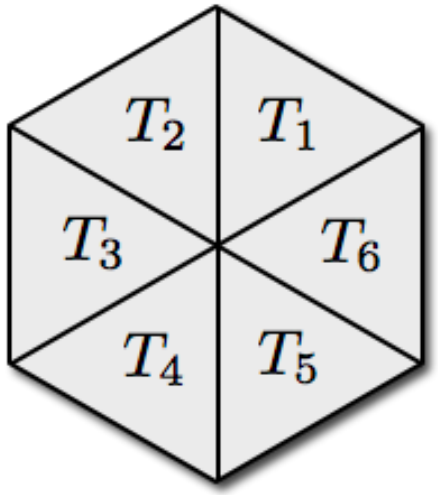
- Triangle normal

$$\mathbf{n}(T) = \frac{(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})}{\|(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})\|}$$



- Vertex normal
  - Average of incident triangles' normals $\mathbf{n}(T_i)$
  - Weighted by area or opening angle $w(T_i)$

$$\mathbf{n}(V) = \frac{\sum_{T_i \ni V} w(T_i)\,\mathbf{n}(T_i)}{\left\|\sum_{T_i \ni V} w(T_i)\,\mathbf{n}(T_i)\right\|}$$



How to transform normal vectors?

- Point $\mathbf{x} = (x, y, z, 1)^\mathsf{T}$ with normal $\mathbf{n} = (n_x, n_y, n_z, 0)^\mathsf{T}$ lies on its tangent plane

$$n_x x + n_y y + n_z z + d = 0 \quad \Leftrightarrow \quad \mathbf{p}^\mathsf{T}\mathbf{x} = 0$$

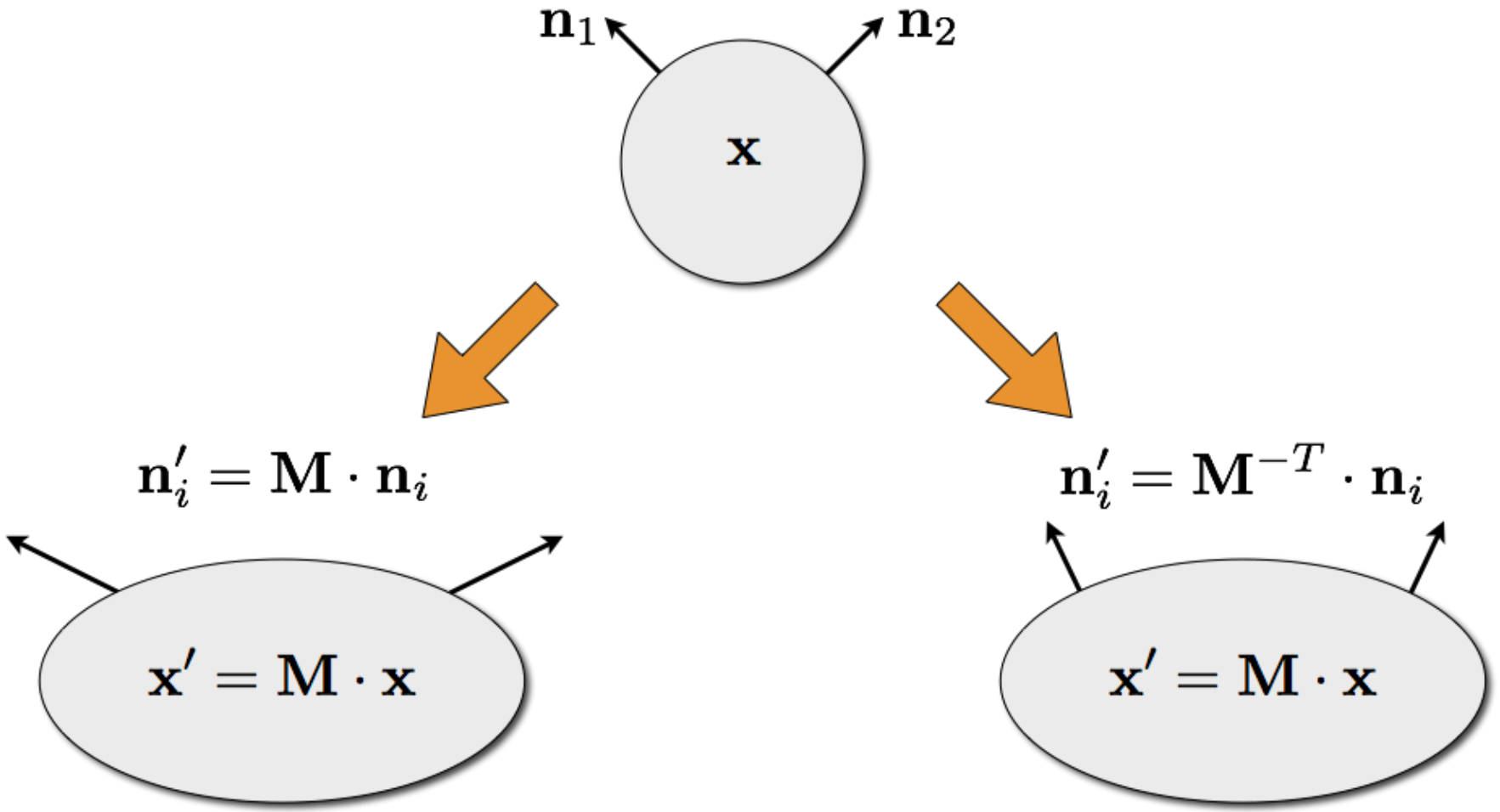
  with $d = \mathbf{n}^\mathsf{T}\mathbf{x}$ and $\mathbf{p} = (n_x, n_y, n_z, d)^\mathsf{T}$.

- The same equation should be satisfied after an affine transformation, represented by 4×4 matrix $\mathbf{M}$. Transformation maps $\mathbf{x}$ to $\mathbf{x}'$, $\mathbf{n}$ to $\mathbf{n}'$, and $\mathbf{p}$ to $\mathbf{p}'$:

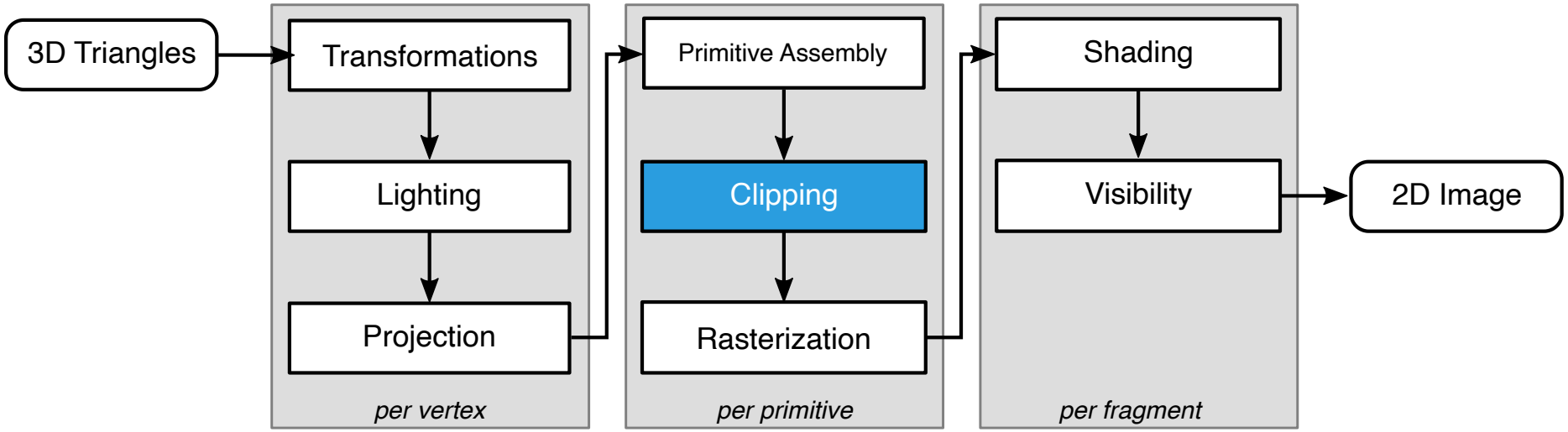
$$0 \;=\; \mathbf{p}'^\mathsf{T}\mathbf{x}' \;=\; \mathbf{p}'^\mathsf{T}\left(\mathbf{M}\mathbf{x}\right) \;=\; \left(\mathbf{M}^\mathsf{T}\mathbf{p}'\right)^\mathsf{T}\mathbf{x}$$

- Comparing the two equations yields $\mathbf{p}' = \mathbf{M}^{-\mathsf{T}}\mathbf{p}$, and therefore $\mathbf{n}' = \mathbf{M}^{-\mathsf{T}}\mathbf{n}$
  - In practice use the upper-left $\mathbf{3} \times \mathbf{3}$ block of $\mathbf{M}$ and build its transposed inverse
  - Then transform the 3D normal vector and re-normalize it

How to transform normal vectors?



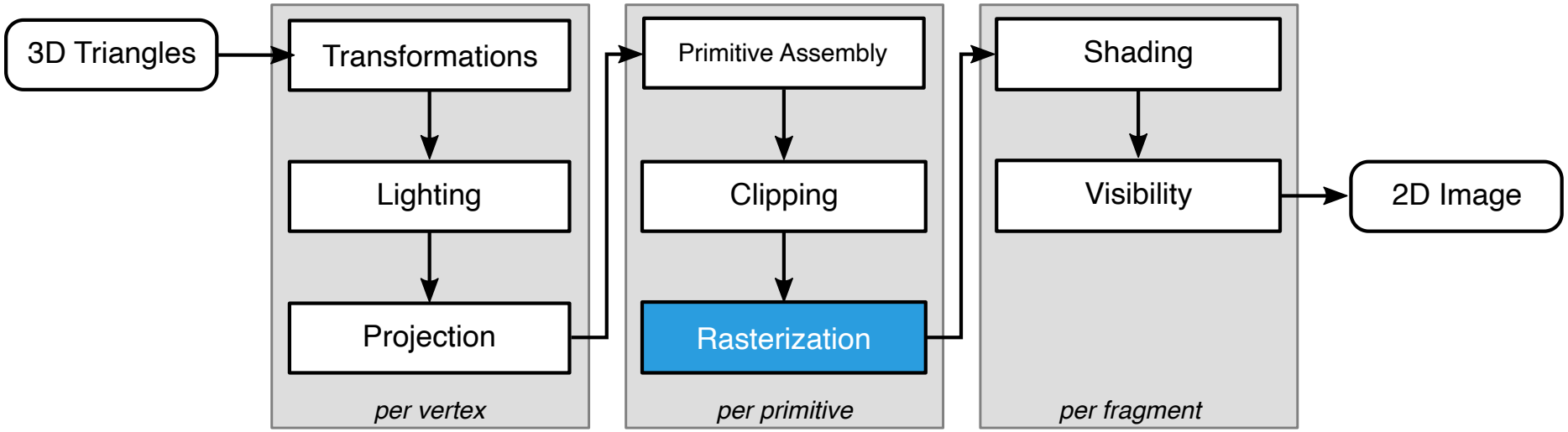
Clipping



Clipping removes everything outside the viewing frustum.

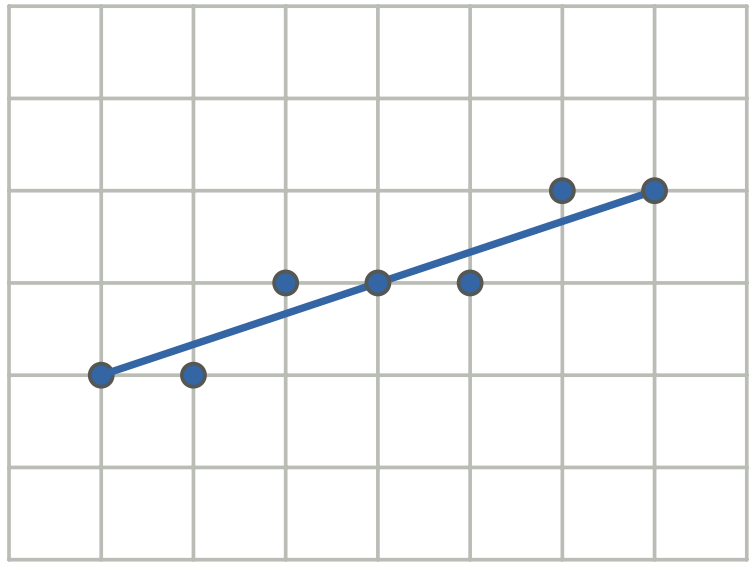
We don't discuss clipping here.

Rasterization



Line Rasterization

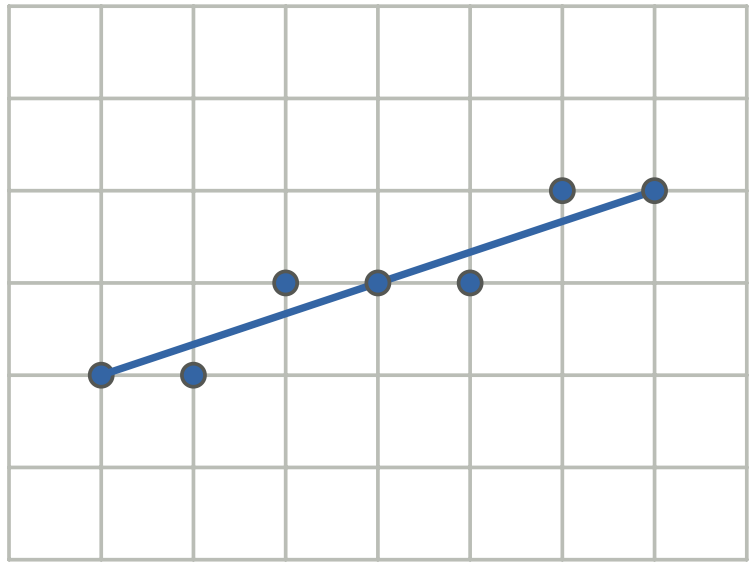
- Discretize line from (x_0, y_0) to (x_1, y_1) to pixel grid
  - Endpoints are integer coordinates (pixel positions)
  - Assume slope is in $[0, 1]$ and $x_0 < x_1$
  - Other cases follow by symmetry



How would you implement this?

Line Representation

- **Explicit:** Range of a function $y(x) = mx + t$ with
  - $m = \frac{y_1 - y_0}{x_1 - x_0} =: \frac{\Delta y}{\Delta x}$
  - $t = y_0 - m\,x_0$



First Guess

```
for (x=x0; x<=x1; ++x)
        set_pixel(x, round(m*x + t));
```

- ⊖ Multiplication
- ⊖ Rounding

Digital Differential Analysis

- Use an incremental algorithm!
- If the current pixel is (x_i, y_i) then
  - $x_{i+1} = x_i + 1$
  - $y_{i+1} = y_i + m$

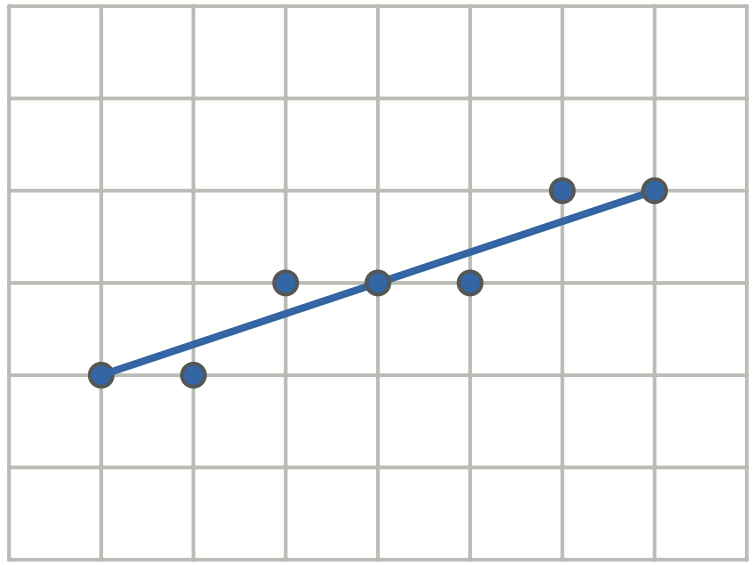
```
for (x=x0, y=y0; x<=x1; ++x, y+=m)
  set_pixel(x, round(y));
```

⊕ no multiplication

⊖ rounding, since y and m are floats

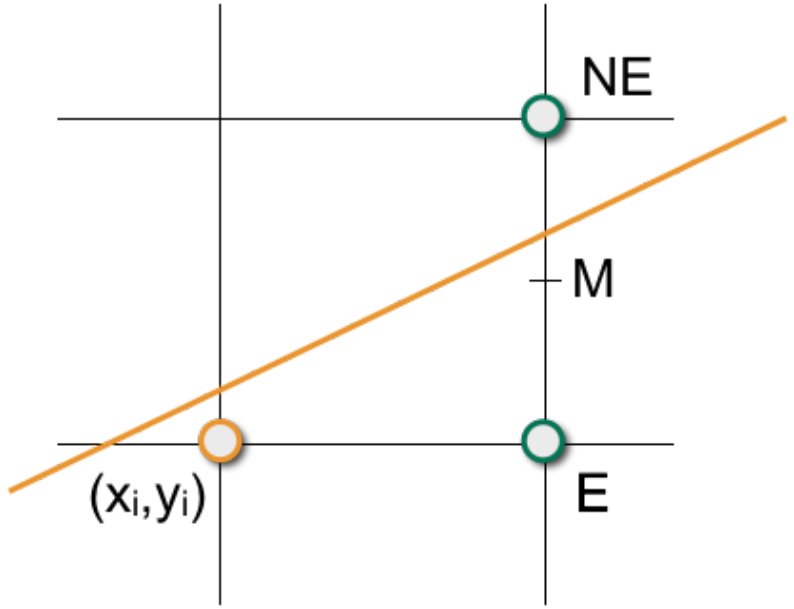
Line Representations

- **Explicit:** Range of a function $y(x) = mx + t$ with
  - $m = \frac{y_1 - y_0}{x_1 - x_0} =: \frac{\Delta y}{\Delta x}$
  - $t = y_0 - m\,x_0$
- **Implicit:** Zero-set of $F(x, y) = ax + by + c$ with
  - $a = (y_1 - y_0) = \Delta y$
  - $b = (x_0 - x_1) = -\Delta x$
  - $c = t\,(x_1 - x_0)$
  - $F(x, y) = 0 \;\Rightarrow$ point on line
  - $F(x, y) > 0 \;\Rightarrow$ point below line
  - $F(x, y) < 0 \;\Rightarrow$ point above line



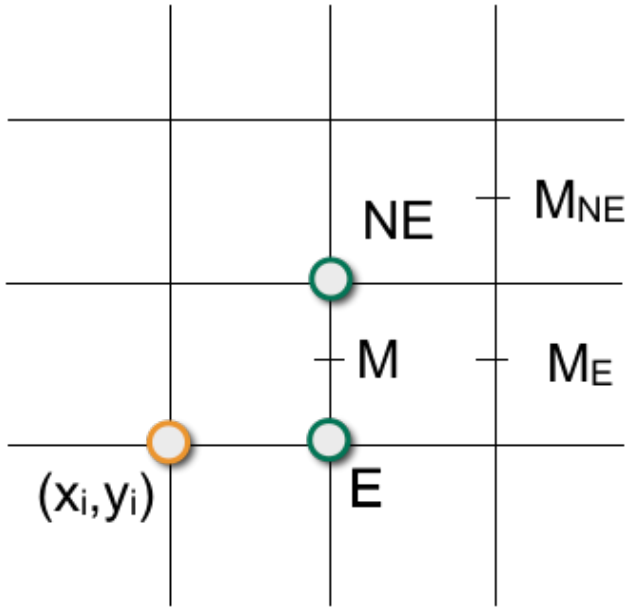
Bresenham Algorithm

- Next pixel can only be east (E) or north-east (NE)
  - Is the line below/above midpoint $\mathbf{M}$?
- Use a *decision variable* d:
  - $d = F(M) = F(x_i + 1, y_i + 0.5)$
  - If $d \leq 0$ then go east
  - If $d > 0$ then go north-east



Bresenham Algorithm

- Incrementally update decision variable d!
- If east was chosen:
  - $d_{\text{new}} = F(x_i + 2, y_i + 0.5) = d_{\text{old}} + a$
  - $\Delta\text{E} := a = \Delta y$
- If north-east was chosen:
  - $d_{\text{new}} = F(x_i + 2, y_i + 1.5) = d_{\text{old}} + a + b$
  - $\Delta\text{NE} := a + b = \Delta y - \Delta x$



Bresenham Algorithm

- Initialization of d
  - $d = F(x_0 + 1, y_0 + 0.5) = \Delta y - \Delta x/2$
  - Bad: $\Delta x/2$ may not be integer
- Use $2 \cdot F(x, y)$ instead of $F(x, y)$
  - Values of d, $\Delta\mathrm{E}$, $\Delta\mathrm{NE}$ are doubled
  - But sign of F and d remain unchanged

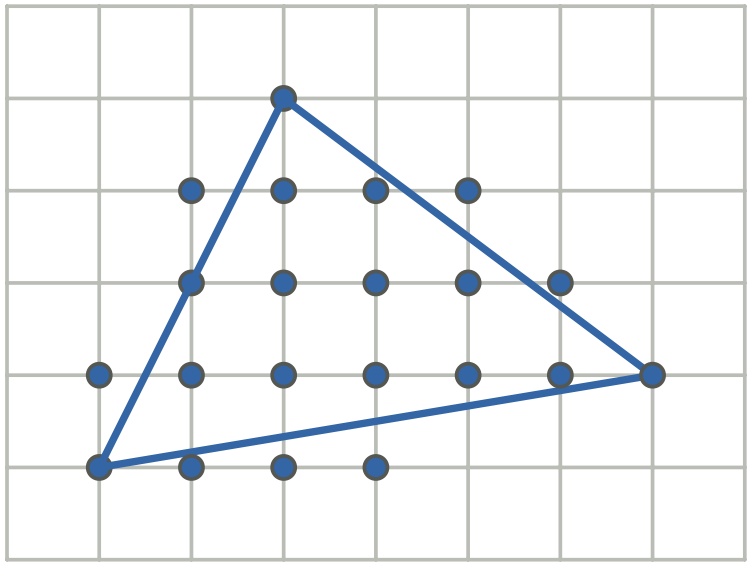
Bresenham Algorithm

```
Δx  = x1-x0;
Δy  = y1-y0;
d   = 2*Δy - Δx;
ΔE  = 2*Δy;
ΔNE = 2*(Δy - Δx);

set_pixel(x0, y0);

for (x = x0, y = y0; x <= x1;)
{
    if (d <= 0) { d += ΔE;  ++x;      }
    else        { d += ΔNE; ++x; ++y; }
    set_pixel(x, y);
}
```

⊕ Only integer arithmetic

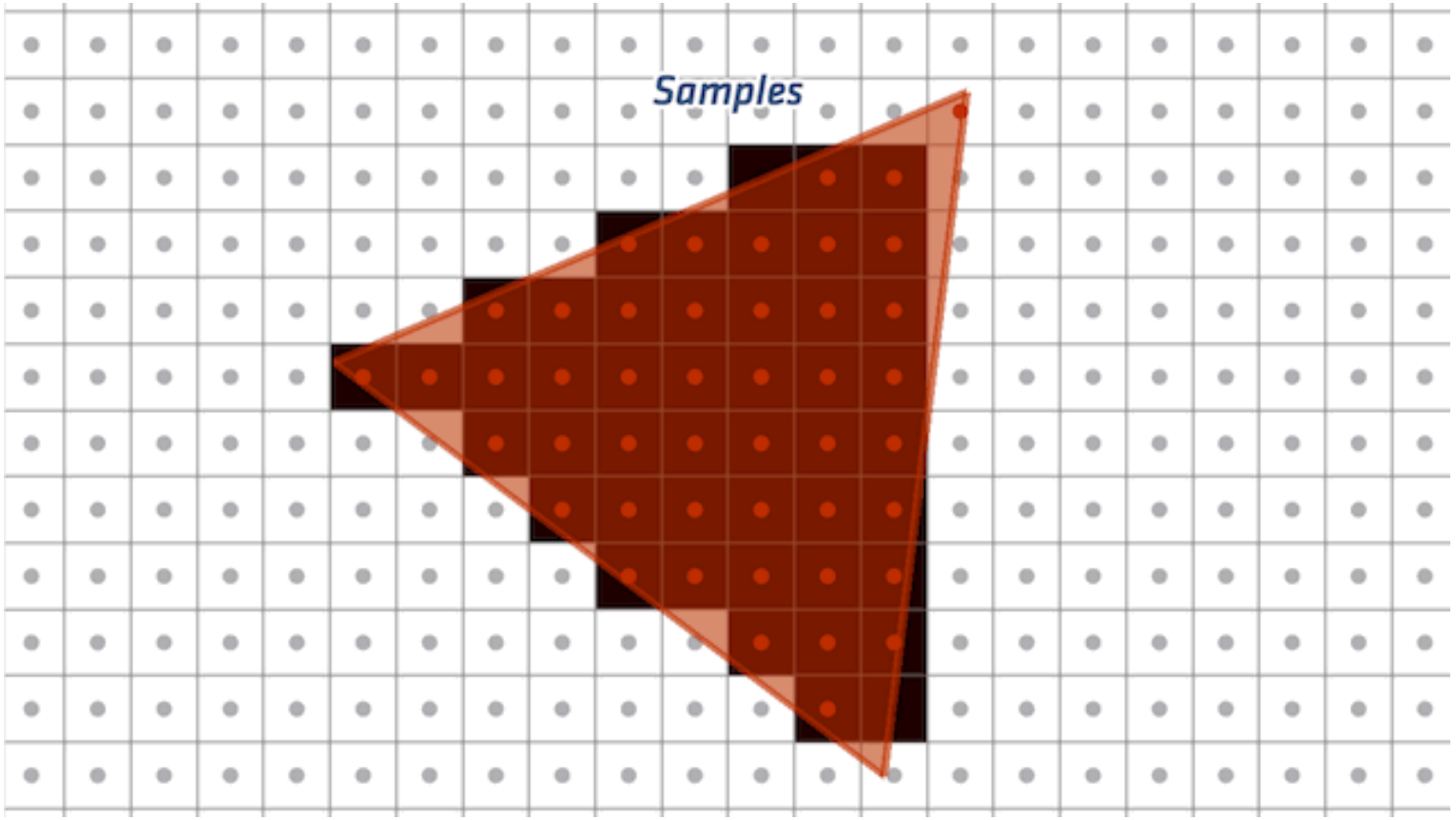
Triangle Rasterization

- Enumerate all pixels inside a 2D triangle
  - Inside test for every pixel is too expensive
- Compute horizontal spans in each scanline
  - Compute the intersections with triangle edges, fill all pixels inbetween

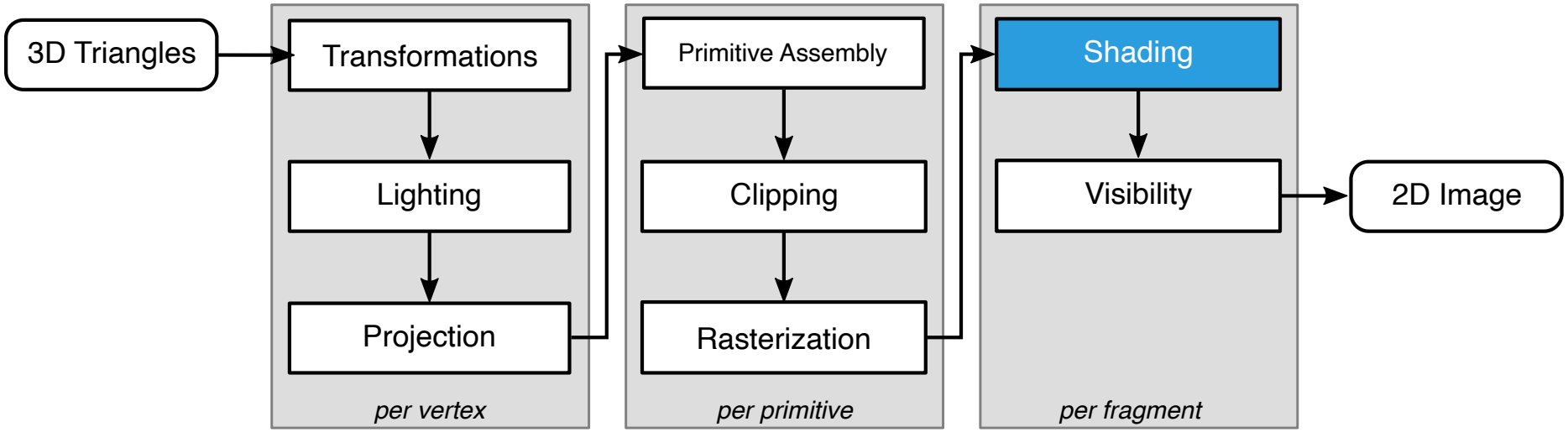


Many special cases to consider!

Triangle Rasterization



Shading


3D Triangles
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per vertex
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per fragment
2D Image

Shading: Fill the Interior

- **Flat Shading**
  - Compute lighting per face
  - Facetted appearance

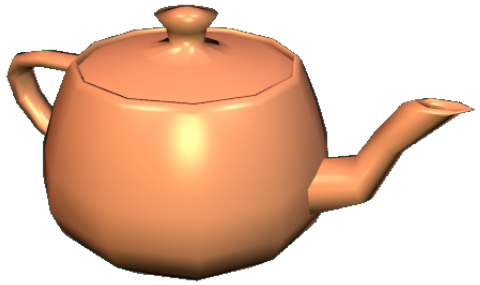
Mach band effect

Shading: Fill the Interior

- **Flat Shading**
  - Compute lighting per face
  - Facetted appearance

- **Gouraud Shading**
  - Compute lighting per vertex
  - Linear interpolation of colors
  - Might loose small highlights

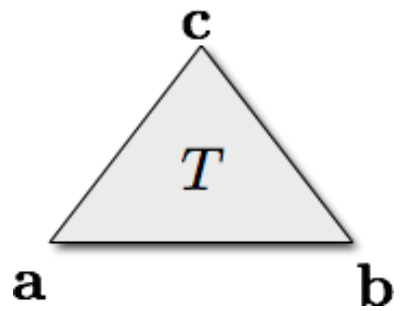
- **Phong Shading**
  - Linear interpolation of vertex normals
  - Compute lighting per pixel
  - Captures small highlights



Normal Vectors

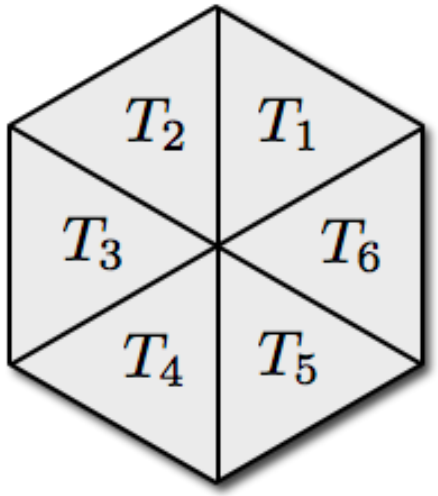
- Triangle normal

$$\mathbf{n}(T) = \frac{(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})}{\|(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})\|}$$



- Vertex normal
  - Average of incident triangles' normals $\mathbf{n}(T_i)$
  - Weighted by area or opening angle $w(T_i)$

$$\mathbf{n}(V) = \frac{\sum_{T_i \ni V} w(T_i)\,\mathbf{n}(T_i)}{\left\|\sum_{T_i \ni V} w(T_i)\,\mathbf{n}(T_i)\right\|}$$



Interpolate Vertex Normals

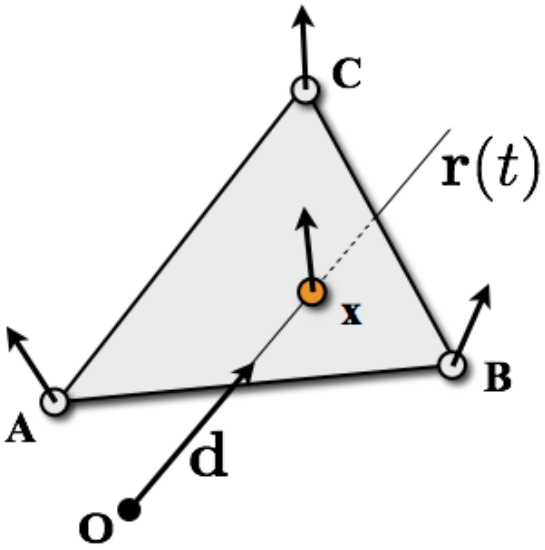
- Intersection point with barycentric coordinates

$$\mathbf{x} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

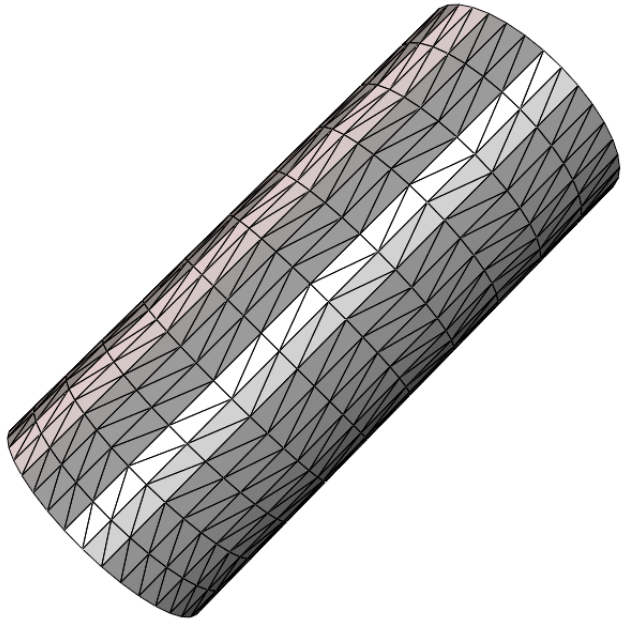
- Linearly interpolate vertex normals

$$\mathbf{n}(\mathbf{x}) = \alpha\mathbf{n}(\mathbf{A}) + \beta\mathbf{n}(\mathbf{B}) + \gamma\mathbf{n}(\mathbf{C})$$

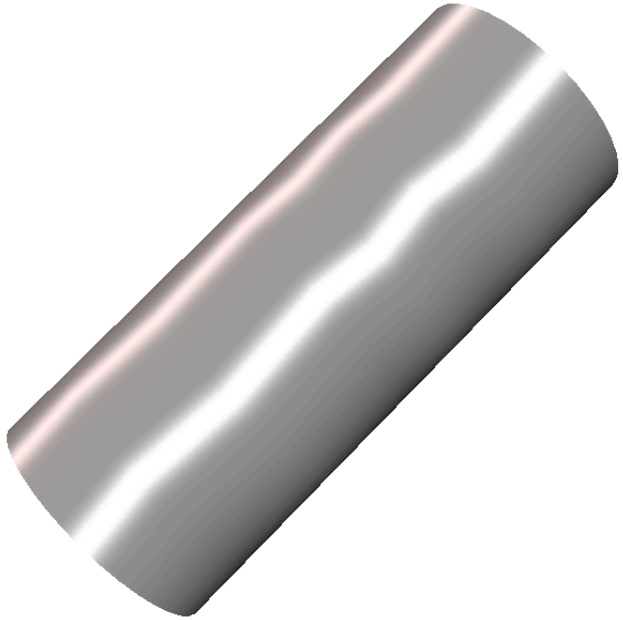
- Use $\mathbf{n}(\mathbf{x})$ to light point $\mathbf{x}$



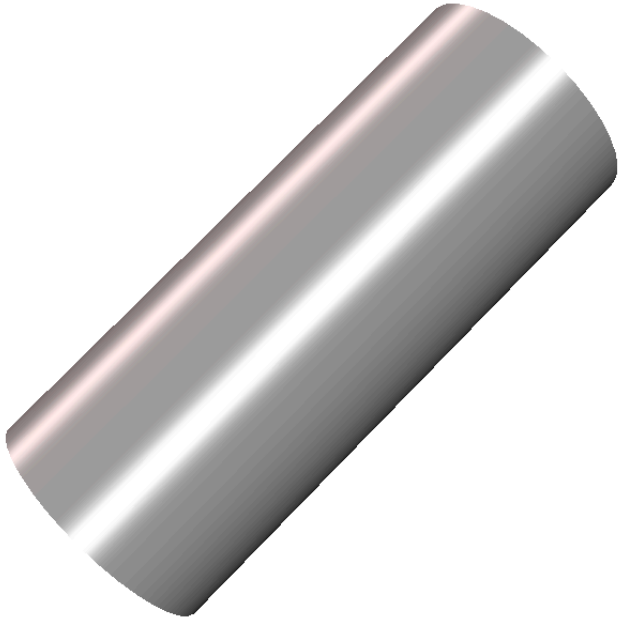
Influence of Vertex Normals



triangulation (flat shading)



no weighting



angle-weighted

Shading: Fill the Interior

Flat Shading

Gouraud Shading

Phong Shading

Shading: Fill the Interior

Quiz: Barycentric Interpolation

During rasterization normal vectors and texture coordinates for each pixel are computed by barycentric interpolation.

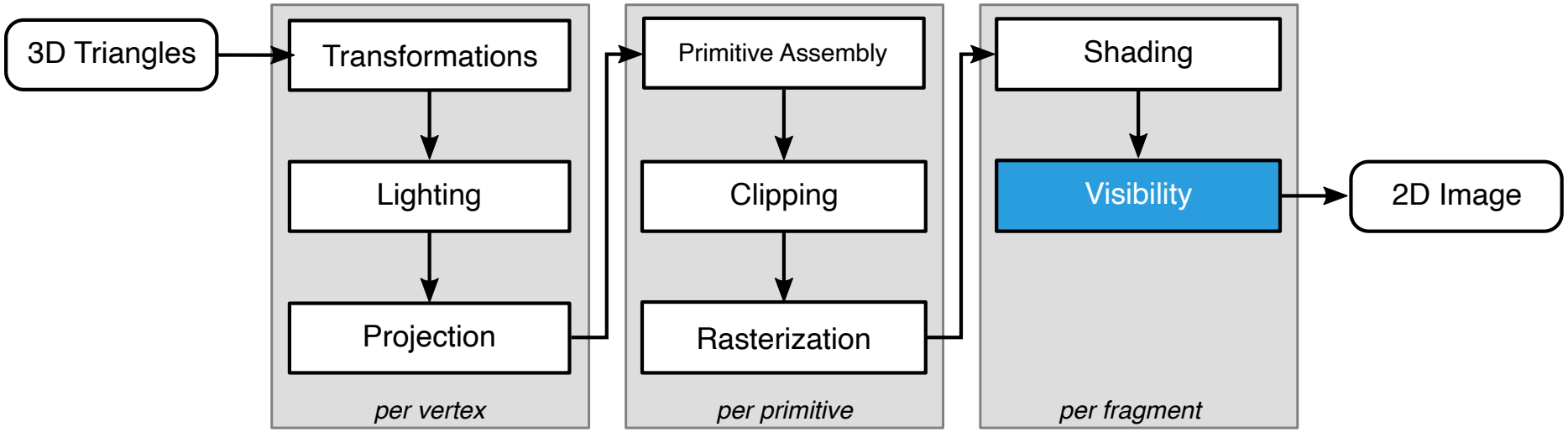
Where should we compute the barycentric coordinates of the pixels?

A: In 2D window coordinates

B: In 3D camera coordinates

C: Makes no difference

Visibility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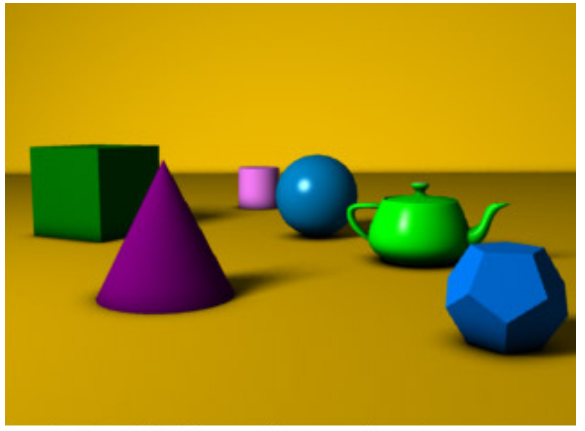
2D Image

Z-Buffer

- Store current minimum z-value for each pixel
  - After model transformation, view transformation, projection transformation, and viewport transformation
- Additional buffer for depth values
  - Framebuffer stores RBG color values
  - Depth buffer (z-buffer) stores depth values
  - Storage: additional 16 to 32 bits per pixel



Images from Wikipedia

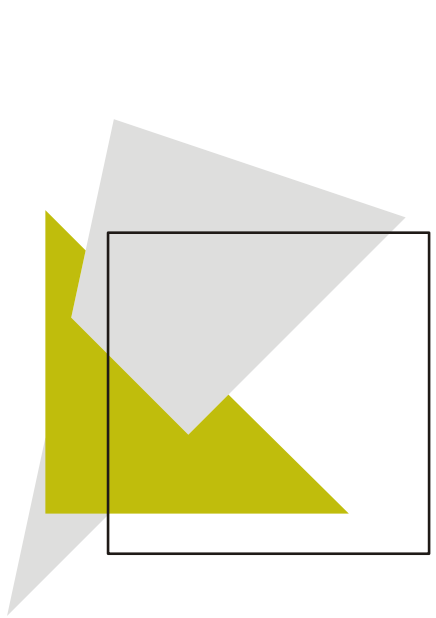
Z-Buffer

- Initialize depth buffer to ∞
- Rasterize triangles, interpolate depth value per fragment
  - each fragment (x,y) still has a depth value z
- Depth buffer test & update

```
// draw color rgb to pixel (x,y)
void set_pixel(x, y, z, rgb)
{
    // is current pixel closer than stored z-value?
    if (z < zbuffer[x,y])
    {
        // write pixel's color to frame buffer
        framebuffer[x,y] = rgb;

        // update depth buffer
        zbuffer[x,y] = z;
    }
}
```

Z-Buffer



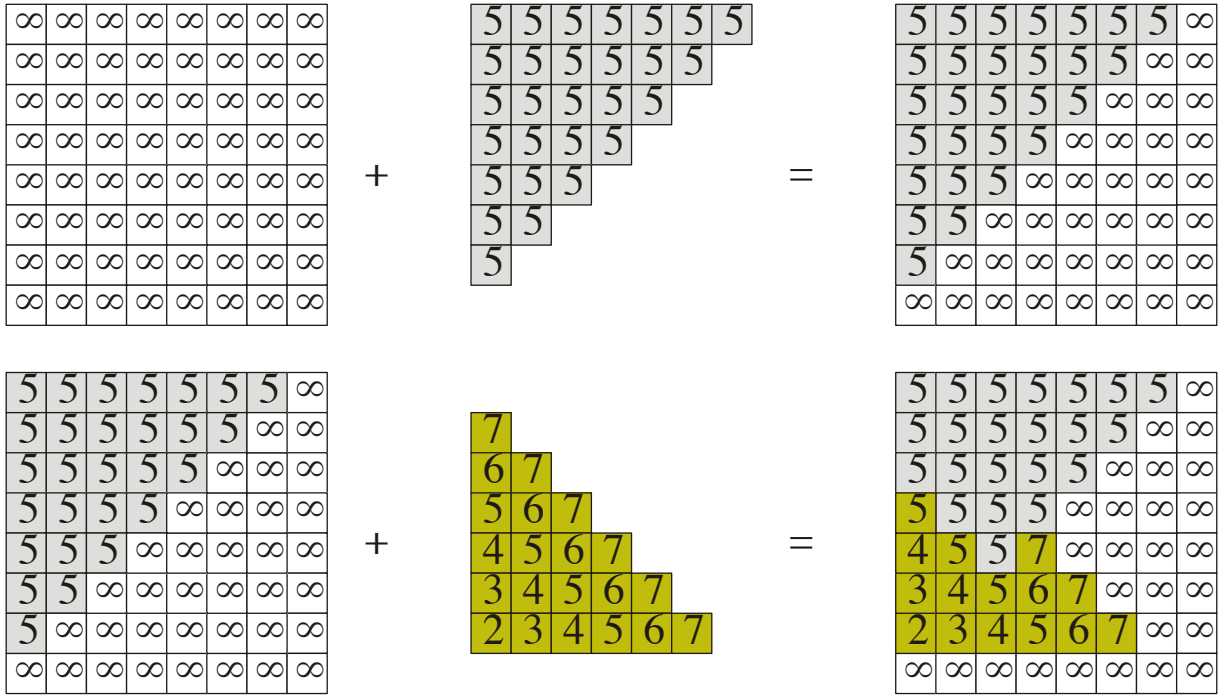


Image from Wikipedia

Quiz: Z-Buffer

- How will the z-buffer resolve visibility when two triangles A and B lie in the same plane and overlap? Assume triangle A is drawn before triangle B.

A: A will be completely visible.

B: B will be completely visible.

C: Some pixels from A and some pixels from B will be visible in the overlap region.

D: One triangle is completely visible, but it is random which one.

Precision Issues

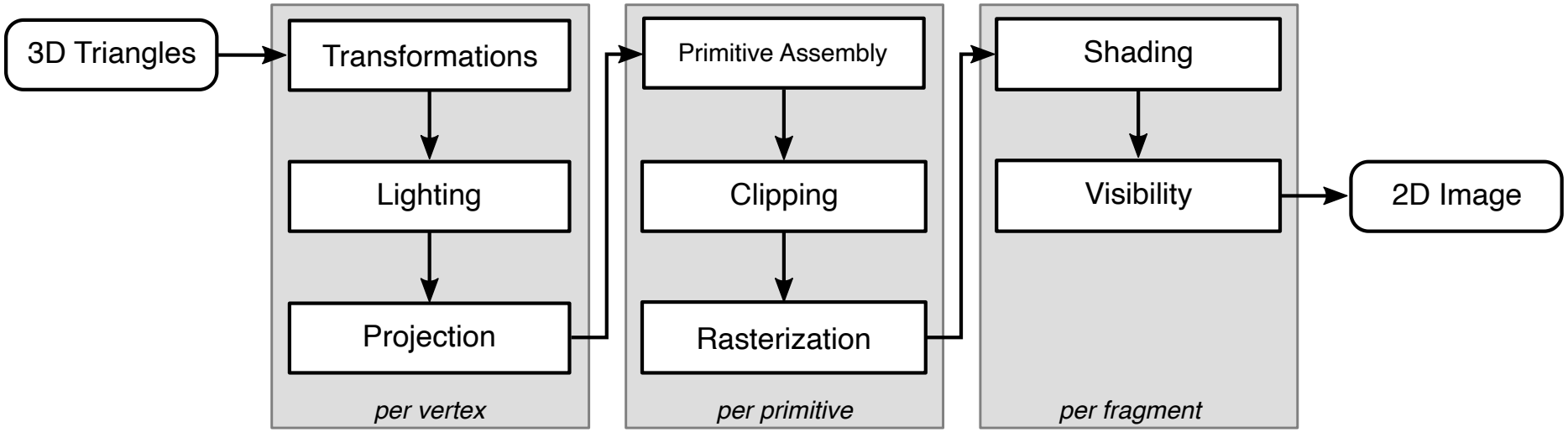
Logarithmic Z-Buffer

Quiz: Transform Normal Vectors

- If an object is affinely transformed by a matrix $\mathbf{M}$, which matrix is used to transform its normal vectors?

A: $\mathbf{M}$ B: $\mathbf{M}^{-1}$ C: $\mathbf{M}^{\mathsf{T}}$ D: $\mathbf{M}^{-\mathsf{T}}$

Rasterization Pipeline



Literature

- Marschner & Shirley: *Fundamentals of Computer Graphics*, 5th Edition, AK Peters, 2021.
  - Chapter 9