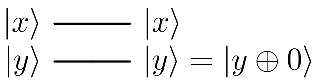
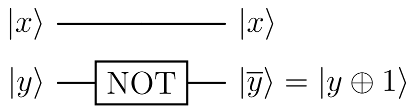
Exercise 1 *Deutsch's algorithm*

(a) The 4 oracle gates U_f are given respectively by:

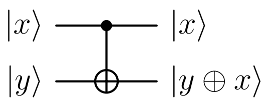
(1) For $f_1(x) = 0$:



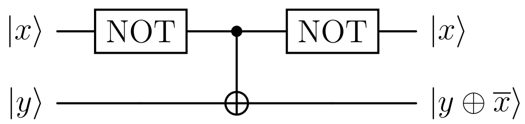
(2) For $f_2(x) = 1$:



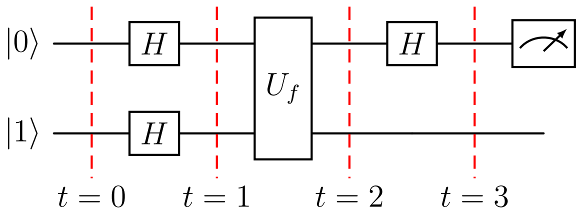
(3) For $f_3(x) = x$:



(4) For $f_4(x) = \overline{x}$:



(b) The Deutsch circuit is the following:



Let us analyze the various states:

- Initially, the state of the 2 qubits is $|\psi_0\rangle = |0\rangle \otimes |1\rangle$.
- After passage through the first Hadamard gates, the state becomes

$$|\psi_1\rangle = H|0\rangle \otimes H|1\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

- After passage through the quantum oracle U_f, the state becomes

$$|\psi_2\rangle = U_f|\psi_1\rangle = \frac{1}{2}\left(|0, f(0)\rangle - \left|0, \overline{f(0)}\right\rangle + |1, f(1)\rangle - \left|1, \overline{f(1)}\right\rangle\right)$$

- Then, after passage of the first qubit through the Hadamard gate on the right, the state becomes:

$$|\psi_3\rangle = (H \otimes I)\,|\psi_2\rangle = \frac{1}{2^{3/2}}\Big(|0, f(0)\rangle + |1, f(0)\rangle - \left|0, \overline{f(0)}\right\rangle - \left|1, \overline{f(0)}\right\rangle$$
$$+ |0, f(1)\rangle - |1, f(1)\rangle - \left|0, \overline{f(1)}\right\rangle + \left|1, \overline{f(1)}\right\rangle \Big)$$
$$= \frac{1}{2^{3/2}}\Big(|0, f(0)\rangle - \left|0, \overline{f(0)}\right\rangle + |0, f(1)\rangle - \left|0, \overline{f(1)}\right\rangle$$
$$+ |1, f(0)\rangle - \left|1, \overline{f(0)}\right\rangle - |1, f(1)\rangle + \left|1, \overline{f(1)}\right\rangle \Big)$$

after some reordering.

- Let us now analyze the state $|\psi_3\rangle$ in the two cases $f(0) = f(1)$ and $f(0) \neq f(1)$:
  - In the case where $f(0) = f(1) = x$, say, we get:

$$|\psi_3\rangle = \frac{1}{2^{3/2}}\Big(|0, x\rangle - |0, \overline{x}\rangle + |0, x\rangle - |0, \overline{x}\rangle \Big) = \frac{1}{\sqrt{2}}\left(|0, x\rangle - |0, \overline{x}\rangle\right)$$

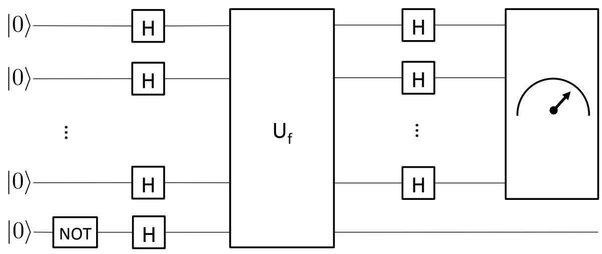
  - In the case where $f(0) = x$ and $f(1) = \overline{x}$, say, we get:

$$|\psi_3\rangle = \frac{1}{2^{3/2}}\Big(|1, x\rangle - |1, \overline{x}\rangle - |1, \overline{x}\rangle + |1, x\rangle \Big) = \frac{1}{\sqrt{2}}\left(|1, x\rangle - |1, \overline{x}\rangle\right)$$

- So finally, measuring the value of the first qubit, we obtain either $|0\rangle$ or $|1\rangle$ (each time with probability 1), which allows us to decide between the two alternatives.

Exercise 2 *Bernstein-Vazirani's algorithm*

(a) We reuse here the same circuit as in the lecture for the Deutsch-Josza algorithm:



The only thing that changes here is the prior information we have on the function f. The output state of the circuit (before the measurement) is given by

$$|\psi_4\rangle = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{y\in\{0,1\}^n} (-1)^{f(x)+x\cdot y}\,|y\rangle \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big)$$
$$= \sum_{y\in\{0,1\}^n} \left(\frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{x\cdot(a+y)} \right) |y\rangle \otimes \frac{1}{\sqrt{2}}\big(\,|0\rangle - |1\rangle\,\big)$$

So after the measurement of the first n qubits, the outcome is state $|y\rangle$ with probability

$$\text{prob}(|y\rangle) = \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a+y)} \right|^2$$

which is equal to 1 if $y = a$ and 0 in all the other cases. Therefore the result.

(b) When adding bit b to the picture, we obtain

$$\begin{aligned} \text{prob}(|y\rangle) &= \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{b \oplus x \cdot (a+y)} \right|^2 \\ &= \left| \frac{1}{2^n} (-1)^b \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a+y)} \right|^2 = \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot (a+y)} \right|^2 \end{aligned}$$

(i) The probabilities remain therefore the same as in the absence of b (which just adds a global phase), so the vector a can be equally determined.

(ii) On the contrary, b remains unknown with this scheme.

Exercise 3 *Construction of the Toffoli gate with C-NOT, H, T and S gates*

Using the first hint, we see that the circuit outputs the tensor product state $|\psi\rangle$ given by

$$|\psi\rangle = T\,|c_1\rangle \otimes SX^{c_1}T^\dagger X^{c_1}T^\dagger\,|c_2\rangle \otimes HTX^{c_1}T^\dagger X^{c_2}TX^{c_1}T^\dagger X^{c_2}H\,|t\rangle\,.$$

We then verify explicitly all the cases of c_1 and c_2. The calculation largely uses the fact that all the quantum gates here are unitary (*e.g.*, $TT^\dagger = T^\dagger T = I$); in particular, the gates X and H are involutory, *i.e.*, $X^2 = H^2 = I$.

For $c_1 = 0$, we have

$$\begin{aligned} |\psi\rangle &= T\,|0\rangle \otimes ST^\dagger T^\dagger\,|c_2\rangle \otimes HTT^\dagger X^{c_2}TT^\dagger X^{c_2}H\,|t\rangle \\ &= |0\rangle \otimes |c_2\rangle \otimes H\left(TT^\dagger\right)\left(X^{c_2}\left(TT^\dagger\right)X^{c_2}\right)H\,|t\rangle = |0\rangle \otimes |c_2\rangle \otimes |t\rangle \end{aligned}$$

For $c_1 = 1$ and $c_2 = 0$, let us follow the second hint:

$$XT^\dagger X = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix} = e^{-i\pi/4}T \tag{1}$$

and use this to compute

$$\begin{aligned} |\psi\rangle &= T\,|1\rangle \otimes SXT^\dagger XT^\dagger\,|0\rangle \otimes HTXT^\dagger TXT^\dagger H\,|t\rangle \\ &= e^{i\pi/4}\,|1\rangle \otimes S\left(XT^\dagger X\right)T^\dagger\,|0\rangle \otimes H\left(T\left(X\left(T^\dagger T\right)X\right)T^\dagger\right)H\,|t\rangle \\ &= e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/4}STT^\dagger\,|0\rangle \otimes |t\rangle \\ &= e^{i\pi/4}\,|1\rangle \otimes e^{-i\pi/4}\,|0\rangle \otimes |t\rangle = |1\rangle \otimes |0\rangle \otimes |t\rangle \end{aligned}$$

Finally, for $c_1 = c_2 = 1$, we compute, using repeatedly (1):

$$\begin{aligned}
|\psi\rangle &= T|1\rangle \otimes SXT^\dagger XT^\dagger |1\rangle \otimes HTXT^\dagger XTXT^\dagger XH|t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{-i\pi/4}STT^\dagger |1\rangle \otimes e^{-i\pi/2}HT^4H|t\rangle \\
&= e^{i\pi/4}|1\rangle \otimes e^{i\pi/4}|1\rangle \otimes e^{-i\pi/2}X|t\rangle
\end{aligned}$$

as

$$T^4 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and therefore

$$HT^4H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

Finally, this gives

$$|\psi\rangle = |1\rangle \otimes |1\rangle \otimes |\bar{t}\rangle$$

as expected.