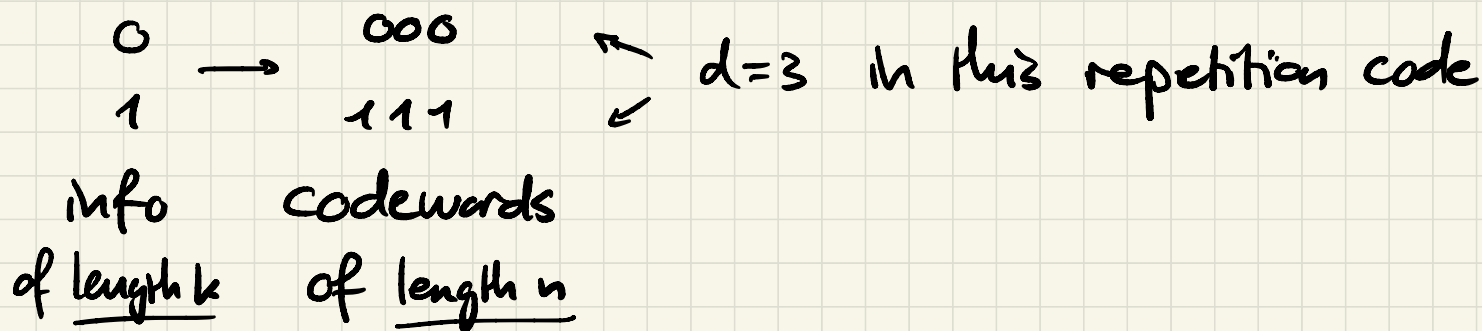


Quantum computation : lecture 12

Reminder : classical codes



These codewords go through a channel:

codeword x — channel — y output

Simple error model: (bit-flip) $y = x + e$, $e = 010$ eg.

bit flip
↓

This repetition code can correct up to $\lfloor \frac{d-1}{2} \rfloor = 1$ error

If probability (bit flip) = $p < \frac{1}{2}$, then the majority rule outputs the most likely info bit 0 or 1.

General linear (n, k, d) code:

$$\left. \begin{array}{l} \begin{array}{c} n \\ \boxed{} \end{array} = \text{generator matrix } G \\ \begin{array}{c} n-k \\ \boxed{} \end{array} = \text{parity-check matrix } H \end{array} \right\} G \cdot H^T = 0$$

$$C = \{x \in \mathbb{F}_2^n : x = u \cdot G, u \in \mathbb{F}_2^k\} = \{x \in \mathbb{F}_2^n : H x^T = 0\}$$

Ex: Hamming code $n=7, k=4, d=3$

$$H = \left(\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right) \begin{matrix} n=7 \\ n-k=3 \end{matrix}$$

all non-zero columns of length 3

$$G = \left(\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right) \begin{matrix} n=7 \\ k=4 \end{matrix}$$

$2^4 = 16$ codewords

$d=3$ here: no column in H is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

& no two columns in H are linearly dep.

$\Rightarrow d \geq 3$ (and $d=3$ because 111000 is a codeword)

This code can therefore correct 1 error

Syndrom decoding:

Codeword $x \rightarrow$ output $y = x + e$

$$H y^T = H(x^T + e^T) = \underbrace{H x^T}_{=0} + H e^T = H e^T$$

assume $e = 0000\underset{\substack{\uparrow \\ \text{position } i \\ (\text{ex: } i=5)}}{1}00 \Rightarrow H e^T = \text{column which is the binary rep. of } i \text{ (ex: } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix})$

Quantum error correction

repetition code : $\begin{cases} |0\rangle \rightarrow |000\rangle \\ |1\rangle \rightarrow |111\rangle \end{cases}$

⚠ this is not cloning!

(cloning would be $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\varphi\rangle \otimes |\varphi\rangle \otimes |\varphi\rangle$)

Here we do:

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\varphi\rangle = \alpha|000\rangle + \beta|111\rangle$$

"codewords"

Error models : $\overset{X}{\text{bit-flip}}$ or $\overset{Z}{\text{phase-flip}}$

(NB: we could think about other types of errors,
like variations of α & β ... later!)

bit-flip: (in position 1, e.g.)

$$\begin{aligned} |\psi\rangle \xrightarrow{X_1} |\psi'\rangle &= \overbrace{(X \otimes I \otimes I)}^{= X_1} |\psi\rangle \\ &= \alpha |000\rangle + \beta |111\rangle \qquad = \alpha |100\rangle + \beta |011\rangle \end{aligned}$$

Measurements ($\rightarrow$ "syndroms")

$$\begin{aligned}\text{Observables: } & \underbrace{Z_1 Z_2} \quad \& \quad \underbrace{Z_2 Z_3} \\ & = Z \otimes Z \otimes I \quad = I \otimes Z \otimes Z\end{aligned}$$

possible eigenvalues $+1$ & -1

NB: remember the parity-check matrix of the classical repetition code:

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\xleftarrow{\quad} Z_1 Z_2$
 $\xleftarrow{\quad} Z_2 Z_3$

Assume no error happened: $|\psi'\rangle = \alpha|000\rangle + \beta|111\rangle$

$$Z_1 Z_2 |\psi'\rangle = \alpha|000\rangle + \underbrace{(-1) \cdot (-1)}_{=+1} \beta|111\rangle = (+1) \cdot |\psi'\rangle$$

$$Z_2 Z_3 |\psi'\rangle = (+1) \cdot |\psi'\rangle \quad (\text{likewise})$$

Assume bit 1 was flipped: $|\psi'\rangle = \alpha|100\rangle + \beta|011\rangle$

$$Z_1 Z_2 |\psi'\rangle = -\alpha|100\rangle - \beta|011\rangle = (-1) |\psi'\rangle$$

$$Z_2 Z_3 |\psi'\rangle = +\alpha|100\rangle + \beta|011\rangle = (+1) |\psi'\rangle$$

In summary: measurement of $z_1 z_2$ & $z_2 z_3$ gives:

= "stabilizers"

$(+1, +1) \iff$ no bit flip

$(-1, +1) \iff$ bit-flip in position 1

$(-1, -1) \iff$ bit-flip in position 2

$(+1, -1) \iff$ bit-flip in position 3

↑

our new syndromes

(observe that $|\psi'\rangle$ an eigen-vector of z_1, z_2, z_3 in all cases: no state perturbation!)

Error detection is done; now we need to do error correction:

- if $(+1, +1) \rightarrow$ do nothing
- if $(-1, +1) \rightarrow$ apply $X_1 \rightarrow$ ^{back to} state $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$

Note X_1 need to be applied after the Z 's to $|\psi'\rangle$, which is not an eigenvector of X_1 (and the two other cases are similar)

In summary:

channel
(bit-flip)

- $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle \rightarrow |\psi'\rangle = X_1 |\psi\rangle$
 $= \alpha|100\rangle + \beta|011\rangle$
- error detection: $Z_1 Z_2 |\psi'\rangle = (-1) |\psi'\rangle$
and $Z_2 Z_3 |\psi'\rangle = (+1) |\psi'\rangle$, $|\psi'\rangle$ unchanged
- error correction: $X_1 |\psi'\rangle = X_1 X_1 |\psi\rangle = |\psi\rangle$

Other possible error: phase-flip $\begin{cases} Z|0\rangle = |0\rangle \\ Z|1\rangle = (-1)|1\rangle \end{cases}$

$\Rightarrow$ new code $\begin{cases} |0\rangle \rightarrow |+++ \rangle \\ |1\rangle \rightarrow |-- \rangle \end{cases} \quad |\psi\rangle = \alpha|+++ \rangle + \beta|-- \rangle$

$Z|+\rangle = |- \rangle, Z|- \rangle = |+\rangle \Rightarrow$ same as before!

phase-flip: $Z_1|\psi\rangle = \alpha|-++ \rangle + \beta|+-- \rangle$

error detection: observables X_1X_2 & X_2X_3

error correction: if $(-1, +1)$, apply Z_1 to
recover the original state!

How to handle bit & phase-flips together?

→ Shor's code (= concatenation of the two previous codes)

Step 1: (good for phase-flips)

$$|0\rangle \rightarrow |+++ \rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \rightarrow |-- -- \rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Step 2: (good for bit-flips)

$$|0\rangle \rightarrow \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} = |0\rangle_{\text{Shor}}$$

$$|1\rangle \rightarrow \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} = |1\rangle_{\text{Shor}}$$

Observe $k=1$ & $n=9$ here.

Claim: This code protects against a bit-flip
and/or a phase-flip.

• Initial state: $|\psi\rangle = \alpha|0\rangle_{\text{shen}} + \beta|1\rangle_{\text{shen}}$
 $\Rightarrow$ output state $|\psi'\rangle$ (bit-flip and/or phase-flip)

• Stabilizers: $Z_1 Z_2, Z_2 Z_3, Z_4 Z_5, Z_5 Z_6, Z_7 Z_8, Z_8 Z_9$

These measurements will not perturb the state and will provide info on the bit-flip.

• Next stabilizers: $X_1 X_2 X_3 X_4 X_5 X_6, X_4 X_5 X_6 X_7 X_8 X_9$

These will provide info on the phase-flip.

Fact: All these operators commute

and $|4'\rangle$ is an eigenvector of all of them

Error correction: apply the correct Z or X !

Ex 1: bit-flip error on bit 3

In this case, only $Z_2 Z_3$ has eigenvalue -1

$\Rightarrow$ one applies X_3 to correct the error

Ex 2: phase-flip error on bit 5

In this case, both $X_1 X_2 X_3 X_4 X_5 X_6$ and $X_4 X_5 X_6 X_7 X_8 X_9$ have eigenvalue -1 ; one does not know where the phase-flip occurred among bits 4, 5 or 6, but one can still correct the error by applying $Z_4 Z_5 Z_6$!

Ex 3: bit-flip and phase-flip error on bit 4

In this case, $Z_4 Z_5$, $X_1 X_2 X_3 X_4 X_5 X_6$ and $X_4 X_5 X_6 X_7 X_8 X_9$ have eigenvalue -1

This bit-phase flip can be corrected by applying both X_4 & Z_4 (note that depending on the order, this might generate a global (-1) phase, as $X_4 Z_4 = -Z_4 X_4$, but the error will be corrected).

Steane's code: $k=1$, $n=7$ (more efficient)

$$\begin{cases} |0\rangle \rightarrow \frac{1}{\sqrt{8}} (|0000000\rangle + |1010101\rangle + |0110011\rangle + |0001111\rangle \\ \quad + |0111100\rangle + \dots) \\ |1\rangle \rightarrow \frac{1}{\sqrt{8}} (|1111111\rangle + |0101010\rangle + |1001100\rangle + |1110000\rangle \\ \quad + |1000011\rangle + \dots) \end{cases}$$

Stabilizers:

$X_4 X_5 X_6 X_7$	$Z_4 Z_5 Z_6 Z_7$
$X_2 X_3 X_6 X_7$	$Z_2 Z_3 Z_6 Z_7$
$X_1 X_3 X_5 X_7$	$Z_1 Z_3 Z_5 Z_7$

Claim: All these measurements commute

and $|\psi'\rangle$ = state after one bit-flip and/or phase-flip is an eigenvector.

$\Rightarrow$ error correction: next week!