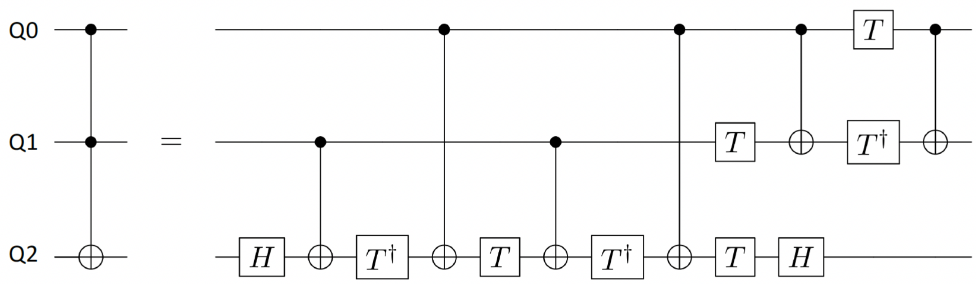
Exercise Set 4
Quantum Computation

Exercise 1 *IBM Q practice: Implementation and tests with the Toffoli gate*

In Homework 3, you proved the following circuit identity:



Note that with respect to Homework 3, we have removed the end gates T and S on the first two qubits. These qubits just add some phase factors with respect to the standard Toffoli gate which make no difference (at least for this exercise). You are asked to test this identity with Qiskit and IBM Q platform for multiple entries of your choice from the computational basis, for example $|000\rangle$, $|110\rangle$, $|111\rangle$. Concretely: produce and extract pdf pictures of histograms of measurements for *the circuit on the right hand-side of the identity* with 1024 shots.

(a) Run the circuit on an ideal simulator.

(b) Choose a real machine (tell which one). Transpile your circuit to match this device. What are the depths of the original circuit and the new one?

Note: Transpilation is a process of rewriting and optimizing a given circuit to match a quantum device. You can see an example in the tutorial.

(c) Obtain the noise model from the real machine (see the example from this link). Construct a simulator with this noise model by passing it as an argument `noise_model` when constructing `AerSimulator`. Run the circuit on this noisy simulator.

(d) Run the circuit on the real machine.

Warning: the job can take several hours before launching, please estimate your time accordingly!

(e) Shortly comment on the nature of fluctuations you observe for the ideal simulator, the noisy simulator, and the real machine.

Exercise 2 *Square-root of the NOT gate*

The aim of the present exercise is to compute, for a given one-qubit gate U, a corresponding gate V such that $V^2 = U$ (cf. Ex 2, Hw 3) in the particular case where $U = X$ (the NOT gate). Here is first a description of the generic procedure.

First observe that since a one-qubit gate U is a 2×2 unitary matrix ($UU^\dagger = U^\dagger U = I$), it is in particular a *normal* matrix satisfying $UU^\dagger = U^\dagger U$. The *spectral theorem* then asserts that such a U is *unitarily diagonalizable*, i.e., there exists Λ a 2×2 diagonal matrix (with possibly complex entries) and W another 2×2 unitary matrix, such that $U = W \, \Lambda \, W^\dagger$.

In order to compute Λ and W, it suffices to compute the two solutions to the eigenvalue-eigenvector equation:

$$U w^{(i)} = \lambda_i \, w^{(i)}, \quad i = 0, 1$$

with the added constraint that $(w^{(i)})^\dagger w^{(j)} = \delta_{i,j}$. Then $\Lambda = \text{diag}(\lambda_0, \lambda_1)$ and $W = (w^{(0)}, w^{(1)})$, i.e., $w^{(i)}$ is the i-th column of W.

Finally, consider $V = W \sqrt{\Lambda} \, W^\dagger$, where $\sqrt{\Lambda} = \text{diag}(\sqrt{\lambda_0}, \sqrt{\lambda_1})$, with square roots being taken in the complex plane $\mathbb{C}$ (! two options for each of them !). You can check that $V^2 = U$.

(a) Compute the 2×2 matrices Λ and W corresponding to $U = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(b) Deduce an explicit expression for a matrix V such that $V^2 = X$. Check now directly that $V^2 = X$.

(c) Is V also unitary? Justify your answer.