Lecture 9: NP-completeness, Reductions

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Recap: NP

The class **NP**

Definition: A verifier for a language A is a TM V, where

$$A = \{w \mid \exists C \text{ s.t. } V \text{ accepts } \langle w, C\rangle\}.$$

A polynomial time verifier runs in polynomial time in $|w|$.

Definition: **NP** is the class of languages with polynomial time verifiers

Non-deterministic Turing Machines

Recall: In a Turing machine, $\delta : (Q \times \Gamma) \longrightarrow Q \times \Gamma \times \{L, R\}$.

In a **Nondeterministic Turing Machine (NTM)**,

$$\delta : (Q \times \Gamma) \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

(several possible transitions for a given state and tape symbol)

Definition: A **nondeterministic decider** for language L is an NTM N such that for each $x \in \Sigma^*$, **every computation of N on x halts**, and moreover,

- If $x \in L$, then some computation of N on x accepts.
- If $x \notin L$, then every computation of N on x rejects.

An NTM is a **polynomial time** NTM if the running time its longest computation on x is **polynomial in $|\mathbf{x}|$**.

Nondeterministic deciders $\Longleftrightarrow$ Verifiers

Theorem: For any language $L \subseteq \Sigma^*$,

L has a nondeterministic poly-time decider $\Longleftrightarrow$ L has a poly-time verifier.

Proof Sketch ($\Longleftarrow$):

Let V be the verifier. NTM N on input x does the following:

1. Write a certificate C nondeterministically.
2. Run V on $\langle x, C \rangle$.

Proof Sketch ($\Longrightarrow$):

Let N be the nondeterministic decider. Verifier V on $\langle x, C \rangle$ computes:

- Simulate N on x, choosing transitions given by C.

V accepts $\langle x, C \rangle$ iff C is the accepting path of N on x.

Non-deterministic Polynomial-time

Theorem: For any language $L \subseteq \Sigma^*$,

L has a nondeterministic poly-time decider $\iff$ L has a poly-time verifier.

Definition: **NP** is the class of languages which have poly-time nondeterministic deciders, or equivalently, have poly-time verifiers.

Definition:

$$\mathsf{NTIME}(t(n)) = \{L \ : \ L \text{ has a nondeterministic } O(t(n)) \text{ time decider}\}$$

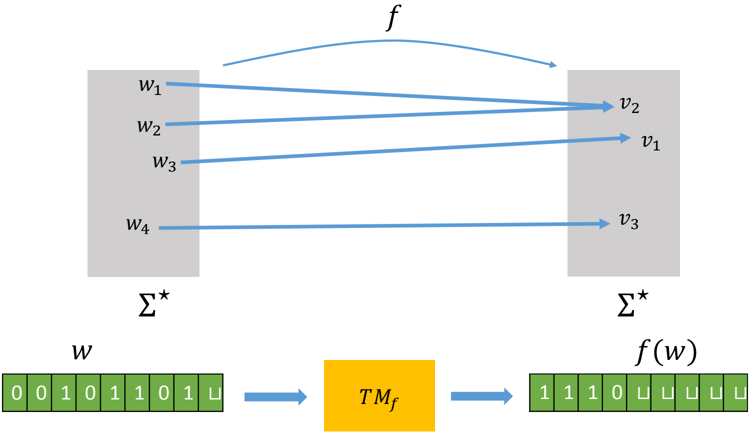
Then

$$\mathsf{NP} = \bigcup_{k=1}^{\infty} \mathsf{NTIME}(n^k)\,.$$

$\therefore$ SAT, GI, INDSET are all in NP.

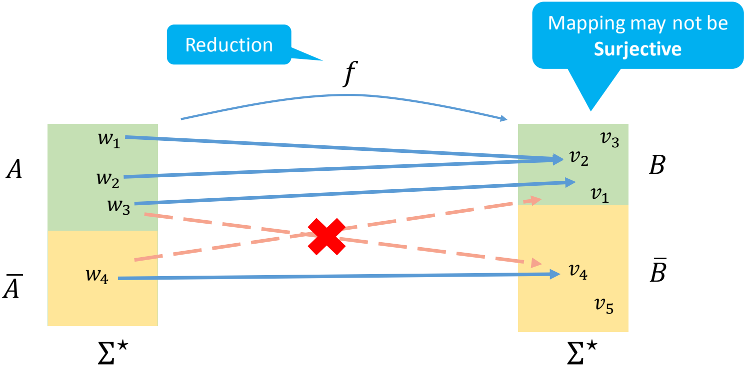
NP-completeness

Poly-time Reductions (1): Poly-time Computability



Definition: A function $f : \Sigma^* \to \Sigma^*$ is a *polynomial time computable function* if some poly-time TM M, on every input w halts with just $f(w)$ on its tape.

Poly-time Reductions (2): Correctness



Definition: Language A is **poly-time mapping reducible** to language B, written $A \leq_P B$, if there is a poly-time computable function $f : \Sigma^* \to \Sigma^*$, such that for every $w \in \Sigma^*$:

$$w \in A \Leftrightarrow f(w) \in B$$

NP-completeness

Definition: A language L is said to be **NP-complete** if

- L is in **NP**.
- For every language L' in **NP**, $L' \leq_P L$.

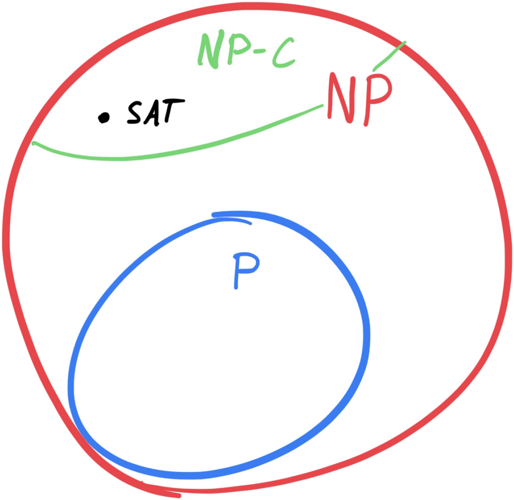
Observe: If **one NP**-complete language has a polynomial time decider, then **every** language in **NP** has a polynomial time decider, i.e. $\mathbf{P} = \mathbf{NP}$.

The Cook–Levin Theorem: SAT is **NP**-complete.

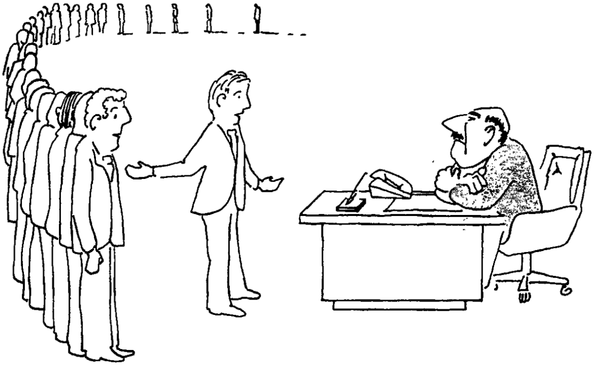
To show L is **NP**-complete:

- **[NP membership]** Give a poly-time verifier for L.
- **[NP hardness]** Show $C \leq_P L$ for some **NP**-complete language C (not the other way around)

Big picture



Takehome message



"I can't find an efficient algorithm, but neither can all these famous people."

3SAT is NP-complete

kSAT $= \{\langle\varphi\rangle \;:\; \varphi$ is satisfiable and each clause of φ contains $\leq k$ literals$\}$

Verifier for 3SAT: Just use the verifier for SAT.

Claim: SAT $\leq_P$ 3SAT

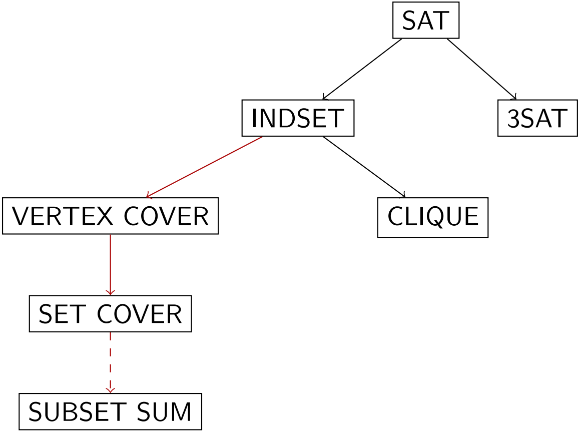
Reduction: Given φ,

- While φ contains a clause $K = (\ell_1 \vee \ell_2 \vee \ell_3 \vee \cdots \vee \ell_m)$ with > 3 literals

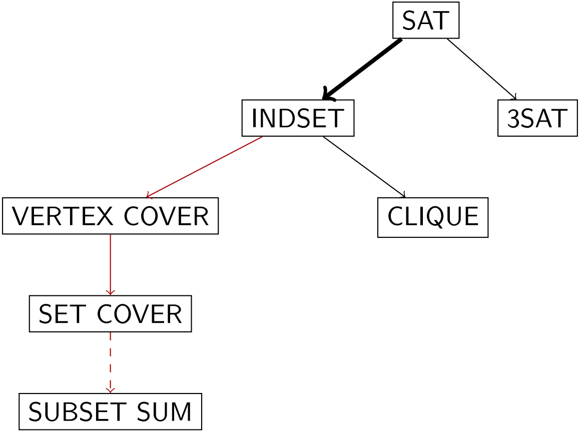
$$\left.\begin{array}{c}\text{Replace } K \text{ with the following two clauses}\\ K_1 = (\ell_1 \vee \ell_2 \vee z)\\ K_2 = (\overline{z} \vee \ell_3 \vee \cdots \vee \ell_m)\end{array}\right\} \text{Preserves satisfiability (check!)}$$

What is the runtime?

Does SAT $\leq_P$ 2SAT analogously?



and many more . . .


SAT
INDSET
3SAT
VERTEX COVER
CLIQUE
SET COVER
SUBSET SUM

INDSET is NP-complete

We have already seen that

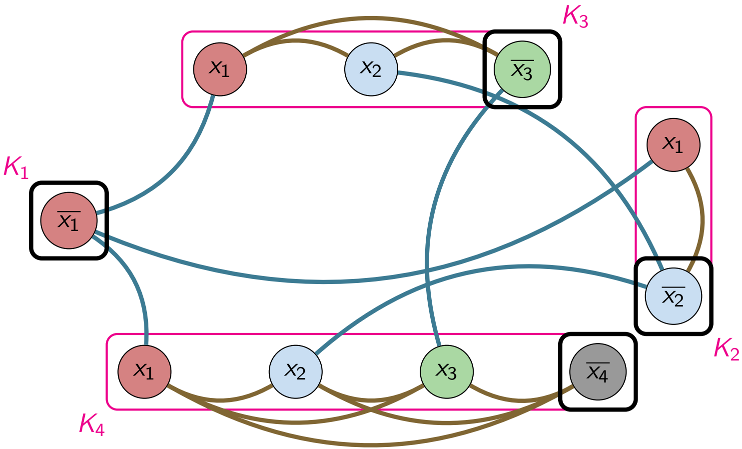
$$\text{INDSET} = \{\langle G, k\rangle : G \text{ has an independet set of size } k\}$$

is in **NP** and so it remains to give a poly-time reduction from an **NP**-complete language

INDSET is NP-complete

Claim: SAT $\leq_P$ INDSET

$$\varphi = \underbrace{\overline{x_1}}_{K_1} \wedge \underbrace{(x_1 \vee \overline{x_2})}_{K_2} \wedge \underbrace{(x_1 \vee x_2 \vee \overline{x_3})}_{K_3} \wedge \underbrace{(x_1 \vee x_2 \vee x_3 \vee \overline{x_4})}_{K_4}$$



INDSET is NP-complete

Claim: SAT $\leq_P$ INDSET

Reduction f: On input φ,

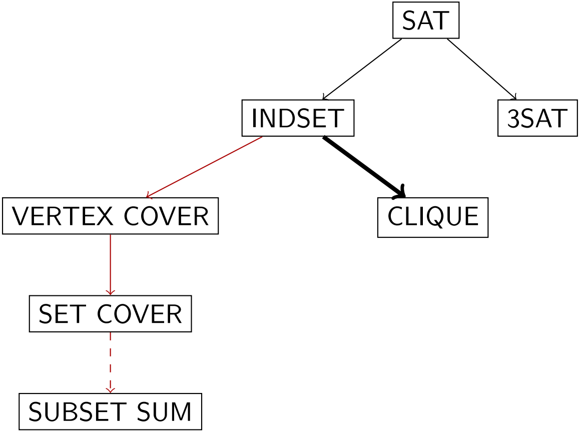
1. Let G be the graph generated as follows.
    1. Take a vertex for each literal of each clause.
    2. Add edges for pairs of conflicting literals.
    3. Add edges for pairs of literals from the same clause.
2. Let m be the number of clauses in φ.
3. Output (G, m).

Claim: $\varphi \in \text{SAT} \implies f(\varphi) \in \text{INDSET}$

Proof: C: satisfying assignment of φ. Pick one `true` literal from each clause. The corresponding vertices form a independent set.

Claim: $f(\varphi) \in \text{INDSET} \implies \varphi \in \text{SAT}$

Proof: C: independent set in G, $|C| = m$. C contains one vertex from each group. Set the corresponding literals to `true` to get a satisfying assignment.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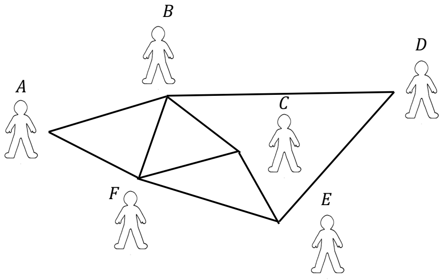
CLIQUE



A,B,F is a 3-clique

Definition: A k-clique is a subset of k pairwise connected vertices

$$\text{CLIQUE} = \{\langle G, k\rangle \mid G \text{ has a clique of size } k\}$$

$\langle G, 3\rangle \in CLIQUE$? Yes

$\langle G, 4\rangle \in CLIQUE$? No

CLIQUE is **NP**-Complete

Theorem: INDSET $\leq_p$ CLIQUE

Corollary: CLIQUE is **NP**-complete

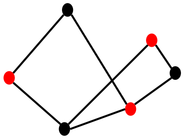
Goal find a poly-time reduction from INDSET to CLIQUE

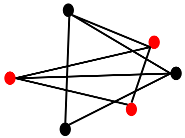
INDSET vs CLIQUE

For a graph G:

Independent set: A subset of pairwise **non-adjacent** vertices

Clique: A subset of pairwise **adjacent** vertices





Def (Complement): The *complement* of a graph $G = (V, E)$ is a graph $\bar{G} = (V, \bar{E})$ with same vertex set and edge set $\bar{E}$ s.t. $uv \in \bar{E}$ iff $uv \notin E$

Observation: If G is a graph and $\bar{G}$ its complement, then a subset S of the vertices of G is an independent set iff S is a clique of $\bar{G}$

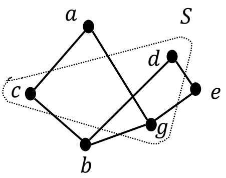
INDSET $\leq_p$ CLIQUE

Reduction: $f(\langle G = (V, E), k\rangle) := \langle \bar{G} = (V, \bar{E}), k\rangle$

Efficiency: The reduction is polynomial time

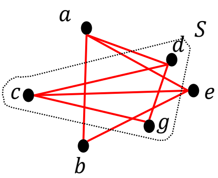
Correctness: $\langle G, k\rangle \in \text{INDSET} \Leftrightarrow \langle \bar{G}, k\rangle \in \text{CLIQUE}$

Proof of correctness:

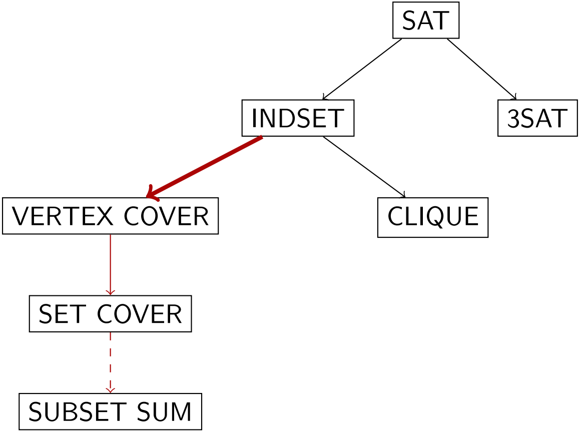


$\langle G, k\rangle \in INDSET, |S| = k$

$\Leftrightarrow$



$\langle \bar{G}, k\rangle \in CLIQUE$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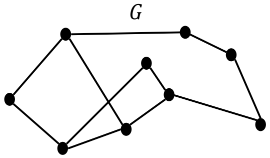
Vertex Cover

Definition: For a graph $G = (V, E)$, a *vertex cover* is a subset S of V such that every edge of G is incident to at least one vertex in S

Example



Definition:

$$\text{VERTEX COVER} = \{\langle G, k\rangle : G \text{ is a graph that has a vertex cover of size } k\}$$

- $\langle G, 4\rangle \in$ VERTEX COVER? Yes
- $\langle G, 3\rangle \in$ VERTEX COVER? No

VERTEX COVER is **NP**-Complete

STEP 1: VERTEX COVER $\in$ **NP**

TM V: "On input $\langle G, k, C\rangle$

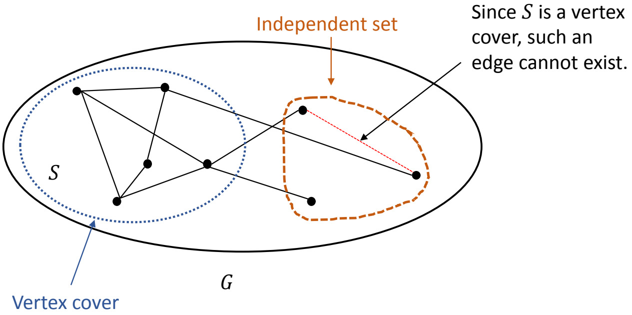
1. IF $|C| \neq k$, THEN REJECT
2. FOR every pair $u \neq v$ of vertices DO
    - IF uv is an edge AND $u \notin C$ AND $v \notin C$ THEN REJECT
3. ACCEPT"

STEP 2: INDSET $\leq_p$ VERTEX COVER

How to reduce INDSET to VERTEX COVER?

INDSET vs VERTEX COVER

Lemma: For every graph G, a subset S of the vertices is a vertex cover if and only if $\bar{S}$ is an independent set where $\bar{S} = V \setminus S$



Corollary: For every graph G and a positive integer k, G has an independent set of size k iff G has a vertex cover of size $n - k$

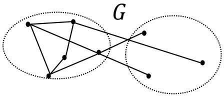
INDSET $\leq_p$ VERTEX COVER

Reduction: $f(\langle G, k\rangle) := \langle G, n-k\rangle$, where n is the number of vertices of G

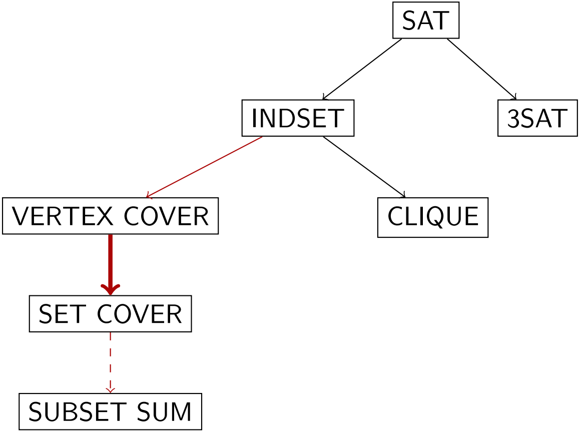
Efficiency: The reduction f is polynomial time

Correctness: Lemma/Corollary on previous slide!

$\langle G, k\rangle \in INDSET \iff$



$\iff \langle G, n-k\rangle \in VERTEX\ COVER$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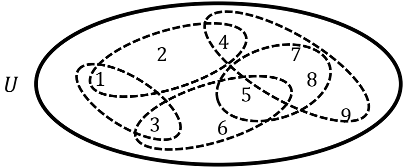
Set Cover

Definition: Let $U = \{1, \ldots, n\}$ and $\mathcal{F} = \{T_1, \ldots, T_m\}$ be a family of subsets $\forall i, T_i \subseteq U$

A subset $\{T_{i_1}, \ldots, T_{i_k}\} \subseteq \mathcal{F}$ is called a set cover of size k if $\bigcup_{j=1}^{k} T_{i_j} = U$



$$\text{SET COVER} = \{\langle U, \mathcal{F}, k\rangle \mid \mathcal{F} \text{ contains a set cover of size } k\}$$

Example:

- $\langle U, \mathcal{F}, 3\rangle \in$ SET COVER? Yes $\{\{1,2,4\}, \{3,6,5\}, \{4,7,8,9\}\}$
- $\langle U, \mathcal{F}, 2\rangle \in$ SET COVER? No

SET COVER is **NP**-Complete

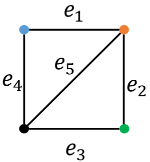
Theorem: VERTEX COVER $\leq_p$ SET COVER

Corollary: SET COVER is **NP**-complete

VERTEX COVER $\leq_p$ SET COVER

Key idea: VERTEX COVER is a *special case* of SET COVER

Example:



$k = 2$

$U = \{e_1, e_2, e_3, e_4, e_5\}$

$\mathcal{F} = \{\{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_3\}, \{e_3, e_4, e_5\}\}$

$k' = k = 2$

Reduction:

- Let $\langle G = (V, E), k \rangle$ be an instance of VERTEX COVER
- Set $U := E$
- For every vertex $v \in V$ create a set S_v

$$S_v := \{e \in E \mid e \text{ is incident to } v\}$$

- Let $\mathcal{F} := \{S_v \mid v \in V\}$ and set $k' = k$

Efficiency: Obvious from construction

Correctness ($\Rightarrow$): If G has a vertex cover of cardinality k, then U can be covered by k sets

Proof:

- Suppose $C \subseteq V$ is a vertex cover of G and $|C| = k$
- Every edge e_i is adjacent to at least one vertex in C

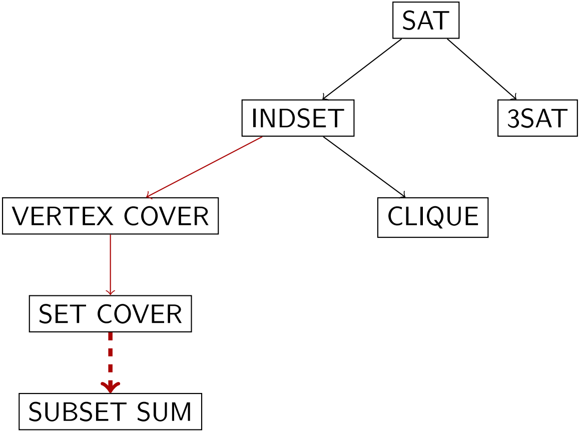
$$\bigcup_{v \in C} S_v = E$$

- Hence U can be covered by k sets

Correctness ($\Leftarrow$): If U can be covered by k sets, then G has a vertex cover of cardinality k

Proof:

- Let $S_{v_1}, S_{v_2}, \dots, S_{v_k}$ be a collection of sets which cover $U = E$
- We claim that $C = \{v_1, v_2, \dots, v_k\}$ is a vertex cover of G
- Indeed, every edge e in G belongs to S_{v_i} for some $i \in \{1, 2, \dots, k\}$
- Hence, every edge e in G is incident to some vertex $v_i \in C$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SUBSET-SUM

Let X denote a (multi) set of positive integers

Definition: SUBSET-SUM
$= \{\langle X, s\rangle : X \text{ contains a subset whose elements sum to } s\}$

Example: $X = \{1, 3, 4, 6, 13, 13\}$

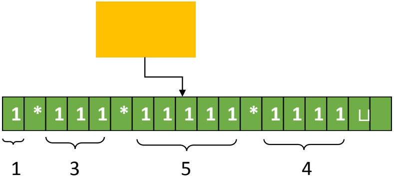
- $\langle X, 8\rangle \in$ SUBSET-SUM? Yes $T = \{1, 3, 4\}$
- $\langle X, 12\rangle \in$ SUBSET-SUM? No

Question: SUBSET-SUM $\in P$?

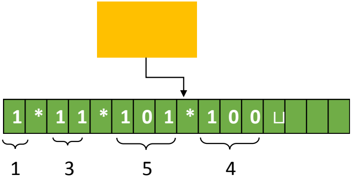
It depends on the input length

In $SUBSET\text{-}SUM$ we use binary encoding

Unary vs Binary



$\langle\{1,3,5\},4\rangle$



$\langle\{1,3,5\},4\rangle$

Input length: $x_1 + x_2 + \cdots + x_m + s$ $\gg$ Input length: $\lceil\log(x_1)\rceil + \cdots + \lceil\log(x_m)\rceil + \lceil\log(s)\rceil$

SUBSET-SUM: An Algorithm

- Let $X = \{x_1, x_2, \dots, x_m\}$, $\quad sum \coloneqq \sum_{j=1}^{m} x_j$
- FOR $i = 1 \,..\, m$ DO
-     FOR $j = 0 \,..\, sum$ DO
-         $A[i,j] := False$
- FOR $i = 0$ to m DO
-     $A[i,0] \coloneqq True$
- FOR $i = 1 \,..\, m$ DO
-     FOR $j = 1 \,..\, sum$ DO
-         IF $A[i-1, j-x_i] = True$ OR $A[i-1, j] = True$ THEN $A[i,j] := True$
- **IF $A[m,s] = True$, ACCEPT; ELSE, REJECT**

Running time: For the input $\langle\{x_1, \dots, x_m\}, s\rangle$, the above algorithm takes $O\left(m \cdot \left(\sum_{i=1}^{m} x_i\right)\right)$ steps to output whether or not $\langle X, s\rangle \in SUBSET\text{-}SUM$

Unary representation: The length of the input $x_1 + x_2 + \cdots + x_m + s$. In this case the running time is **polynomial** in the input length

Binary representation: The length of the input $\lceil\log(x_1)\rceil + \cdots + \lceil\log(x_m)\rceil + \lceil\log(s)\rceil$. In this case, the running time is **exponential** in the input length

SUBSET-SUM is **NP**-Complete

Theorem: SET COVER $\leq_p$ SUBSET-SUM

Proof: Next lecture

Corollary: SUBSET-SUM is **NP**-Complete