Lecture 5: Decidability and Undecidability

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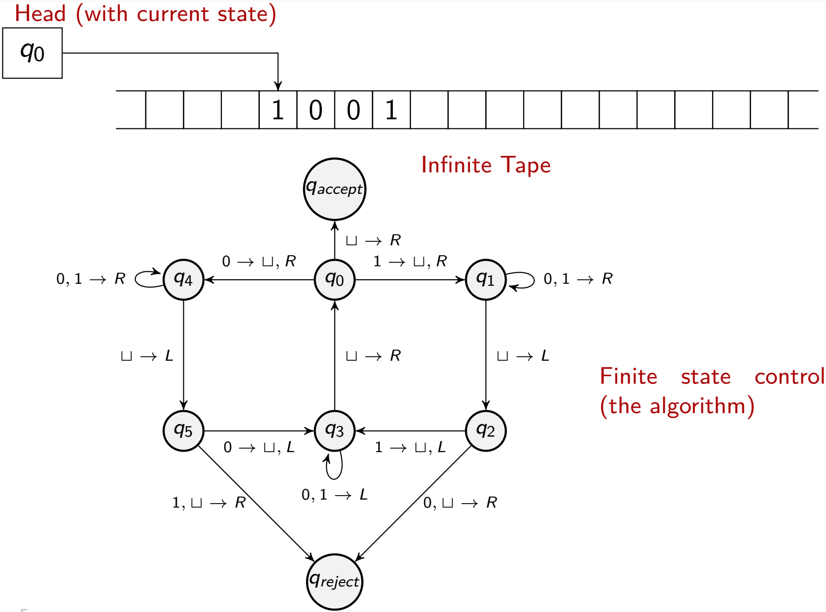
Lecture 5

And thus the Turing Machine was born.

Recall

The Turing Machine



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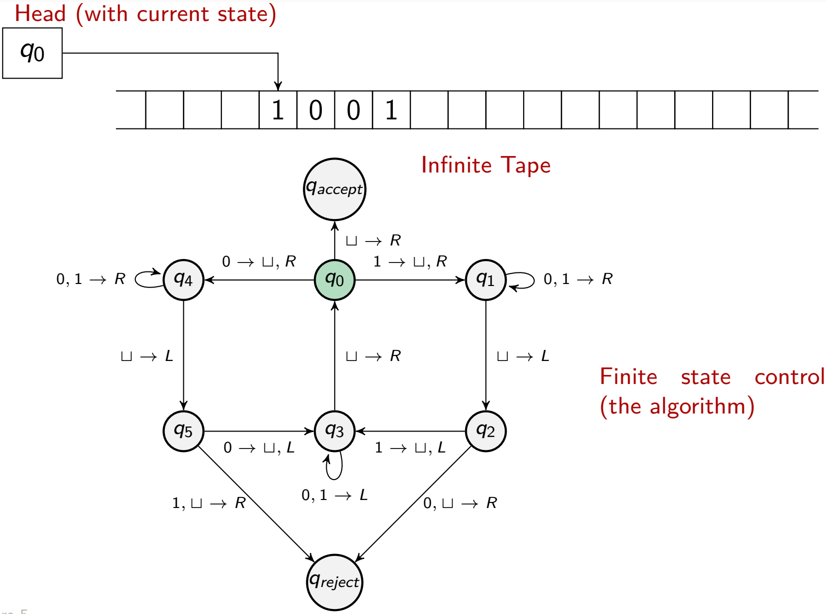

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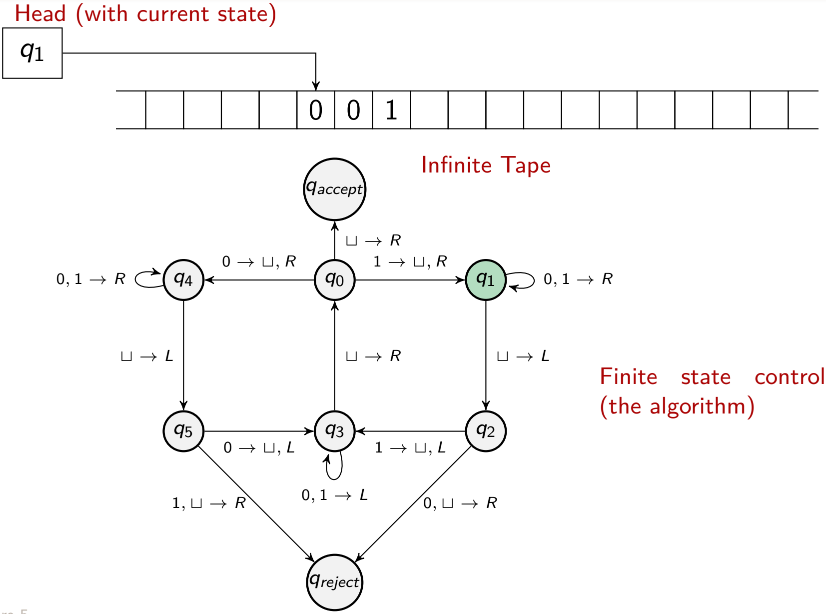

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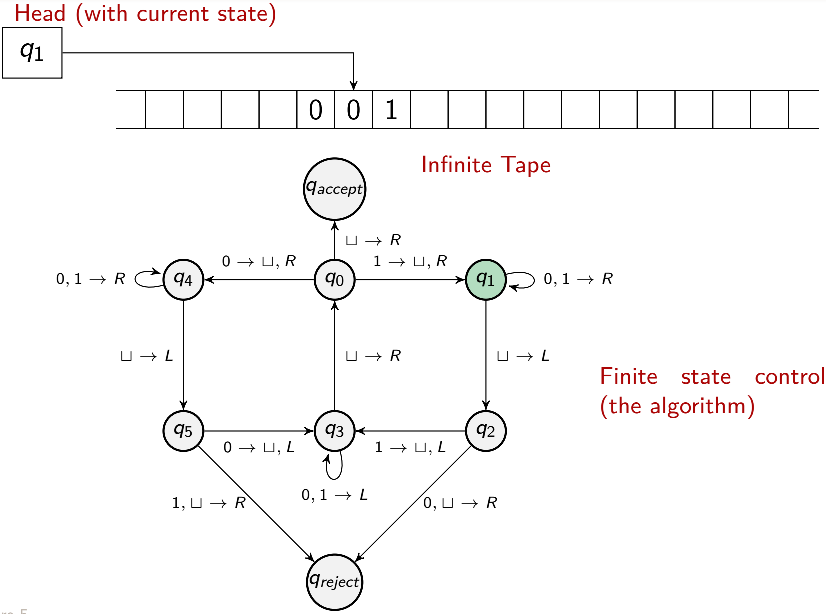

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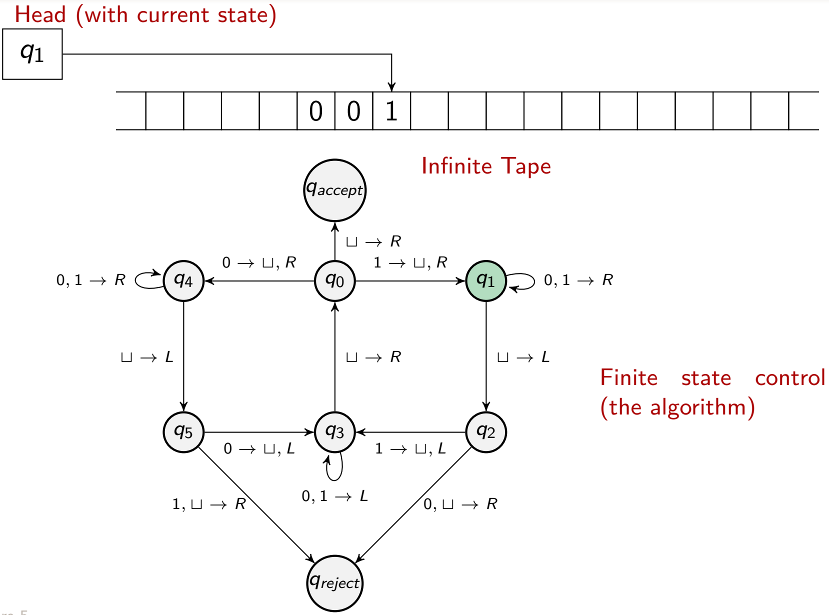

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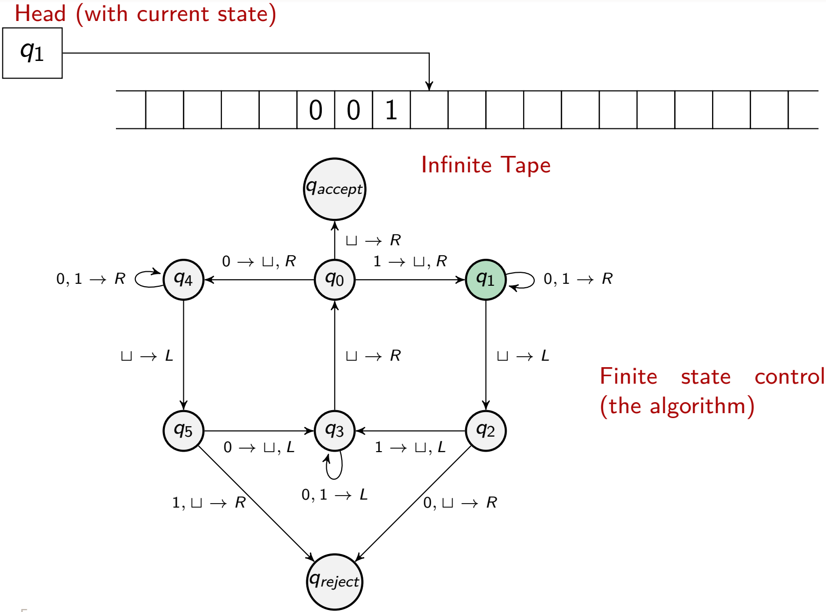

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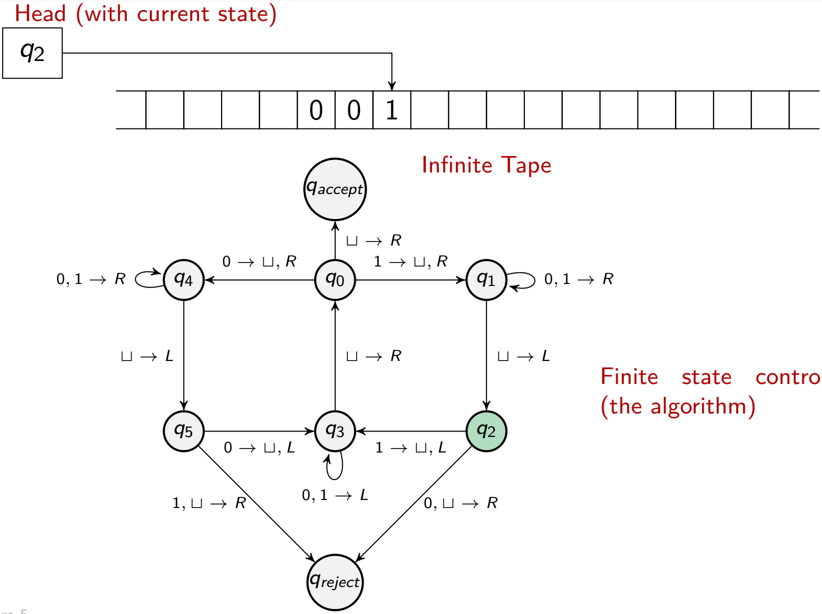
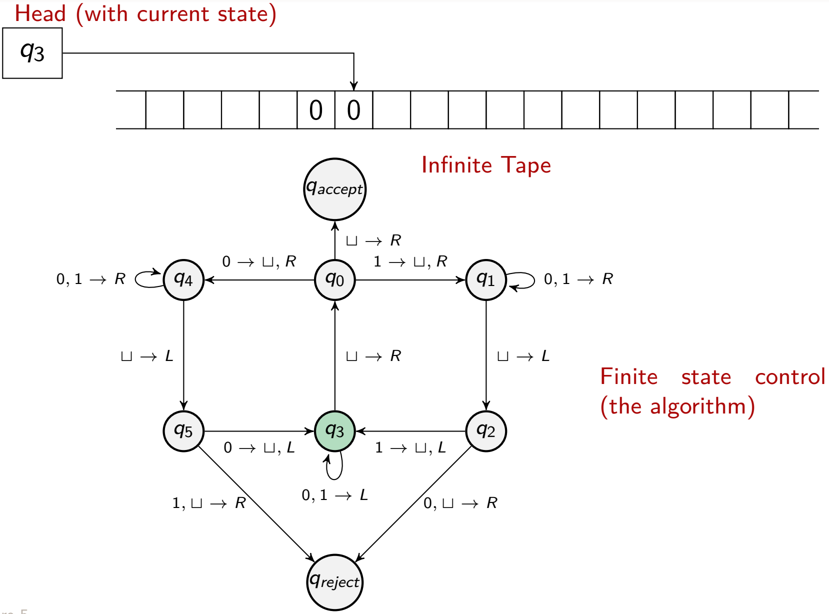


Finite state control (the algorithm)

The Turing Machine



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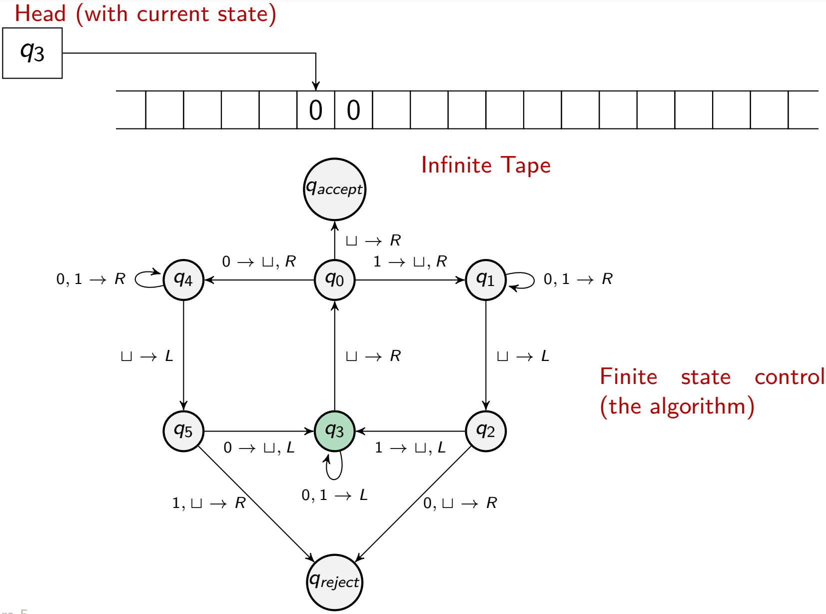

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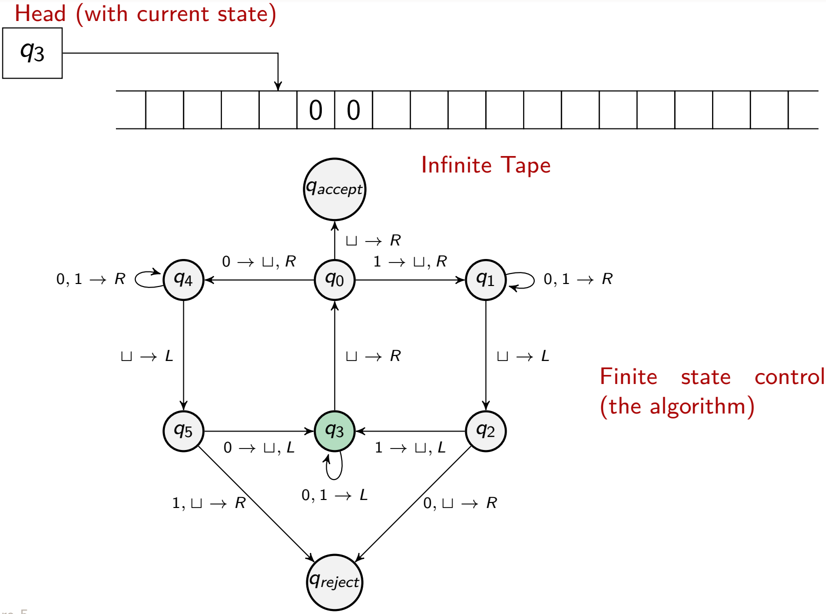

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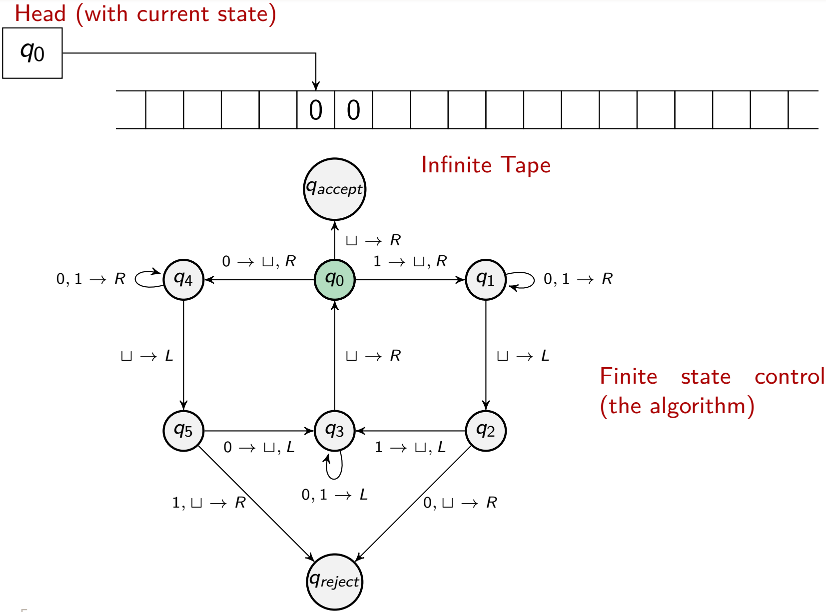

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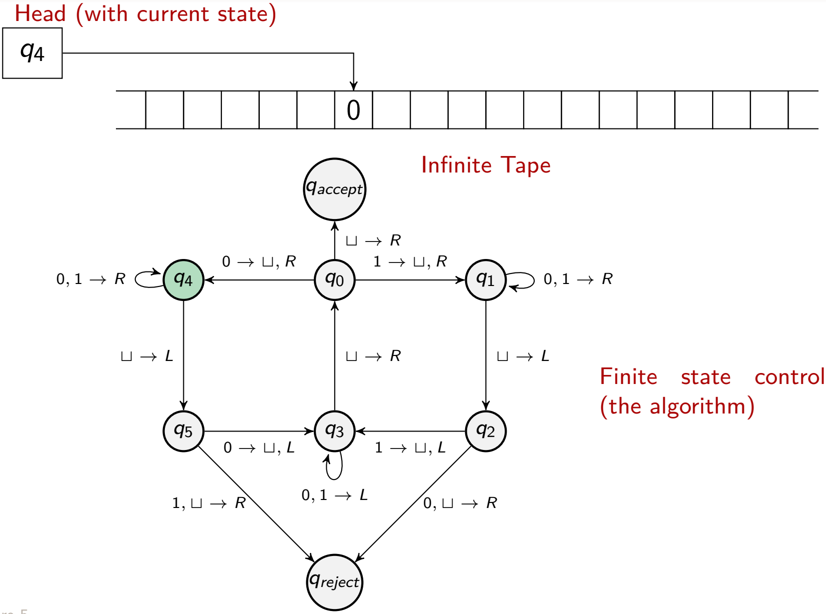

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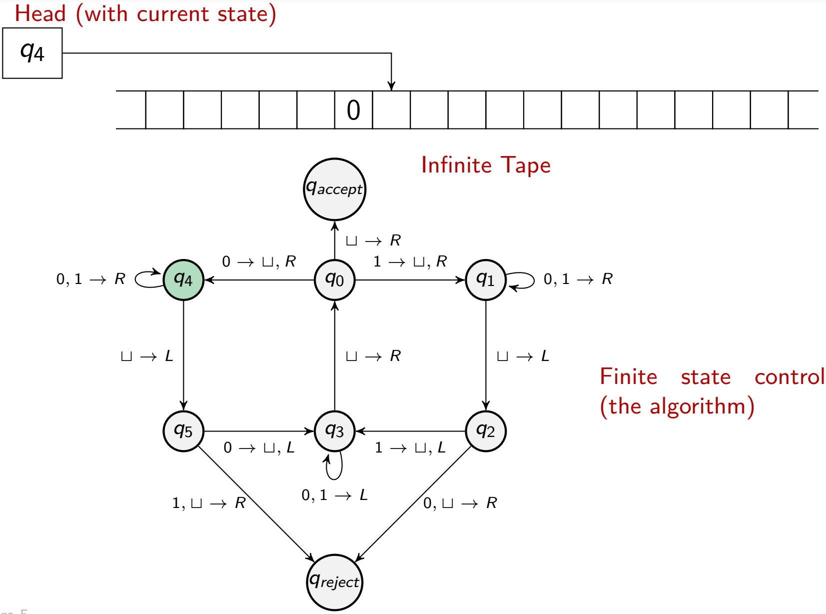

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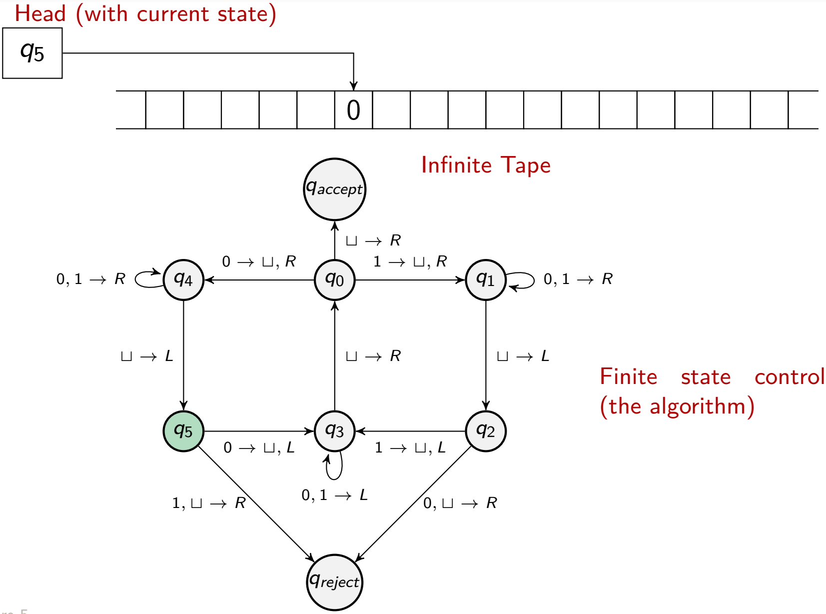

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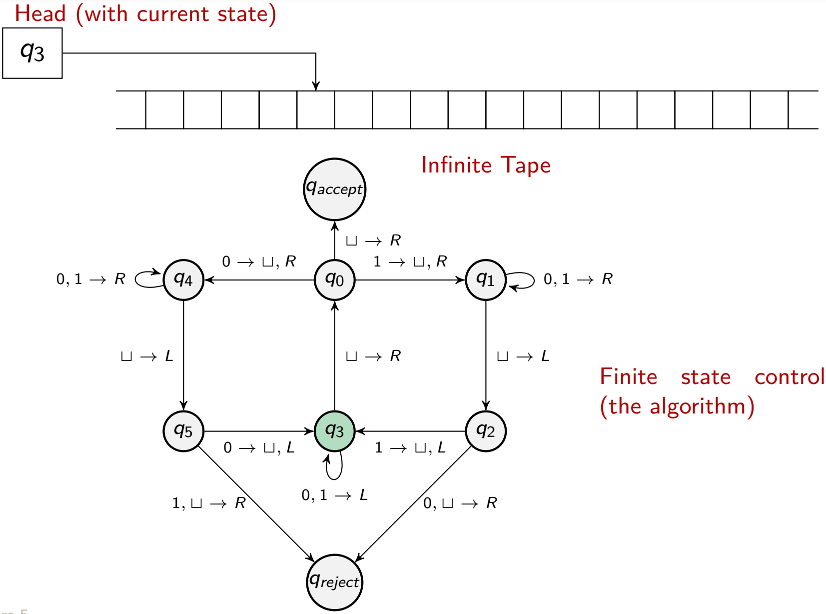

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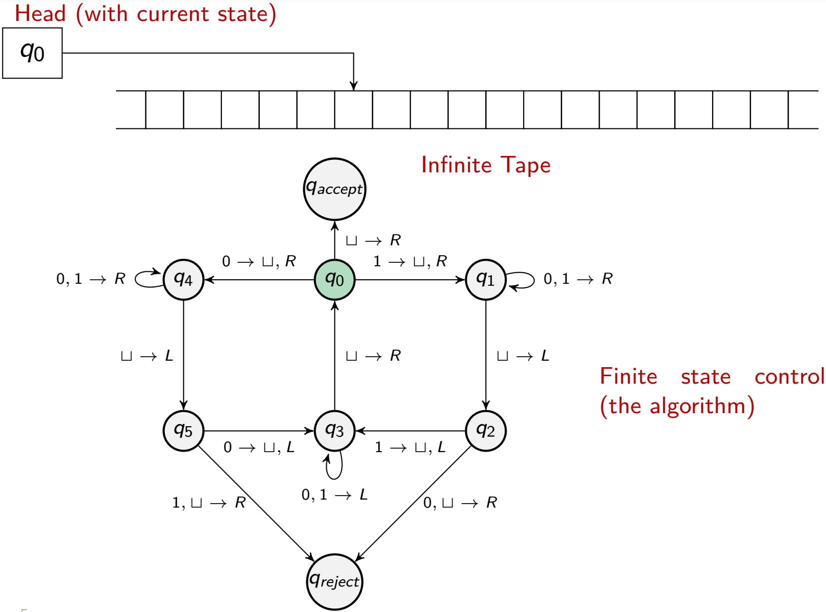

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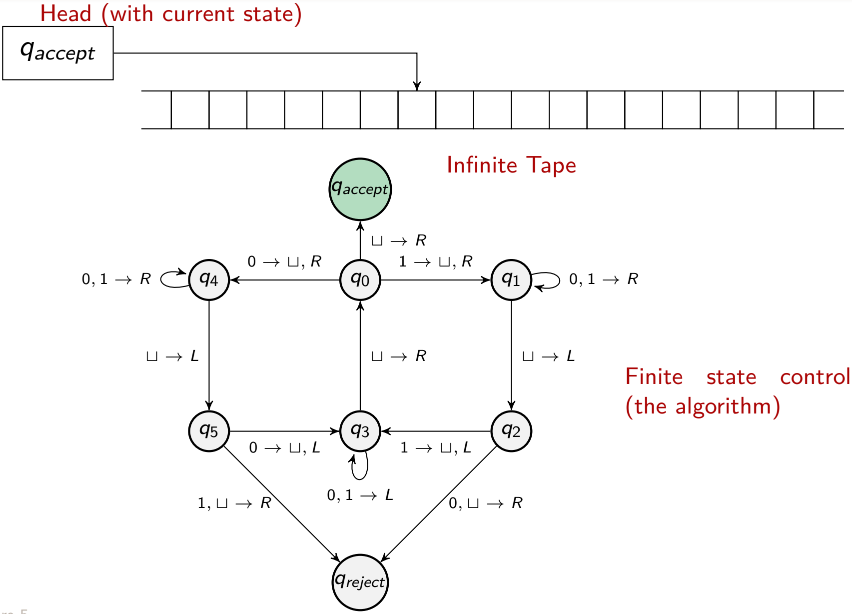

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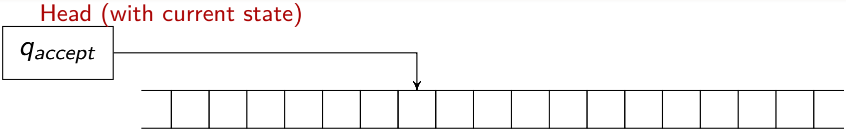
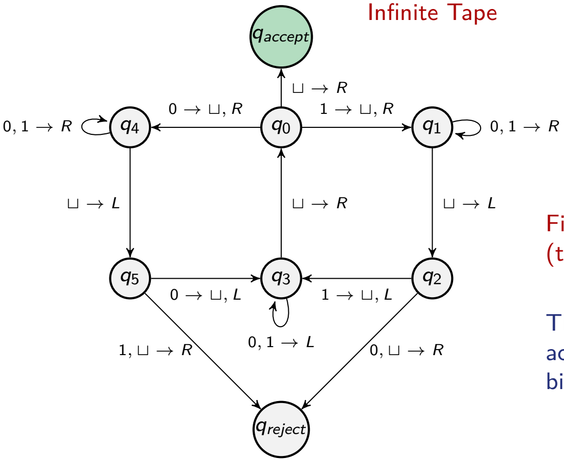
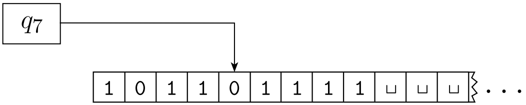




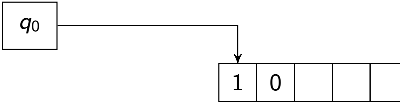
Finite state control (the algorithm)

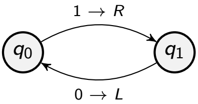
This Turing machine accepts even length binary palindromes

Turing Machine



- Infinite tape
- Tape alphabet contains input alphabet plus ⊔ (blank symbol) plus maybe more symbols
- Head has states (corresponding to the finite control automata)
- Exactly **one** Accept state and exactly **one** Reject state *(where computation immediately ends)*
- Remaining states *"computation in progress"*
- May never reach an accept state. May never halt!





Formal Definition of a TM

A **Turing Machine** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are all finite sets and

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6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{accept} \neq q_{reject}$.

Turing-Recognizable/Decidable Languages

A TM machine M recognizes a language $L \subseteq \Sigma^*$ iff for all inputs $w \in \Sigma^*$:

1. If $w \in L$ then M accepts w and
2. If $w \notin L$ then M either rejects w or never halts

Such languages are called (Turing)-Recognizable

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A TM machine M decides a language $L \subseteq \Sigma^*$ iff for all inputs $w \in \Sigma^*$:

1. M halts on w, and
2. M accepts w iff $w \in L$

Such languages are called (Turing)-Decidable

Church-Turing Thesis

Intuitive notion of algorithms	equals	*Turing machine algorithms*

Church-Turing Thesis

Intuitive notion of algorithms	equals	*Turing machine algorithms*

- All algorithms we know of can be executed on TMs
- Anything you write in C, Java, Scala, Python and so on
- The definition is also robust to variations: if we allow for many tapes instead of one, then nothing changes

To show language decidable or recognizable it sufficient to describe an algorithm in a high level language (Since by above any such algorithm can be run on a TM)

Decidable Languages

What can TMs compute?

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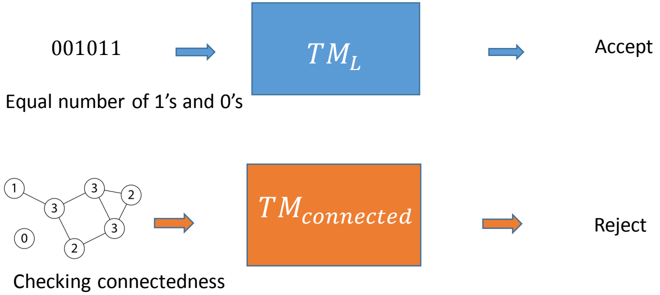
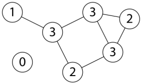


What can TMs compute?

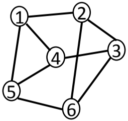
001011 → TM_L → Accept

Equal number of 1's and 0's

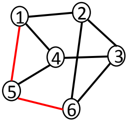


→ $TM_{connected}$ → Reject

Checking connectedness



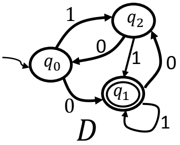
1,6 → $TM_{Shortestpath}$ →



Shortest path problem

And more…

How to encode inputs



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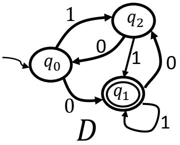

$\boldsymbol{\Sigma}$	$\boldsymbol{Q}$	$\boldsymbol{F}$	$\boldsymbol{q_0}$	*
q_0	1	q_2	*	
q_0	0	q_1	*	
q_1	0	q_2	*	
⋮	...			

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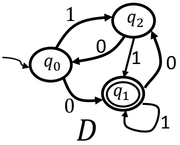

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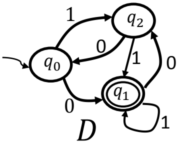
Q	Σ	F	q_0	*	q_0	1	q_2	*	q_0	0	...

How to encode inputs



Σ	Q	F	q_0	*
q_0	1	q_2	*	
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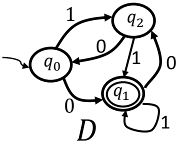
⋮	...			

Q	Σ	F	q_0	*	q_0	1	q_2	*	q_0	0	...

$\langle D \rangle \in \{0,1\}^\star$:

0	1	...	1	1	...	0	0	1	0	1	1	0	0	0	1	0	...	0	0	1	...

How to encode inputs



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q_1	0	q_2	$*$	
$\vdots$	…			

Q	Σ	F	q_0	$*$	q_0	1	q_2	$*$	q_0	0	…

$\langle \boldsymbol{D} \rangle \in \{\mathbf{0}, \mathbf{1}\}^{\star}$:

0	1	…	1	1	…	0	0	1	0	1	1	0	0	0	1	0	…	0	0	1	…

Anything can be encoded using binary strings!

Checking Emptiness of a DFA

$$E_{DFA} = \{\langle D \rangle : L(D) = \emptyset\}$$

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First approach:

1. For each string $s \in \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
2. Simulate D on s
   - If D accepts s, THEN *reject*
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If $L(D) = \emptyset$, then this TM will never halt!

Checking Emptiness of a DFA (2nd attempt)

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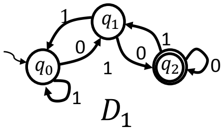

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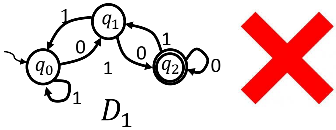

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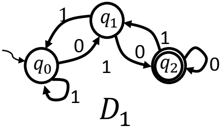

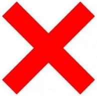

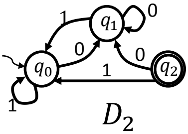

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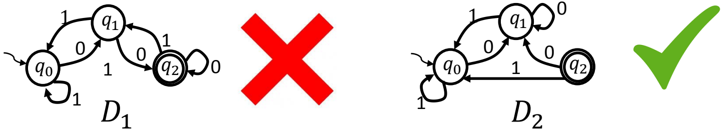

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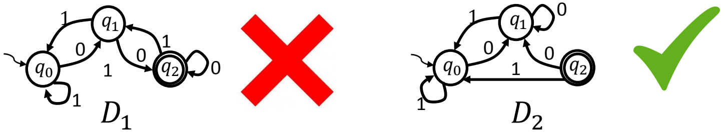

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Given $\langle D \rangle$ with $D = (Q, \Sigma, \delta, q_0, F)$

1. Initialize, $R = \{q_0\}$
2. For each $q \in R$ and $q' \in Q \setminus R$, check if there exists a transition of the form $\delta(q, a) = q'$ for some $a \in \Sigma$
3. If at least one such q' is found, add q' to R and go back to Step 2
4. Accept iff $R \cap F = \emptyset$

Two DFAs recognize the same language?

$$EQ_{DFA} = \{\langle D, D' \rangle : L(D) = L(D')\}$$

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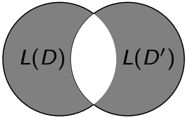

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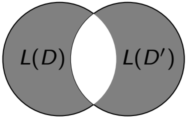

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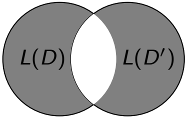

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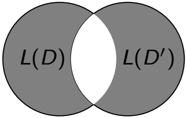

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- Thus $L(D) \oplus L(D')$ is regular
- Let $D^{\oplus}$ be DFA accepting $L(D) \oplus L(D')$

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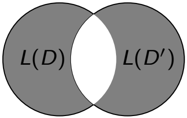
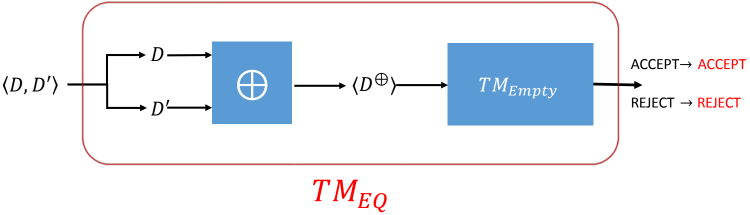
Fact: $L(D) = L(D')$ if and only if $L(D) \oplus L(D') = \emptyset$



- Here $L(D) \oplus L(D') = \left(L(D) \cap \overline{L(D')}\right) \cup \left(\overline{L(D)} \cap L(D')\right)$
- Thus $L(D) \oplus L(D')$ is regular
- Let $D^{\oplus}$ be DFA accepting $L(D) \oplus L(D')$



Mika, we get it: Turing machines can compute a lot

but can they compute anything?

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but can they compute anything? NO

Theorem (Turing 1936): Halting problem

$$HALT = \{\langle M, w\rangle : M \text{ is a TM and } M \text{ halts on input } w\}$$

is undecidable

Intuitively the difficulty of is to conclude in a finite number of steps whether M loops forever or is just slow to halt . . .

Halting problem is recognizable

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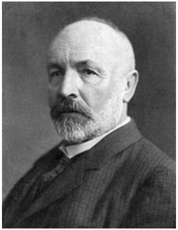
- Since we will show that $HALT$ is undecidable this shows that recognizers *are* more powerful than deciders



George Cantor (1845–1918)

Diagonalization

Comparing the size of infinite sets

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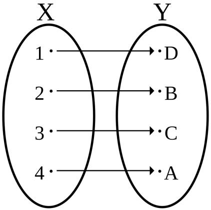
Comparing the size of infinite sets

Recall that a function $f : X \to Y$ is bijective if it is

- **one-to-one** it never maps two different elements to the same place – that is, $f(a) \neq f(a')$ whenever $a \neq a'$
- **onto** it hits every element of Y – that is, for every $b \in Y$ there is an $a \in X$ such that $f(a) = b$



Definition. A set A is countable if either it is finite or it has the same size as $\mathbb{N}$ (i.e., there is a bijection between A and $\mathbb{N}$)

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$\mathbb{N}$ and the set of even numbers are of the same size by the bijection $f(n) = 2n$:

n	$f(n)$
1	2
2	4
3	6
$\vdots$	$\vdots$

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May seem bizzarre at first but incredibly influential!

> *"No one shall expel us from the paradise which Cantor has created for us."*
>
> *– David Hilbert*

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Diagonalization

The reals are uncountable

- Suppose toward contradiction that there is a bijection $f : \mathbb{N} \to \mathbb{R}$
- We reach a contradiction by defining an $x \in \mathbb{R}$ such that no number $a \in \mathbb{N}$ maps to x, i.e., $f(a) = x$
- For every $n \in \mathbb{N}$, the n-th fractional digit of x is selected to be different from the n-th digit of $f(n)$
- Example:

n	$f(n)$
1	$3.\underline{1}4159\ldots$
2	$55.5\underline{5}555\ldots$
3	$0.12\underline{3}45\ldots$
4	$0.500\underline{0}0\ldots$
$\vdots$	$\vdots$

$$x = 0.4641\ldots$$

- By definition there is no $n \in \mathbb{N}$ such that $f(n) = x$ which contradicts that f is onto.

We now use Diagonalization to

- First prove that there is a language that is not decidable
- Then prove that the specific language

$$HALT = \{\langle M, w\rangle : M \text{ is a TM and } M \text{ halts on } w\}$$

  is not decidable

There are undecidable languages

- **Turing machines are countable:** enumerate all encodings

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	$\langle M_6 \rangle$	$\dots$
M_1	A	∞	R	A	A	R	$\dots$
M_2	R	R	A	A	∞	A	$\dots$
M_3	R	∞	A	∞	R	R	$\dots$
M_4	A	∞	R	R	R	∞	$\dots$
M_5	∞	∞	A	A	A	A	$\dots$
M_6	R	A	R	∞	A	∞	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

There are undecidable languages

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M_2	R	R	A	A	∞	A	...
M_3	R	∞	A	∞	R	R	...
M_4	A	∞	R	R	R	∞	...
M_5	∞	∞	A	A	A	A	...
M_6	R	A	R	∞	A	∞	...
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Let $DIAG = \{\langle M_i \rangle : M_i \text{ doesn't accept } \langle M_i \rangle\} = \{\langle M_2 \rangle, \langle M_4 \rangle, \langle M_6 \rangle, \dots\}$

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Theorem *DIAG is undecidable*

Proof by contradiction

- Suppose M decided $DIAG$
- Then $M = M_i$ for some $i \in \mathbb{N}$
- We have $L(M_i) = DIAG$

Does $\langle M_i \rangle \in L(M_i)$? Two cases

- If $\langle M_i \rangle \in L(M_i)$ then by def. of $DIAG$, $\langle M_i \rangle \notin DIAG$ – contradiction
- If $\langle M_i \rangle \notin L(M_i)$ then by def. of $DIAG$, $\langle M_i \rangle \in DIAG$ – contradiction

Thm: $HALT = \{\langle M, w\rangle : M \text{ is a TM and } M \text{ halts on } w\}$ is undecidable

Proof by contradiction

Assume on the contrary that H is a decider for $HALT$

We construct a decider D for $DIAG = \{\langle M\rangle : M \text{ doesn't accept } \langle M\rangle\}$

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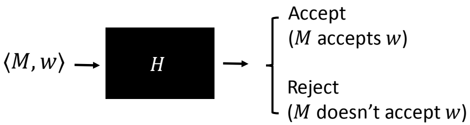

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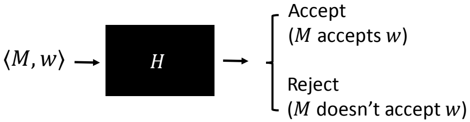

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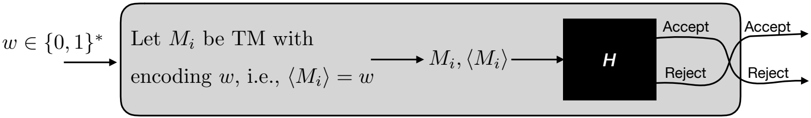

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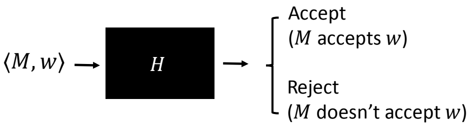

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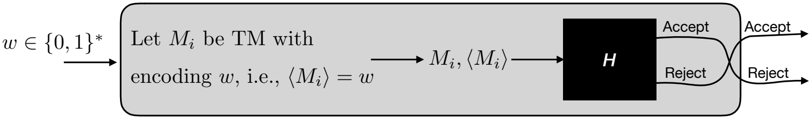

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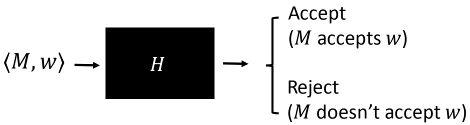

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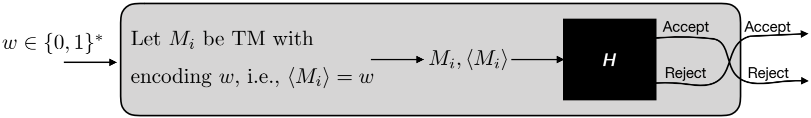

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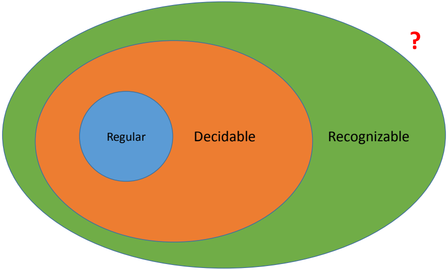

Unrecognizable languages?

Unrecognizable languages exist!

Thm: $\overline{HALT}$ is not recognizable

Step 1

Thm: A language L is decidable iff it is recognizable and its complement is also recognizable

Proof:

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Formally: we alternate between running 1 step of $\boldsymbol{M_1}$ and 1 step of $\boldsymbol{M_2}$

$\boldsymbol{M}$ = "On input **w**:
1- Run **both** $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ on input $\boldsymbol{w}$, **in parallel**
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Formally: we alternate between running 1 step of $\boldsymbol{M_1}$ and 1 step of $\boldsymbol{M_2}$

- M decides L
  - For all w, either $w \in L$ or $w \in \bar{L}$
  - Either M_1 accepts w or M_2 accepts w
  - M halts once one of them stops $\Rightarrow$ M must always halt!

Step 2

Corollary: Language $\overline{HALT}$ is unrecognizable

- Recall that
  - $HALT$ is undecidable
  - $HALT$ is recognizable
- So by previous Thm we must have that $\overline{HALT}$ is unrecognizable

Next week: **Reductions**