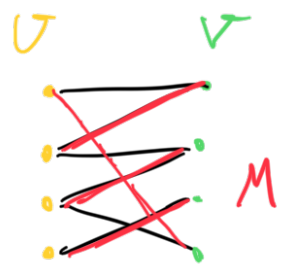
Plan: SAT $\le_p$ P3M $\le_p$ SUBSET-SUM

Perfect matchings in bipartite graphs

$G = (U \cup V, E), \quad E \subseteq U \times V$



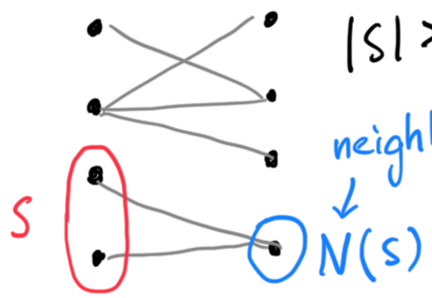
$M \subseteq E$ is a **matching** if each $v \in U \cup V$ has at most one incident egde in M

M is **perfect** if

$$|M| = |U| = |V|$$

Claim: $PM \in NP$

$\overline{PM} \in NP$



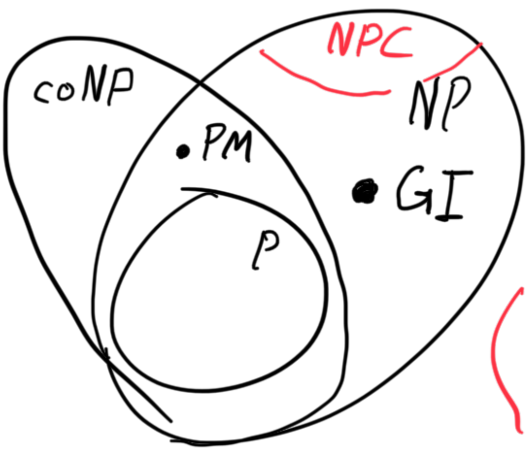
$|S| > |N(S)|$ ← Obstruction ⇒ no PM

neighbourhood

Thm (Hall) $G \in PM$ iff no such obstruction

Thm $PM \in P$
(algo course)

$coNP = \{L : \overline{L} \in NP\}$



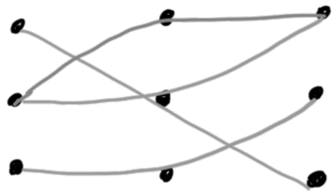
Belief: $NP \neq coNP$

Perfect-3-Matching

U V W

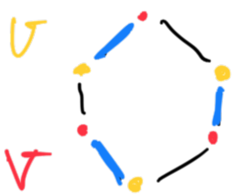
Input: $E \subseteq U \times V \times W$

$|U| = |V| = |W| = n$



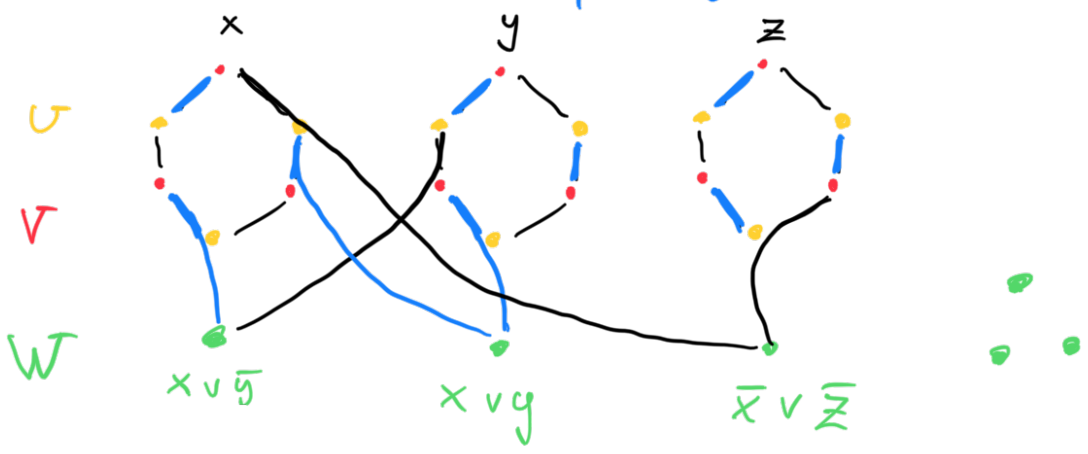
Decide: Is there $M \subseteq E$ where each $v \in U \cup V \cup W$ appears <u>once</u>

"Gadget": consider even length cycle



<u>Key idea:</u> for $SAT \leq P3M$
add a cycle for each variable x.
Two PMs correspond to setting $x \in \{1, 0\}$

<u>Reduction</u> f: Given φ, say, $(x \vee \bar{y}) \wedge (x \vee y) \wedge (\bar{x} \vee \bar{z})$



1. Add a cycle for each variable (U & V)
2. Add a node for each clause (W)
3. Connected literals to cycles

④ Add dummy nodes $(|U| = |V| = |W|)$

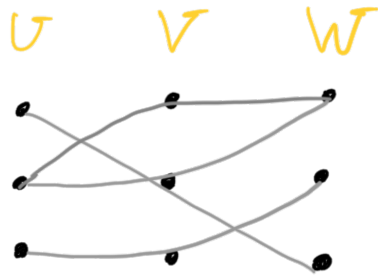
Claim: $\varphi \in \text{SAT} \underset{\Leftarrow}{\Rightarrow} f(\varphi) \in \text{P3M}$

Subset-Sum

Input: Set $S \subseteq \mathbb{N}$ (binary encoded)
target $k \in \mathbb{N}$

Decide: Is there $S' \subseteq S$: $\sum_{s \in S'} s = k$?

Perfect-3-Matching

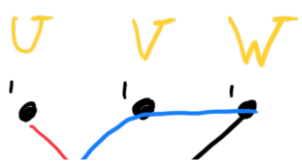


Input: $E \subseteq U \times V \times W$
$|U| = |V| = |W| = n$

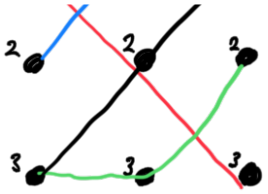
Decide: Is there $M \subseteq E$ where
each $v \in U \cup V \cup W$ appears once

Thm $\text{P3M} \leq_p \text{SUBSET-SUM}$

Idea: Encode each edge $e \in E \subseteq V \times U \times W$
as 3n-bit binary number. Subset sum target
$k = 1^{3n}$



$$\overbrace{010}^{U}\ \overbrace{100}^{V}\ \overbrace{100}^{W}$$



$$
\mapsto
\begin{array}{rccc}
 & 100 & 010 & 001 \\
+ & 001 & 001 & 010 \\
\hline
 & 111 & 111 & 111
\end{array}
$$

Claim: $x \in P3M \Rightarrow f(x) \in SS$

$\Leftarrow$

Problem: carry bits

$$
\begin{array}{r}
\overbrace{001}^{v} \\
100 \\
001 \\
001 \\
\hline
101
\end{array}
\qquad\qquad
\begin{array}{r}
\overbrace{001}^{v} \\
100 \\
001 \\
001 \\
\hline
103
\end{array}
$$

(binary) (base 10)

Solution: Use base-b numbers $b \geq |E|$