Final Exam, Theory of Computation 2022

- Books, notes, communication, calculators, cell phones, computers, etc... are not allowed.
- Your explanations and proofs should be clear enough and in sufficient detail so that they are easy to understand and have no ambiguities.
- You are allowed to refer to material covered in lectures (but not exercises) including theorems without reproving them.
- **Do not touch until the start of the exam.**

Good luck!

Name: ____________________ **N° Sciper:** ____________________

Problem 1 / 15 points	Problem 2 / 15 points	Problem 3 / 15 points	Problem 4 / 20 points	Problem 5 / 20 points	Problem 6 / 15 points

Total / 100

1 *(15 pts)* **Quick-fire round.** Consider the following statements.

1. If A is regular and $B \leq_p A$, then B is regular.
2. If A can be recognised by an NFA, then $\overline{A}$ can also be recognised by an NFA.
3. If A is decidable and $B \subseteq A$, then B is decidable.
4. If A is **NP**-complete and $A \leq_p B$, then B is **NP**-complete.
5. If A is recognisable and $A \leq_m \overline{A}$, then A is undecidable.
6. If A and B are recognisable, then $A \cap B$ is recognisable.
7. The language $\{0^n 1^m : n - m \text{ is divisible by } 2022\}$ is regular.
8. The language $\{\langle D \rangle : D \text{ is a DFA and } L(D) \text{ is infinite}\}$ is decidable.
9. The language $\{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is infinite}\}$ is recognisable.
10. The language $\{\langle G \rangle : G \text{ is a graph that contains a clique of size } 2022\}$ is in **P**.
11. 3-SAT $\leq_p$ 2-SAT.
12. SAT $\leq_m$ HALT.
13. If **P** = **NP**, then GRAPH-ISOMORPHISM $\in$ **P**.
14. If **NP** $\neq$ **coNP**, then SAT $\notin$ **coNP**.
15. Every function $f\colon \{0,1\}^n \to \{0,1\}$ can be computed by a CNF with $O(2^n)$ clauses.

For each box below, write one of the following symbols:

- **T** if the statement is known to be true.
- **F** if the statement is false *or not known to be true.* E.g., both **P** = **NP** and **P** $\neq$ **NP** should be marked **F**.
- or leave the box empty.

A correct **T**/**F** answer is worth +1 point, an incorrect answer is worth −1 point, and an empty answer is worth 0 points.

1.		6.		11.	
2.		7.		12.	
3.		8.		13.	
4.		9.		14.	
5.		10.		15.	

2 *(15 pts)* **Pumping Lemma.**

2a Write down the statement of the Pumping Lemma. *(Only the statement, not its proof!)*

2b For a string $w = w_1 w_2 \cdots w_n \in \{0,1\}^n$ define its *reverse* $w^{\mathcal{R}}$ as the string written backwards, that is, $w^{\mathcal{R}} = w_n w_{n-1} \cdots w_1$. Show that the following language is not regular:

$$L = \{w \in \{0,1\}^* : w^{\mathcal{R}} = w\}.$$

(A word w such that $w^{\mathcal{R}} = w$ is called a palindrome.*)*

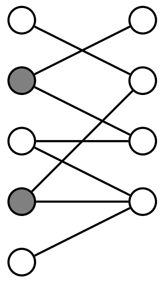
3 *(15 pts)* **Regular languages.** For any language $A \subseteq \{0,1\}^*$, define $\textsc{Drop}(A)$ to be the language containing all strings that can be obtained by removing one bit from a string in A. That is,

$$\textsc{Drop}(A) \;=\; \{xy : xby \in A \text{ where } b \in \{0,1\}\}.$$

Show that if A is regular, then so is $\textsc{Drop}(A)$.

(Here it suffices that you describe clearly how to construct a DFA/NFA for $\textsc{Drop}(A)$*. You do* ***not*** *need to formally prove the correctness of your construction.)*

4 *(20 pts)* **NP-completeness.** Let $G = (L \cup R, E)$ be a bipartite graph with left vertices L, right vertices R (where $L \cap R = \emptyset$), and where edges exist only between left and right vertices, that is, $E \subseteq L \times R$. We say that a subset $D \subseteq L$ is a *left-dominating set* iff for every right vertex $v \in R$ there is some $u \in D$ such that $(u, v) \in E$. For example, the highlighted vertices in the bipartite graph below form a left-dominating set of size 2.



Show that the following problem is **NP**-complete:

$$\text{LeftDom} = \{\langle G, k\rangle : G \text{ is a bipartite graph with a left-dominating set of size } k\}.$$

(In your proof, you may assume the **NP***-completeness of any of the problems discussed in the lectures, but* ***not*** *exercises/homework. This includes* SAT, Independent-Set, Clique, Vertex-Cover, Set-Cover, Subset-Sum, *etc. Make sure to prove that your reduction is correct!)*

5 *(20 pts)* **Undecidability.** Recall the notation $w^{\mathcal{R}}$ from **2b**. Define

$$T = \{\langle M \rangle : M \text{ is a TM such that for all } w,\ M \text{ accepts } w \text{ iff it accepts } w^{\mathcal{R}}\}.$$

Show that both T and $\overline{T}$ are unrecognisable.

6 *(15 pts)* **CNF Equivalence.** We say that two CNF formulas φ and ψ, both defined over the same set of n variables $x = (x_1, \ldots, x_n)$, are *equivalent* if they compute the same boolean function, that is, $\varphi(x) = \psi(x)$ for all $x \in \{0,1\}^n$. Consider the language

$$\text{CNF-E}{\scriptstyle\text{Q}} \;=\; \{\langle \varphi, \psi \rangle : \varphi \text{ and } \psi \text{ are a pair of equivalent CNFs}\}.$$

Classify this problem into as small a complexity class as possible: is it in **P**, **NP**, **coNP**, or merely decidable? Is it complete for any complexity class? Justify your answer with a proof.

(Note that, in general, for any class of languages **C***, we say a language A is* **C**-complete *iff $A \in \mathbf{C}$ and we have $L \leq_p A$ for all $L \in \mathbf{C}$.)*