Final Exam, Theory of Computation 2021

- Books, notes, communication, calculators, cell phones, computers, etc... are not allowed.
- Your explanations and proofs should be clear enough and in sufficient detail so that they are easy to understand and have no ambiguities.
- You are allowed to refer to material covered in lectures (but not exercises) including theorems without reproving them.
- **Do not touch until the start of the exam.**

Good luck!

Name: ______________________ **N° Sciper:** ______________

Problem 1 / 15 points	Problem 2 / 20 points	Problem 3 / 20 points	Problem 4 / 15 points	Problem 5 / 15 points	Problem 6 / 15 points

Total / 100

1 *(15 pts)* **Quick-fire round.** Which of the following statements are true?

1. If A is a finite set, then A is regular.
2. The language $\{0^n : n \text{ is divisible by } 2021\}$ is regular.
3. If A is regular and $B \subseteq A$ then B is regular.
4. Let A be undecidable but recognisable. Then $\overline{A}$ is unrecognisable.
5. The language $\{\langle N, w\rangle : N \text{ is an NFA and } N \text{ accepts } w\}$ is undecidable.
6. If A and B are decidable, then $A \cap \overline{B}$ is decidable.
7. If $A \in \mathbf{P}$ then $A^* \in \mathbf{P}$ where $A^* = \{a_1 a_2 \cdots a_n : n \geq 0,\ a_i \in A\}$ is the Kleene star.
8. HALT $\leq_m$ SAT.
9. If $A \leq_m B$ then $\overline{B} \leq_m \overline{A}$.
10. k-SAT $\leq_p$ 3-SAT for any $k \geq 1$.
11. If $\mathbf{P} = \mathbf{NP}$ then $\mathbf{NP} = \mathbf{coNP}$.
12. If SAT $\in \mathbf{P}$ then $\mathbf{P} = \mathbf{NP}$.
13. If A and B are both $\mathbf{NP}$-hard, then $A \leq_p B$.
14. The language $\{\langle n\rangle : n \in \mathbb{N} \text{ and } n \text{ is not a prime number}\}$ is in $\mathbf{NP}$.
15. A circuit of size m can be written equivalently as a CNF formula of size $O(m^2)$.

For each box below, indicate whether the statement is either **T**rue, **F**alse, or if you are uncertain, leave the box empty. A correct answer is worth +1 point, an incorrect answer is worth −1 point, and an empty answer is worth 0 points.

Your answers:

1.	
2.	
3.	
4.	
5.	

6.	
7.	
8.	
9.	
10.	

11.	
12.	
13.	
14.	
15.	

2 *(20 pts)* **Basics of NP.**

2a Write down the definition of the class **NP** and the definition a given problem to be **NP**-complete. If you use the concept of a verifier or an NTM, then explain what it is.

2b The class of problems solvable in deterministic exponential time is defined by

$$\mathbf{EXP} = \bigcup_{k \geq 1} \mathbf{TIME}(2^{n^k}).$$

Show that $\mathbf{NP} \subseteq \mathbf{EXP}$.

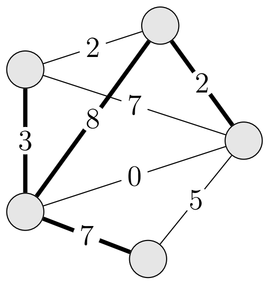
3 *(20 pts)* **Regular languages.** For the following two languages, determine, with proof, whether they are regular. If a language is regular it suffices to draw a DFA or an NFA. You do not need to prove that your automaton is correct.

3a $A = \{0^n : n \text{ is a power of } 2\}$

3b $B = \{w \in \{0,1\}^* : w \text{ has an equal number of occurrences of } 01 \text{ and } 10 \text{ as substrings}\}$.

(For example, 010 *has one occurrence of* 01 *and one occurrence of* 10*, so* $010 \in B$*. On the other hand,* 01101 *has two occurrences of* 01 *and one occurrence of* 10*, so* $01101 \notin B$*.)*

4 *(15 pts)* **NP-completeness.** Let $G = (V, E)$ be a graph and $w\colon E \to \mathbb{N}$ an assignment of non-negative integer weights to the edges. A subset of edges $E' \subseteq E$ is a *spanning tree* if the subgraph (V, E') is a tree (no cycles) that connects all vertices. The weight of the spanning tree is $\sum_{e \in E'} w(e)$. For example, the bold edges below form a spanning tree of weight $3+7+8+2 = 20$.



In the Exact Spanning Tree problem (ESP for short), the input consists of a graph $G = (V, E)$, edge weights $w\colon E \to \mathbb{N}$, and a target $k \in \mathbb{N}$. The goal to decide whether G contains a spanning tree of weight exactly k. That is,

$$\text{ESP} \;=\; \{\langle G, w, k\rangle : G \text{ contains a spanning tree of weight exactly } k\}.$$

Show that ESP is **NP**-complete.

(You may use, without proof, the **NP***-completeness of any problem discussed in lectures.)*

5 *(15 pts)* **Decidability.** Consider the language

$$L = \{\langle M \rangle : M \text{ is a Turing machine that halts on every input}\}$$

Classify L as one of (i) decidable, (ii) undecidable but recognisable, (iii) unrecognisable. Justify your answer with a proof.

6 *(15 pts)* **Circuit complexity.** This problem asks you to show that regular languages admit linear-size circuits. That is, let $L \subseteq \{0,1\}^*$ be a regular language. For any input length $n \in \mathbb{N}$, show how to construct a boolean circuit C_n with n input variables, one output wire, and $O(n)$ gates such that

$$\forall x \in \{0,1\}^n : \quad C_n(x) = 1 \iff x \in L.$$