ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exam III, Theory of Computation 2018-2019

- Books, notes, communication, calculators, cell phones, computers, etc... are not allowed.
- Your explanations and proofs should be clear enough and in sufficient detail so that they are easy to understand and have no ambiguities.
- You are allowed to refer to material covered in the class including theorems without reproving them.
- **Do not touch until the start of the exam.**

Good luck!

Name: ______________________ **N° Sciper:** ______________

Problem 1 / 8 points	Problem 2 / 6 points	Problem 3 / 8 points	Problem 4 / 6 points	Problem 5 / 6 points	Problem 6 / 6 points

Total / 40

1 Basic questions. *(8 pts)* Let $A \subseteq \{0,1\}^*$ and $B \subseteq \{0,1\}^*$ be two languages. Which of the following statements are true?

1. $A \leq_m B \implies A \leq_p B$.
2. If $\mathbf{P} = \mathbf{NP}$, then $\{\langle\phi\rangle \mid \phi$ is a satisfiable CNF formula$\}$ is in $\mathbf{P}$.
3. $\{\langle M, w\rangle \mid M$ is a TM and M doesn't halt on input $w\}$ is in $\mathbf{NP}$.
4. A is recognized by a DFA $\iff$ A is recognized by an NFA.
5. A is regular and $A \leq_m B \implies B$ is regular.
6. A is regular $\implies$ A is decidable $\implies$ A is in $\mathbf{P}$ $\implies$ A is in $\mathbf{NP}$.
7. A is in $\mathbf{P}$ and $B \leq_p A \implies B$ is in $\mathbf{P}$.
8. $A \in \text{TIME}(n^{100}) \implies A \in \mathbf{NP}$.
9. Suppose A is $\mathbf{NP}$-hard and B is $\mathbf{NP}$-complete. If B is in $\mathbf{P}$ then A is in $\mathbf{P}$.
10. A is in $\mathbf{P}$ and B is in $\mathbf{P}$ $\implies$ $A \circ B$ is in $\mathbf{P}$.
11. A is recognizable by a TM $\implies$ its complement $\bar{A}$ is recognizable by a TM.
12. $\{w \in \{0,1\}^* \mid$ number of 1's in w is divisible by 2019$\}$ is regular.

*(A complete solution identifies **all** true statements. A fully correct solution is worth 8 points. A solution with one mistake is worth 7 points. A solution with two mistakes is worth 5 point. A solution with three mistakes is worth 3 points. A solution with four mistakes is worth 1 point. Solutions with more mistakes are worth 0 points. A mistake is to either indicate falsely that a false statement is true or to not indicate that a true statement is true.)*

2 Regular languages. *(6 pts)* Consider the following language on alphabet $\Sigma = \{0, 1\}$:

$$L_2 = \{w \in \{0,1\}^* \mid w \text{ has more 0's than 1's}\} .$$

Is L_2 regular? Provide a proof of your claim.

(In this problem you should decide whether L_2 is regular or not, and to give a ***proof*** *of your claim. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)*

3 Graph coloring. *(8 pts, consisting of subproblems (a) and (b))* A *coloring* of an undirected graph is an assignment of colors to its vertices so that no two adjacent vertices are assigned the same color. For each positive integer k, let

$$k\text{-COL} = \{\langle G\rangle \mid \ G \text{ is colorable with } k \text{ colors}\}\,.$$

In Homework 3 you proved that 3-COL is **NP**-complete. In this problem we will consider the related languages 2-COL and 4-COL. Specifically, we will show that 2-COL is in **P** whereas 4-COL is **NP**-hard.

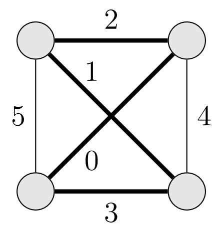
3a *(3 pts)* Show that 2-COL is in **P**.

(In this problem you should show that 2-COL is in **P***. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)*

3b *(5 pts)* Show that 3-COL $\leq_p$ 4-COL.

(In this problem you should describe a polynomial time reduction from 3*-COL to* 4*-COL and prove its correctness. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)*

4 The traveling salesman problem. *(6 pts)* An instance of the classic traveling salesman problem is defined by a tuple $H = (V, d)$ where V is a set of cities and $d : V \times V \to \mathbb{Z}$ defines their pairwise distances. That is for two cities $i \in V$ and $j \in V$, $d(i, j) = d(j, i)$ equals the distance required to travel between cities i and j. Notice that H can be thought of as a complete graph (every possible edge is present) with weights on the edges corresponding to the distances. A tour is a cycle that visits every city *exactly* once, i.e., it is a cycle of length $|V|$. The length of a tour is the total distance travelled which equals the sum of the distances of the edges in the tour. An example is depicted below. The instance has 4 cities and the thick edges depict a tour of length $0 + 2 + 1 + 3 = 6$.



Show that the following language is **NP**-complete:

$$\text{TSP} = \{\langle H = (V, d), k\rangle \mid H \text{ has a tour of length } \textit{at most } k\}.$$

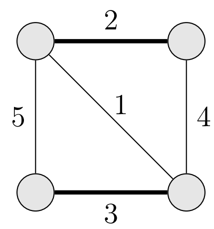
In your proof you may use that the following language is **NP**-complete:

$$\text{HAM} = \{\langle G\rangle \mid G \text{ is an undirected graph that is Hamiltonian}\}.$$

An undirected graph $G = (V, E)$ is said to be Hamiltonian if it contains a cycle of length $|V|$, i.e., a cycle that visits every vertex. Two examples of Hamiltonian graphs are the complete graph and the graph consisting of a single cycle of length $|V|$.

(In this problem you are asked to prove that the language TSP is **NP***-complete. Recall that you are allowed to use that HAM is* **NP***-complete and to refer to material covered in the class including theorems without reproving them.)*

5 Perfect matchings of a given weight. *(6 pts)* Let $G = (V, E, w)$ be an undirected graph with edge-weights $w : E \to \mathbb{Z}$. A subset $M \subseteq E$ of the edges is a *perfect matching* if every vertex is incident to exactly one edge in M. In other words, the edges in M pair up all the vertices. The weight of a matching M is the total weight of its edges, i.e., $\sum_{e \in M} w(e)$. An example is depicted below. The graph consists of 4 vertices, 5 edges with integer weights, and the thick edges indicate a perfect matching of weight $2 + 3 = 5$.



Show that the following language is **NP**-complete:

$$L_5 = \{\langle G = (V, E, w), k\rangle \mid G \text{ has a perfect matching of weight } k\}.$$

(In this problem you are asked to show that the language L_5 *is* **NP***-complete. Recall that you are allowed to refer to material covered in the class including theorems without reproving them. In particular, you are allowed to use any of the* **NP***-complete languages that we saw in class.)*

6 Decidability. *(6 pts)* Consider the following language on alphabet $\Sigma = \{0, 1\}$:

$$L_6 = \{\langle M \rangle \mid M \text{ is a TM and the language recognized by } M \text{ is } \mathbf{NP}\text{-complete}\}.$$

Is L_6 decidable? Provide a proof of your claim.

(In this problem you should decide whether L_6 is decidable or not, and give a ***proof*** *of your claim. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)*