Final Exam, Theory of Computation 2019-2020

- Books, notes, communication, calculators, cell phones, computers, etc... are not allowed.
- Your explanations and proofs should be clear enough and in sufficient detail so that they are easy to understand and have no ambiguities.
- You are allowed to refer to material covered in the class including theorems without reproving them.
- **Do not touch until the start of the exam.**

Good luck!

Name: ____________________ **N° Sciper:** ________________

Problem 1 / 18 points	Problem 2 / 12 points	Problem 3 / 20 points	Problem 4 / 15 points	Problem 5 / 20 points	Problem 6 / 15 points

Total / 100

1 *(consisting of subproblems* ***a-b****, 18 pts)* **Basic questions.**

1a *(8 pts)* Write the formal definition of the class **NP**. If you use the concept of a verifier or a non-deterministic Turing machine, then explain what it is.

Solution:

1b *(10 pts)* Which of the following statements are true?

1. All regular languages are in **NP**.
2. All languages in **P** are regular.
3. All irregular languages are decidable.
4. All decidable languages are in **NP**.
5. All decidable languages are recognizable.
6. If a language L is decidable, then its complement $\bar{L}$ is decidable.
7. If a language L is recognizable, then its complement $\bar{L}$ is recognizable.
8. If a language is **NP**-hard, then it is **NP**-complete.
9. The language $\{w \in \{0,1\}^* \mid \text{number of 0's in } w \text{ is divisible by } 2020\}$ is regular.
10. The language $\{0^n 1^n \mid n \geq 0\}$ is in **NP**.
11. $\{\langle M \rangle \mid M \text{ is a TM that halts on all inputs of length at least } 2020\}$ is recognizable.
12. If languages L_1 and L_2 are unrecognizable then $L_1 \cup L_2$ is unrecognizable.

(A complete solution identifies ***all*** *true statements. A fully correct solution is worth* 10 *points. A solution with one mistake is worth* 9 *points. A solution with two mistakes is worth* 7 *point. A solution with three mistakes is worth* 4 *points. A solution with four mistakes is worth* 1 *point. Solutions with more mistakes are worth* 0 *points. A mistake is to either indicate falsely that a false statement is true or to not indicate that a true statement is true.)*

Solution:

Among the above statements, the following are true ____________________________

(In this problem you do not need to justify your answer.)

2 *(12 pts)* **NP-completeness.** Prove that the following language LONGCYCLE is **NP**-complete:

$$\text{LONGCYCLE} = \{\langle G \rangle \mid G \text{ is an undirected graph with a cycle that visits at least half the vertices}\}.$$

In your proof you may use that the following language is **NP**-complete:

$$\text{HAM} = \{\langle G \rangle \mid G \text{ is an undirected graph that is Hamiltonian}\}.$$

An undirected graph $G = (V, E)$ is said to be Hamiltonian if it contains a cycle of length $|V|$, i.e., a cycle that visits every vertex. Two examples of Hamiltonian graphs are the complete graph and the graph consisting of a single cycle of length $|V|$.

(In this problem you are asked to prove that the language LONGCYCLE is **NP***-complete. Recall that you are allowed to use that HAM is* **NP***-complete and to refer to material covered in the class including theorems without reproving them.)*

3 *(consisting of subproblems **a**-**b**, 20 pts)* **Reductions.** In the next two subproblems, we are going to consider reductions between languages INDSET and ODD. Recall from class that INDSET denotes the ***NP**-complete* language

$$\text{INDSET} = \{\langle G, k\rangle \mid G \text{ is an undirected graph with an independent set of size } k\}$$

and we define ODD to be the following *regular language*

$$\text{ODD} = \{w \in \{0,1\}^* \mid w \text{ has an odd number of 1's}\}\,.$$

3a *(10 pts)* In this subproblem, we are going to show that if INDSET is *poly-time* mapping reducible to ODD then $\mathbf{P} = \mathbf{NP}$. Specifically, your task is to prove the following statement:

If INDSET $\leq_p$ ODD then $\mathbf{P} = \mathbf{NP}$.

(In this problem you are asked to prove that if INDSET $\leq_p$ ODD then $\mathbf{P} = \mathbf{NP}$*. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)*

Solution:

3b *(10 pts)* In this subproblem, we are going to show that INDSET is mapping reducible to ODD. Specifically, your task is to show that INDSET $\leq_m$ ODD.

(In this problem you are asked to give a mapping reduction from INDSET to ODD. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)

Solution:

4 *(15 pts)* **Regular languages.** Given a language $A \subseteq \{0,1\}^*$, let

$$B = \{\text{EVEN}(w) \mid w \in A\},$$

where $\text{EVEN}(w)$ denotes the string with only the even-positioned letters of w. For example, $\text{EVEN}(11101011) = 1001$. Describe an NFA that shows that B is regular if A is regular.

(In this problem you are asked to construct an NFA that shows that if A is regular then so is B.)

Solution:

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5 *(20 pts)* **Computability.** Classify the following language into one of: decidable, undecidable but recognizable, unrecognizable.

$$\text{SLOWHALT} = \{\langle M, x\rangle \mid \ M \text{ is a TM that halts on input } x \text{ after taking at least 2020 steps}\}\,.$$

Justify your answer with a formal proof.

(In this problem, you are asked to identify whether SLOWHALT is (decidable and recognizable), (undecidable and recognizable), or (unrecognizable) and provide a formal correctness proof. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)

Solution:

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6 *(15 pts)* **Large DFAs.** Let LARGE $= \{0^n 1^n \mid 1 \leq n \leq 10^6\}$. That is LARGE contains all strings that have n zeros followed by n ones where n is an integer between 1 and 10^6. As LARGE contains a finite number of strings (10^6 many strings), LARGE is a regular language. However, in this problem you are asked to prove that there is no "small" DFA that recognizes LARGE. More specifically, your task is to prove the following statement:

> Any DFA that recognizes LARGE has at least 10^5 states.

(In this problem you should give a proof of the statement that any DFA that recognizes LARGE has at least 10^5 states. Recall that you are allowed to refer to material covered in the class including theorems without reproving them.)

Solution:

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