Exercise I, Algorithms 2024-2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

Basic Algorithms

1 (Based on Exercise 2.1-3 in the book) Consider the **searching problem:**

Input: A sequence of n numbers $A = \langle a_1, a_2, \ldots, a_n \rangle$ and a value v.

Output: An index i such that $v = A[i]$ or the special value NIL if v does not appear in A.

1a Which of the following are instances of the problem and what are the correct outputs for those instances?

$$\langle 1, 5, 3, 4 \rangle \text{ and a value } v$$
$$\langle 3, 8, 3, 4, 50, 47 \rangle \text{ and a value } 50$$
$$\langle 3, 8, 3, 4, 49, 47 \rangle \text{ and a value } 9$$

Solution. (i) Input v is a variable, not a constant number. Therefore, this is not a valid instance of the problem.
(ii) The input consists of a sequence A of numbers and a constant value of 50 to be searched in A. This is a valid instance of the problem and the output is 5.
(iii) The input consists of a sequence A of numbers and a constant value of 9 to be searched in A. This is a valid instance of the problem and the output is NIL since number 9 is not in the sequence.
□

1b Write pseudocode for **linear search**, which scans through the sequence, looking for v. Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the three necessary properties (Initialization, Maintenance, Termination).

Solution. Pseudocode for linear search is:

```
Linear-Search(A,v)
  for i ← 1 to length(A)
    if A[i] = v then
      return i
  return NIL
```

Loop invariant: At the start of each iteration of the **for** loop we have $A[j] \neq v$ for all $j < i$.

Initialization: Before the first iteration of the **for** loop, the loop invariant is trivially true because the statement is empty.

Maintenance: Suppose that the loop invariant is true before the i-th iteration of the **for** loop and that there is a $(i+1)$-st iteration of the **for** loop. From the former we know that $A[j] \neq v$ for all $j < i$ and from the latter we know that $A[i] \neq v$. We conclude that $A[j] \neq v$ for all $j < i + 1$ before the $(i + 1)$-st iteration of the **for** loop as claimed.

Termination: There are two cases: The first case is that the **for** loop terminates with $i \leq length(A)$ and the algorithm outputs i. In this case the loop invariant ensures that $A[j] \neq v$ for $j < i$ and the **if** condition ensures that $A[i] = v$. Hence the output i is correct. The second case is that $i = length(A) + 1$ and the algorithm outputs NIL. In this case the loop invariant ensures that $A[j] \neq v$ for $j < length(A) + 1$. Hence the output NIL is correct. □

1c Analyze the worst-case running time of your algorithm in terms of Θ-notation.

Solution. There is one **for** loop in the algorithm which always starts from $i = 1$, and, in the worst case, iterates until $i = n + 1$ (i.e., when the value v is not in the list). The **if** condition inside the loop and the increment of i at the end of each iteration both run in constant time, i.e., $O(1)$. The maximum number of iterations is $n + 1$ and this dominates the constant running time of primitive lines of the algorithm. As a result, the running time of the algorithm is lower-bounded by some function $c_1 \cdot n$ and upper-bounded by some other function $c_2 \cdot n$ whenever $n \geq n_0$, where $c_1 < c_2$, and c_1, c_2, n_0 are positive constants. Hence, the worst-case running time of linear search is $\Theta(n)$. □

Asymptotics

2 Show that for any real constants a and b, where $b > 0$,

$$(n + a)^b = \Theta(n^b).$$

Solution. Suppose $a > 0$. Then, since $(n+a)^b \geq n^b$ for all $n > 0$, we obviously have $(n+a)^b = \Omega(n^b)$. Let us fix some $N = a$, then for all $n > N$, we have

$$(n+a)^b \leq (2n)^b = 2^b n^b.$$

Thus, $(n+a)^b = O(n^b)$ as well and therefore $(n+a)^b = \Theta(n^b)$.

Now, if $a < 0$, we have that $n^b = ((n+a)-a)^b = \Theta((n+a)^b)$ by the previous part, which concludes the proof. □

3 Simplify and arrange the following functions in increasing order according to asymptotic growth.

$$3^N, \sqrt{4^N}, \log^2 N, 2^{N\log_2 N}, \sqrt{N}, N^2, \log N, 20N, N!, (N/e)^N$$

Solution.

- First the logs: $\log N, \log^2 N$
- Second the polynomials: $\sqrt{N}, 20N, N^2$
- Then the exponential functions: $\sqrt{4^N} = 2^N, 3^N$
- Finally, $(N/e)^N, N!$ and $2^{N\log_2 N} = N^N$. This is because $N! = \sqrt{2\pi N}(N/e)^N(1+o(1))$ by Stirling's formula, so $(N/e)^N = o(N!)$ and $N! = o(N^N)$

□

4 Express the following functions in terms of Θ-notation:

$$3^N + 2^N, \sqrt{3^N + 4^N}, \log^2(N^3 + 300N^2), \log\binom{N}{2}$$

Solution.

For the first function: $3^N + 2^N = \Theta(3^N)$, as $3^N \leq 3^N + 2^N \leq 2 \cdot 3^N$ for all $N \geq 1$.

For the second function: $\sqrt{3^N + 4^N} = \Theta(2^N)$. This is because $\sqrt{3^N + 4^N} \geq \sqrt{4^N} = 2^N$ for all N, and $3^N \leq 4^N$ for $N \geq 0$, so $3^N + 4^N = O(4^N)$, and hence $\sqrt{3^N + 4^N} = O(2^N)$.

For the third function $\log^2(N^3 + 300N^2) = \Theta(\log^2 N)$. This is because $\log^2(N^3 + 300N^2) \geq \log^2(N^3) = \Omega(\log^2 N)$. At the same time $N^3 + 300N^2 \leq 2N^3$ for all $N \geq 300$, so $N^3 + 300N^2 = O(N^3)$.

For the fourth function $\log\binom{N}{2} = \Theta(\log N^2) = \Theta(\log N)$. Indeed, for all $N \geq 2$ one has $N^2/4 \leq \binom{N}{2} \leq N^2/2$. In particular, this means that $\log\binom{N}{2} \geq 2\log N - O(1) \geq \log N$ for sufficiently large N, so $\log\binom{N}{2} = \Omega(\log N)$. Similarly $\log\binom{N}{2} \leq 2\log N = O(\log N)$. □

Proof by Induction

5 Prove the following equalities using induction:

5a $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution. **Base case:** Clearly, $1^2 = \frac{1(2)(3)}{6}$.

Induction step: Suppose now that $n > 1$ and the equality holds for $n - 1$. On the one hand,

$$
\begin{aligned}
1^2 + 2^2 + ... + (n-1)^2 + n^2 &= \frac{(n-1)n(2n-1)}{6} + n^2 \\
&= \frac{2n^3 - 3n^2 + n + 6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6}.
\end{aligned}
$$

On the other hand, $n(n+1)(2n+1) = 2n^3 + 3n^2 + n$.

Conclusion: We can then conclude with weak induction. □

5b The Fibonacci series is recursively defined for $n \geq 1$ by

$$f_1 = 1, f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, n \geq 2.$$

Show the Binet formula:

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \qquad \text{for all } n \geq 1$$

Hint: $\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$ and $\frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$

Solution. **Base Case:** If $n = 1$,

$$\frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\frac{2\sqrt{5}}{2} = 1 = f_1.$$

If $n = 2$,

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}\frac{2\sqrt{5}}{2} = 1 = f_2.$$

Induction Step: Suppose further that $n > 1$ and the formula holds for all $k < n$. Then, by the definition of a fibonacci number and by the induction step,

$$\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}+1\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{3-\sqrt{5}}{2}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^2\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right].
\end{aligned}$$

Conclusion: By strong induction, we can conclude the proof. Note that at each step we only need the two previous ones; this is also why we have to prove two cases in the base step. □

6 (*) What is wrong with the following proof that all horses have the same color?

> Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.
>
> Base case: Clearly, $P(1)$ is true.
>
> Now assume that $P(n)$ is true. That is, assume that all the horses in any set of n horses are the same color. Consider any $n+1$ horses; number these as horses $1, 2, 3, ..., n, n+1$. Now the first n of these horses all must have the same color, and the last n of these must also have the same color. Since the set of the first n horses and the set of the last n horses overlap, all $n+1$ must be the same color. This shows that $P(n+1)$ is true and finishes the proof by induction.

Solution. The problem is that the "proof" that $P(n) \Rightarrow P(n+1)$ does not hold when $n = 1$. This is because the argument given above is based on the "fact" that given $n+1$ elements, any two subsets of size n overlap or have at least one element in common. While this is true for $n+1 \geq 3$, it is not true when $n+1 = 2$. In this case, there are two subsets of size one and it is easy to see that these subsets are disjoint. While $P(1)$ holds for each of these subsets, i.e. elements, each element can have a different color. □

A Refresh "Proof by Induction"

Main principles. Recall that the principle of weak induction is

1. The statement is true for a base case (say an integer b).
2. Any time the statement is true for $n \geq b$, one can show that the statement is true for $n+1$.

Under these conditions, the statement is true for all integers $\geq b$.

Similarly, the principle of strong induction is:

1. The statement is true for one or several base cases (say integers $b, b+1, \ldots, b+i$).
2. Any time the statement is true for all integers in $[b, n]$ for some $n \geq b+i$, one can show that the statement is true for $n+1$.

Under these conditions, the statement is true for all integers $\geq b$.

This looks rather technical so let us clarify these concepts with two examples[1]

Example A.1 (weak induction) *Prove by induction that for all positive integers $n \in \mathbb{Z}_+$*

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}. \tag{1}$$

Proof. We will prove (**??**) by weak induction.

Base case: When $n = 1$, the left side of (**??**) is $1/(1 \cdot 2) = 1/2$, and the right side is $1/2$, so both sides are equal and (**??**) is true for $n = 1$.

Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose (**??**) is true for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \textit{(by induction hypothesis)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \quad \textit{(by algebra)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \quad \textit{(by algebra)} \\
&= \frac{k+1}{k+2}.
\end{aligned}$$

Thus, (**??**) holds for $n = k+1$, and the proof of the induction step is complete.

Conclusion: By the principle of (weak) induction, (**??**) is true for all $n \in \mathbb{Z}_+$. □

Example A.2 (strong induction) *The fibonacci series is recursively defined for $n \geq 1$ by*

$$f_1 = 1, f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, n \geq 2.$$

We have for all $n \in \mathbb{Z}_+$ that

$$f_n \geq (3/2)^{n-2}. \tag{2}$$

[1] These examples are taken from *http://www.math.uiuc.edu/ hildebr/213/inductionsampler.pdf*. See that page for many more examples.

Proof. **Base cases:** When $n = 1$, the left side of (**??**) is $f_1 = 1$, and the right side is $(3/2)^{-1} = 2/3$, so (**??**) holds for $n = 1$. When $n = 2$, the left side of (**??**) is $f_2 = 1$ and the right side is $(3/2)^0 = 1$, so both sides are equal and (**??**) is true for $n = 2$.

Induction step: Let $k \geq 2$ be given and suppose (**??**) is true for all $n = 1, 2 \ldots, k$. Then

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \quad \textit{(by recurrence for } f_n\textit{)} \\
&\geq (3/2)^{k-2} + (3/2)^{k-3} \quad \textit{(by induction hypothesis with } n = k \textit{ and } n = k-1\textit{)} \\
&= (3/2)^{k-1} \left((3/2)^{-1} + (3/2)^{-2}\right) \quad \textit{(by algebra)} \\
&= (3/2)^{k-1} \left(\frac{2}{3} + \frac{4}{9}\right) \\
&= (3/2)^{k-1} \frac{10}{9} \geq (3/2)^{k-1}.
\end{aligned}$$

Thus, (**??**) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of strong induction, it follows that (**??**) is true for all $n \in \mathbb{Z}_+$.

□

Remarks: Number of base cases: Since the induction step involves the cases $n = k$ and $n = k - 1$, we can carry out this step only for values $k \geq 2$ (for $k = 1, k - 1$ would be 0 and out of range). This in turn forces us to include the cases $n = 1$ and $n = 2$ in the base step. Such multiple base cases are typical in proofs involving recurrence sequences. For a three term recurrence we would need to check three initial cases, $n = 1, 2, 3$, and in the induction step restrict k to values 3 or greater.