Algorithms: Recall Binary Search Trees and a Dynamic Programming

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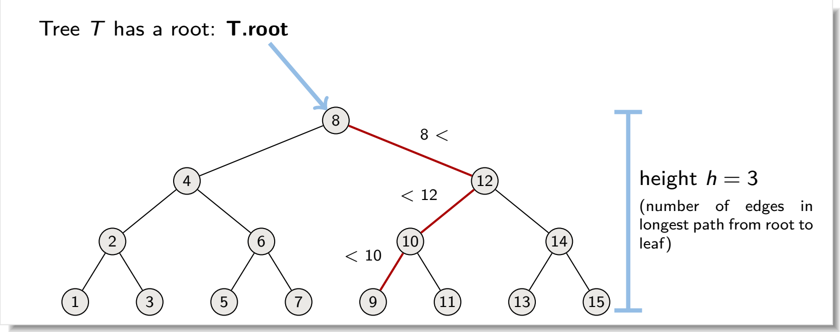
RECALL BINARY SEARCH TREES

Binary Search Trees

Key property:

- If y is in the left subtree of x then $y.key < x.key$
- If y is in the right subtree of x then $y.key \geq x.key$



Binary Search Trees

Encodes a strategy whatever number we look for

Key property:

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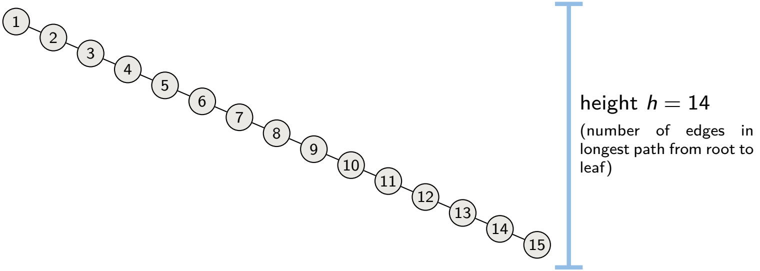
- If y is in the right subtree of x then $y.key \geq x.key$

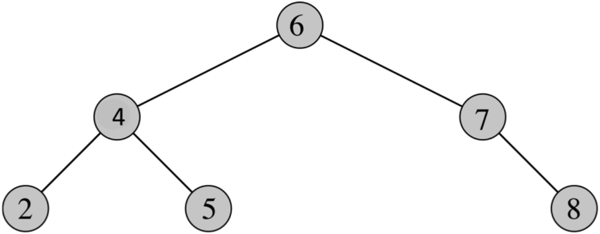


Basic operations take time proportional to height: $O(h)$

QUERYING A BINARY SEARCH TREE

(Searching, Minimum, Maximum, Successor, Predecessor)

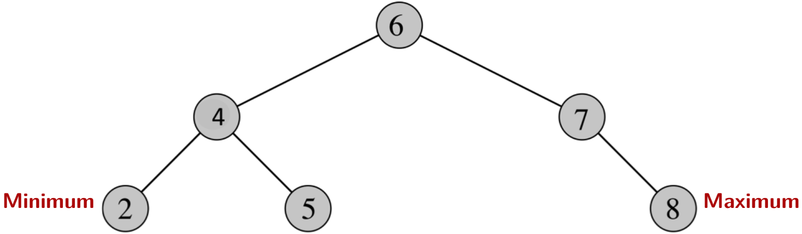
Searching



What is the running time? $O(h)$

```
TREE-SEARCH(x, k)
  if x == NIL or k == key[x]
      return x
  if k < x.key
      return TREE-SEARCH(x.left, k)
  else return TREE-SEARCH(x.right, k)
```

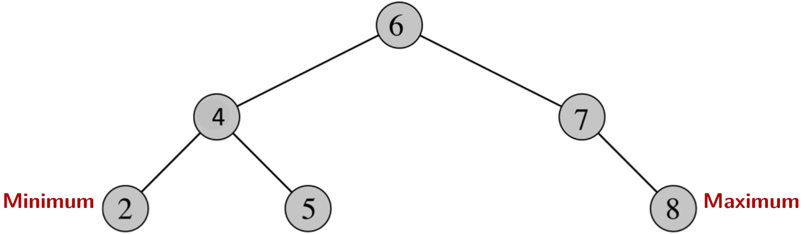
Minimum and Maximum



By key property:

- Minimum is located in leftmost node
- Maximum is located in rightmost node

Minimum and Maximum

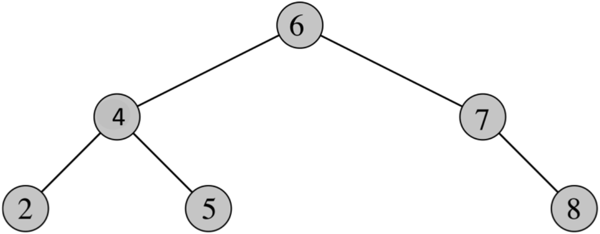


What is the running time? $O(h)$

TREE-MINIMUM(x)
while $x.left \neq$ NIL
$x = x.left$
return x

TREE-MAXIMUM(x)
while $x.right \neq$ NIL
$x = x.right$
return x

Successor



Successor of a node x is the node y such that $y.key$ is the

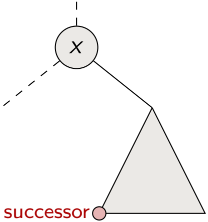
$$\text{“smallest key”} > x.key$$

- What is the successor of 6?
- What is the successor of 5?

Two cases when finding successor of x:

Case 1: x has a non-empty right subtree

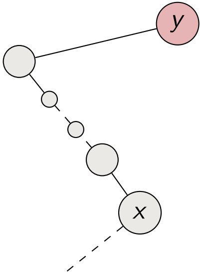
x's successor is the minimum in the right subtree



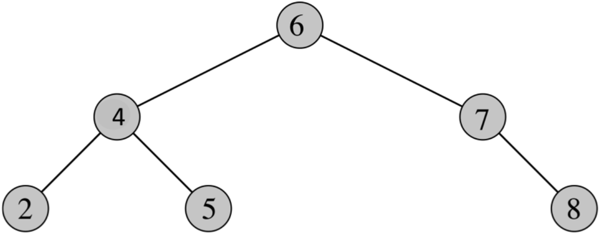
Case 2: x has an empty right subtree

As long as we go to the left up the tree we're visiting smaller keys

x's successor is y is the node that x is the predecessor of (x is the maximum in y's left subtree)



Successor (Predecessor is symmetric)



What is the running time? $O(h)$

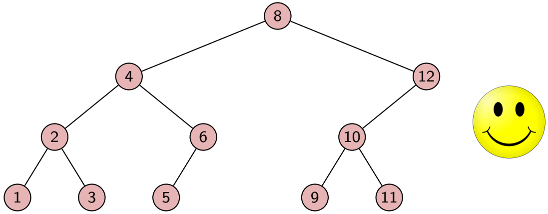
```
TREE-SUCCESSOR(x)
  if x.right ≠ NIL
      return TREE-MINIMUM(x.right)
  y = x.p
  while y ≠ NIL and x == y.right
      x = y
      y = y.p
  return y
```

PRINTING A BINARY SEARCH TREE

(Inorder, Preorder, Postorder)

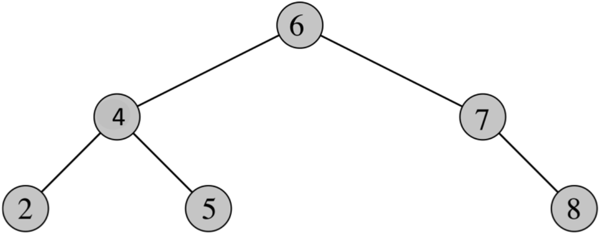
Printing Inorder (Idea)

- Print left subtree recursively
- Print root
- Print right subtree recursively



Output: 1,2,3,4,5,6,8,9,10,11,12

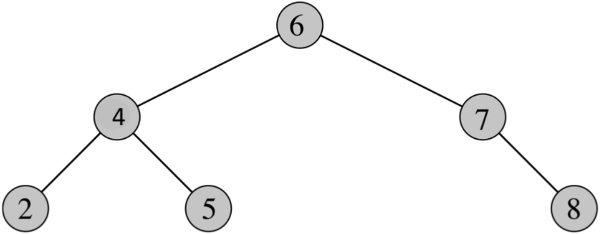
Inorder tree walk



What is the running time? $\Theta(n)$

```
INORDER-TREE-WALK(x)
  if x ≠ NIL
      INORDER-TREE-WALK(x.left)
      print key[x]
      INORDER-TREE-WALK(x.right)
```

Printing Preorder and Postorder



PREORDER-TREE-WALK(x)

1. **if** $x \neq NIL$
2. print $key[x]$
3. PREORDER-TREE-WALK($x.left$)
4. PREORDER-TREE-WALK($x.right$)

POSTORDER-TREE-WALK(x)

1. **if** $x \neq NIL$
2. POSTORDER-TREE-WALK($x.left$)
3. POSTORDER-TREE-WALK($x.right$)
4. print $key[x]$

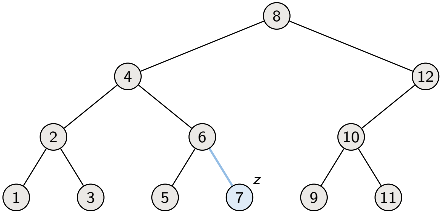
MODIFYING A BINARY SEARCH TREE

(Insertion and Deletion)

Idea of inserting z

- Search for $z.key$
- When arrived at *nil* insert z at that position

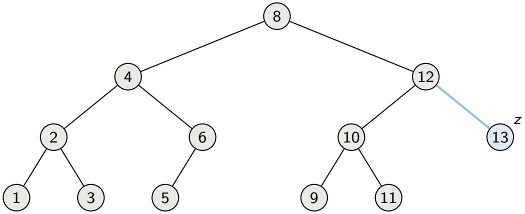
Ex: insert z with key 7



Idea of inserting z

- Search for $z.key$
- When arrived at *nil* insert z at that position

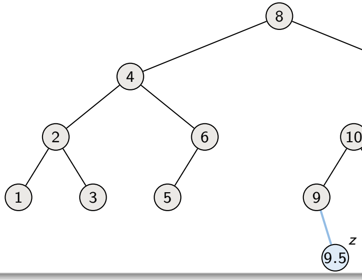
Ex: insert z with key 13



Idea of inserting z

- Search for $z.key$
- When arrived at *nil* insert z at that position

Ex: insert z with key 9.5



Insertion

"search" phase

"insert" phase

```
TREE-INSERT(T, z)
  y = NIL
  x = T.root
  while x ≠ NIL
      y = x
      if z.key < x.key
          x = x.left
      else x = x.right
  z.p = y
  if y == NIL
      T.root = z          // tree T was empty
  elseif z.key < y.key
      y.left = z
  else y.right = z
```

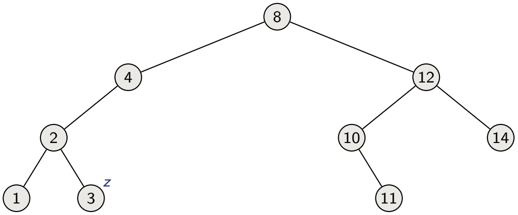
What is the running time? $O(h)$

Idea of deletion

Conceptually 3 cases:

- If z has no children, remove it

Ex: Delete z



Idea of deletion

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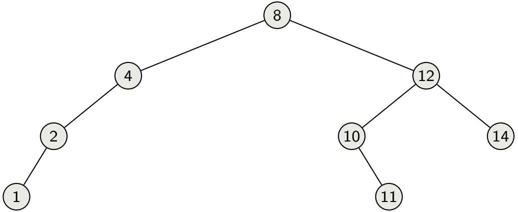
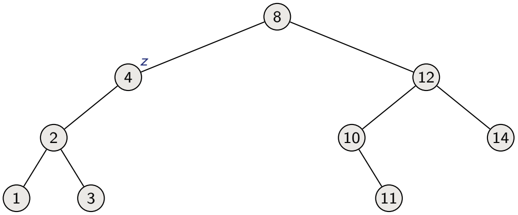


Idea of deletion

Conceptually 3 cases:

- If z has no children, remove it
- If z has one child, then make that child take z's position in the tree

Ex: Delete z



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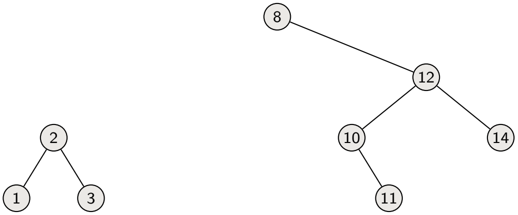

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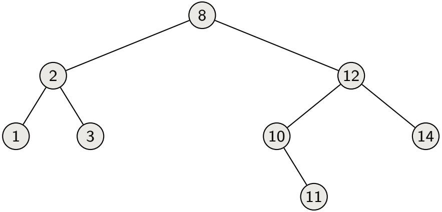
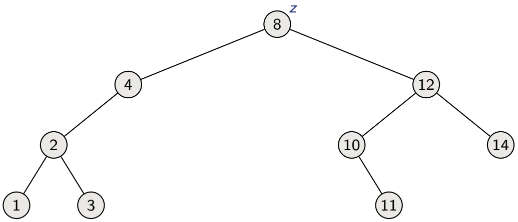


Idea of deletion

Conceptually 3 cases:

- If z has no children, remove it
- If z has one child, then make that child take z's position in the tree
- If z has two children, then find its successor y and replace z by y

Ex: Delete z



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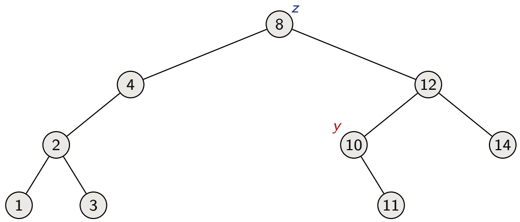

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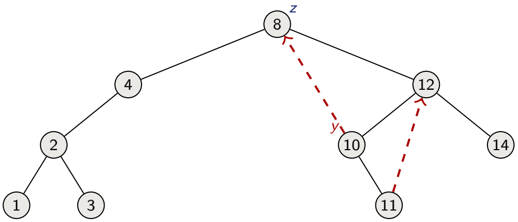

Idea of deletion

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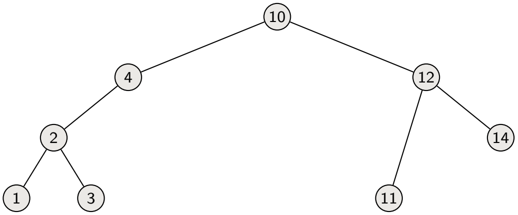
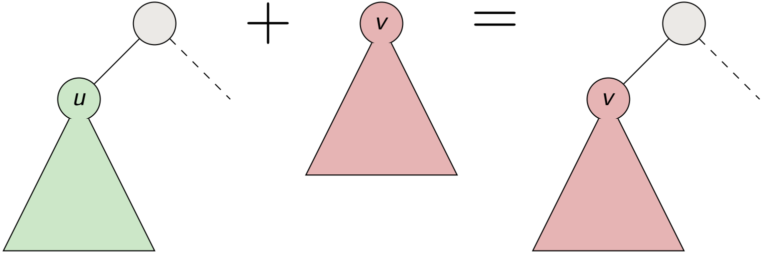
Ex: Delete z



Deletion Implementation: Transplant

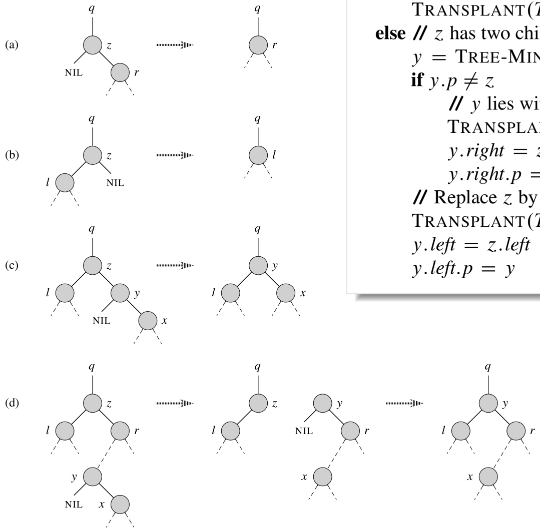
TRANSPLANT(T, u, v)
if $u.p$ == NIL
    $T.root = v$
elseif u == $u.p.left$
    $u.p.left = v$
else $u.p.right = v$
if $v \neq$ NIL
    $v.p = u.p$

TRANSPLANT(T, u, v) replaces subtree rooted at u with that rooted at v



Deletion Procedure

```
TREE-DELETE(T, z)
  if z.left == NIL
      TRANSPLANT(T, z, z.right)          // z has no left child
  elseif z.right == NIL
      TRANSPLANT(T, z, z.left)           // z has just a left child
  else // z has two children.
      y = TREE-MINIMUM(z.right)          // y is z's successor
      if y.p ≠ z
          // y lies within z's right subtree but is not the root of this
          TRANSPLANT(T, y, y.right)
          y.right = z.right
          y.right.p = y
      // Replace z by y.
      TRANSPLANT(T, z, y)
      y.left = z.left
      y.left.p = y
```

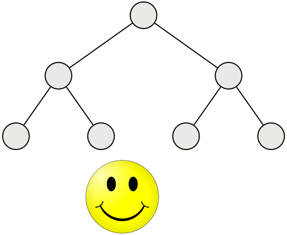


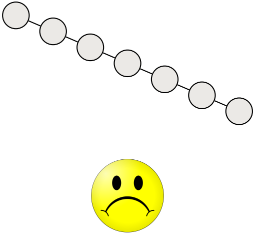
Summary of Binary Search Trees

Query operations: Search, Max, Min, Predecessor, Successor: **O(h)** time

Modifying operations: Insertion, Deletion: **O(h)** time

Exist efficient procedures to keep tree balanced (AVL trees, red-black trees, etc.)





Comparison of Data Structures

Stacks: Last-in-first-out, Insertion and deletion $O(1)$ time,
Array implementation: fixed capacity

Queues: First-in-first-out, Insertion and deletion $O(1)$ time,
Array implementation: fixed capacity

Linked List: No fixed capacity, Insertion and deletion $O(1)$ time, supports search but $O(n)$ time

Binary Search Trees: No fixed capacity, supports most operations (insertion, deletion, search, max, min, ...) in time $O(\text{height of tree})$

DYNAMIC PROGRAMMING

(An algorithmic paradigm not a way of "programming")

What is $2^5 + 3 - \sqrt{16}$?

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What is $2^5 + 3 - \sqrt{16}$?

What is $2^5 + 3 - \sqrt{16}$?

Dynamic Programming (DP)

Main idea:

- Remember calculations already made
- Saves enormous amounts of computation

Allows to solve many optimization problems

- Always at least one question in google code jam needs DP

First application: Fibonacci numbers

Sequence of numbers defined 1000 years ago:

$$F_0 = 1$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

1, 1, 2, 3, 5, 8, 13, 21, ?

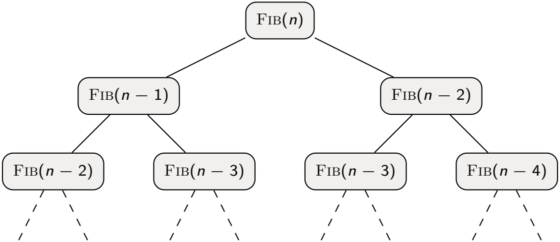
Calculating the n-th Fibonacci number

First idea:

```
Fib(n)

if n = 0 or n = 1
    return 1
else
    return Fib(n − 1) + Fib(n − 2)
```

What is the problem? Same calculations again and again $\Rightarrow$ exponential time!



The solution

Remember what we have done

Two different ways:

1. Top-down with memoization
   - Solve recursively but store each result in a table
   - Memoizing is remembering what we have computed previously
2. Bottom-up
   - Sort the subproblems and solve the smaller ones first
   - That way, when solving a subproblem, have already solved the smaller subproblems we need

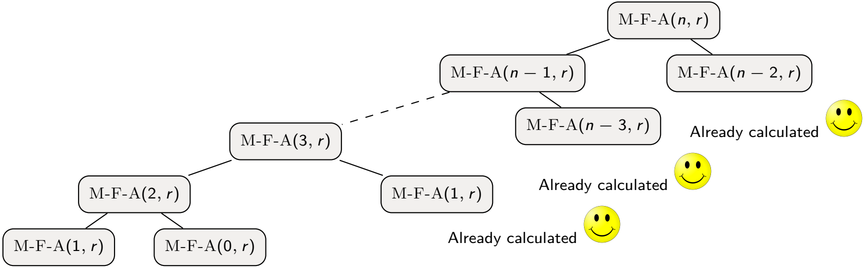
Top-down with memoization: Fibonacci numbers

MEMOIZED-FIB(n)

```
Let r = [0 . . . n] be a new array
for i = 0 to n
    r[i] ← −∞
return MEMOIZED-FIB-AUX(n, r)
```

MEMOIZED-FIB-AUX(n, r)

```
if r[n] ≥ 0
    return r[n]
if n = 0 or n = 1
    ans ← 1
else
    ans ← MEMOIZED-FIB-AUX(n − 1, r)+
                MEMOIZED-FIB-AUX(n − 2, r)
r[n] ← ans
return r[n]
```



Top-down with memoization: Fibonacci numbers

MEMOIZED-FIB(n)

1. Let $r = [0 \ldots n]$ be a new array
2. **for** $i = 0$ **to** n
3. $r[i] \leftarrow -\infty$
4. **return** MEMOIZED-FIB-AUX(n, r)

MEMOIZED-FIB-AUX(n, r)

1. **if** $r[n] \geq 0$
2.   **return** $r[n]$
3. **if** $n = 0$ or $n = 1$
4. $ans \leftarrow 1$
5. **else**
6. $ans \leftarrow$ MEMOIZED-FIB-AUX($n-1, r$)+ MEMOIZED-FIB-AUX($n-2, r$)
7. $r[n] \leftarrow ans$
8. **return** $r[n]$

Time analysis:

- Steps 1-3 in MEMOIZED-FIB take time $\Theta(n)$
- Each call to MEMOIZED-FIB-AUX takes time $\Theta(1)$
- Number of calls to MEMOIZED-FIB-AUX is $\Theta(n)$
- Total time is thus $\Theta(n)$

Bottom-up: Fibonacci numbers

```
BOTTOM-UP-FIB(n)

Let r = [0 . . . n] be a new array
r[0] ← 1
r[1] ← 1
for i = 2 to n
    r[i] ← r[i − 1] + r[i − 2]
return r[n]
```

Example $n = 8$:

$r =$

1	1	2	3	5	8	13	21

Time? $\Theta(n)$

Summary

- We had a recursive formulation of our problem

$$F_0 = 1$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

- Introduced memory (array r)
- Filled in table"top-down with memoization" or with "bottom-up"

Key elements in designing a DP-algorithm

Optimal substructure

- Show that a solution to a problem consists of making a choice, which leaves one or several subproblems to solve and the optimal solution solves the subproblems optimally

Overlapping subproblems

- A naive recursive algorithm may revisit the same (sub)problem over and over.
- Top-down with memoization

  Solve recursively but store each result in a table
- Bottom-up

  Sort the subproblems and solve the smaller ones first; that way, when solving a subproblem, have already solved the smaller subproblems we need

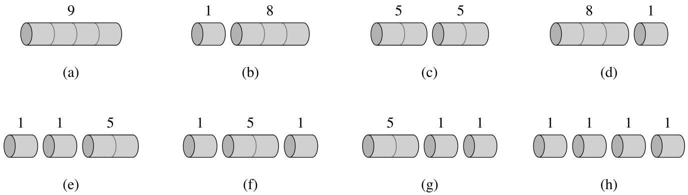
ROD CUTTING

Rod cutting

Instance:
- A length n of a metal rod.
- A table of prices p_i for rods of lengths $i = 1, \ldots, n$.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Objective: Decide how to cut the rod into pieces and maximize the price.



Size of the Problem

- There 2^{n-1} possible solutions—either cut or do not cut after every length unit.
- Need structure for an efficient algorithm.

Theorem

If:

- *the leftmost cut in an optimal solution is after i units.*
- *an optimal way to cut a solution of size $n - i$ is into rods of sizes: $s_1, s_2, \ldots, s_k$.*

Then, an optimal way to cut our rod is into rods of sizes: $i, s_1, s_2, \ldots, s_k$.

Proof of Structural Theorem

Theorem

If:

- *the leftmost cut in an optimal solution is after i units.*
- *an optimal way to cut a solution of size $n - i$ is into rods of sizes: $s_1, s_2, \ldots, s_k$.*

Then, an optimal way to cut our rod is into rods of sizes: $i, s_1, s_2, \ldots, s_k$.

Proof

Feasibility: Since $s_1, s_2, \ldots, s_k$ is a feasible solution for an instance of size $n - i$:

$$\sum_{j=1}^{k} s_j = n - i \ .$$

Hence, $i + \sum_{j=1}^{k} s_j = n$.

Proof of Structural Theorem

Theorem

If:

- *the leftmost cut in an optimal solution is after i units.*
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Then, an optimal way to cut our rod is into rods of sizes: $i, s_1, s_2, \ldots, s_k$.

Proof

Optimality: Let $i, o_1, o_2, \ldots, o_\ell$ be an optimal solution—exists by assumption. Recall that $s_1, s_2, \ldots, s_k$ is an optimal way to cut a rod of size $n - i$, thus,

$$\sum_{j=1}^{k} p_{s_j} \geq \sum_{j=1}^{\ell} p_{o_j} \quad .$$

Hence, $p_i + \sum_{j=1}^{k} p_{s_j} \geq p_i + \sum_{j=1}^{\ell} p_{o_j}$.

First Algorithm

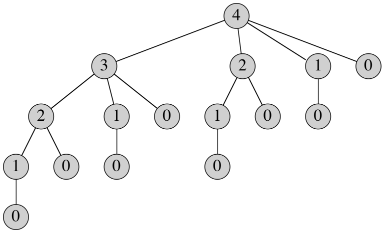
If we let $r(n)$ be the optimal revenue from a rod of length n, then, by the structural theorem, we can express $r(n)$ recursively as follows

$$r(n) = \begin{cases} 0 & \text{if } n = 0 \ , \\ \max_{1 \leq i \leq n} \{p_i + r(n-i)\} & \text{otherwise if } n \geq 1 \ . \end{cases}$$

```
CUT-ROD(p, n)
  if n == 0
      return 0
  q = −∞
  for i = 1 to n
      q = max(q, p[i] + CUT-ROD(p, n − i))
  return q
```

Problem

- The procedure is extremely inefficient—in fact exponential.
- What went wrong?



- The procedure repeatedly calculates the same profits.
- Dynamic programming can save the extra calculations.

Top-Down Dynamic Programming

General Approach

- Keep the recursive structure of the pseudocode.
- Memoize (store) the result of every recursive call.
- At each recursive call, try to avoid work using memoized results.

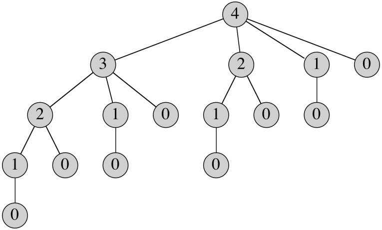
Pseudocode

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
  if r[n] ≥ 0
      return r[n]
  if n == 0
      q = 0
  else q = −∞
      for i = 1 to n
          q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
  r[n] = q
  return q
```

```
MEMOIZED-CUT-ROD(p, n)
  let r[0..n] be a new array
  for i = 0 to n
      r[i] = −∞
  return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

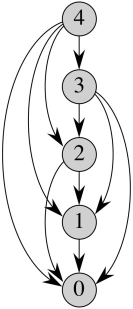
What did we gain?

Memoization helps us avoid recalculations.



$\Rightarrow$

Subproblem Graph



One can think of all the recursive calls using a memoized value as additional parents of the call generating this value.

Time Complexity

- The initialization takes $O(n)$ time.
- Processing each sub-problem takes linear time in the number of sub-problems it evokes.

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
  if r[n] ≥ 0
      return r[n]
  if n == 0
      q = 0
  else q = −∞
      for i = 1 to n
          q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
  r[n] = q
  return q
```

- The time complexity is proportional to the number of nodes and edges in the subproblem graph.

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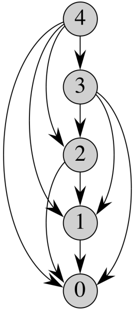

Time Complexity
$O(n^2)$

Bottom-Up Dynamic Programming

General Approach

- Sort the sub-problems by size.
- Solve the smaller ones first.
- When reaching a sub-problem, the smaller ones are already solved.

Pseudocode

```
BOTTOM-UP-CUT-ROD(p, n)
  let r[0..n] be a new array
  r[0] = 0
  for j = 1 to n
      q = -∞
      for i = 1 to j
          q = max(q, p[i] + r[j - i])
      r[j] = q
  return r[n]
```

Time Complexity
$O(n^2)$

Reconstructing an Optimal Solution

- The above algorithms only return the optimal profit.
- Sometimes one needs also to find an optimal solution.

Approach

- Each cell of the memoization table corresponds to a decision: the location of the left most cut.
- Store the decision corresponding to every cell in a separate table.

Reconstructing an Optimal Solution (cont.)

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
  let r[0..n] and s[0..n] be new arrays
  r[0] = 0
  for j = 1 to n
      q = −∞
      for i = 1 to j
          if q < p[i] + r[j − i]
              q = p[i] + r[j − i]
              s[j] = i
      r[j] = q
  return r and s
```

Output

i	0	1	2	3	4	5	6	7	8
r[i]	0	1	5	8	10	13	17	18	22
s[i]	0	1	2	3	2	2	6	1	2

Summary

- We had a recursive formulation for the optimal value for our problem

$$r(n) = \begin{cases} 0 & \text{if } n = 0 \ , \\ \max_{1 \leq i \leq n} \{p_i + r(n-i)\} & \text{otherwise if } n \geq 1 \ . \end{cases}$$

- Speed up the calculations by filling in a table either "top-down with memoization" or with "bottom-up".

- Recovered an optimal solution using an additional table.

Problem Solving: the Change-Making Problem

- How can a given amount of money be made with the least number of coins of given denominations?

Formally:

Input: n distinct coin denominators (integers) $0 < w_1 < w_2 < \ldots < w_n$ and an amount W (the change) which is also a positive integer.

Output: The minimum number of coins needed in order to make the change:

$$\min \left\{ \sum_{j=1}^{n} x_j : \sum_{j=1}^{n} w_j x_j = W \text{ and } x_j\text{'s are integers} \right\}.$$

Example: On input $w_1 = 1, w_2 = 2, w_3 = 5$ and $W = 8$, the output should be 3 since the best way of giving 8 is $x_1 = x_2 = x_3 = 1$.

Summary

- Identify choices and optimal substructure
- Write optimal solution recursively as a function of smaller subproblems
- Use top-down with memoization or bottom-up to solve the recursion efficiently (without repeatedly solving the same subproblems)