Algorithms: Elementary Data Structures and Binary Search Trees

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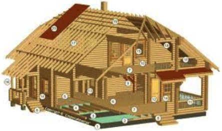
Elementary Data Structures

Algorithm →

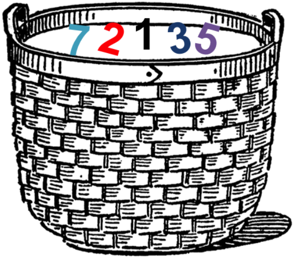


Algorithm →

Data structures = dynamic sets of items

What kind of operations do we want to do?

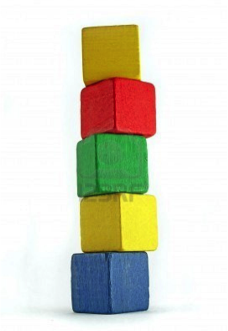
- Modifying operations: insertion, deletion, . . .
- Query operations: search, maximum, minimum, . . .



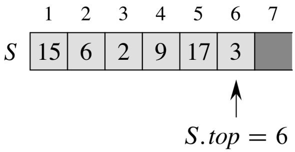
Data structure containing numbers

Stacks (last-in, first-out)

- Insert operation called PUSH(S,x)
- Delete operation called POP(S)



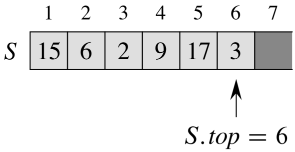
Stacks Implementation



Implementation using arrays: S consists of elements $S[1, \ldots, S.top]$

- $S[1]$ element at the bottom
- $S[S.top]$ element at the top

Stacks Implementation



What is the running time of these operations? $O(1)$

Stack-Empty(S)

```
if S.top = 0
    return TRUE
else return FALSE
```

Push(S,x)

```
S.top ← S.top + 1
S[S.top] ← x
```

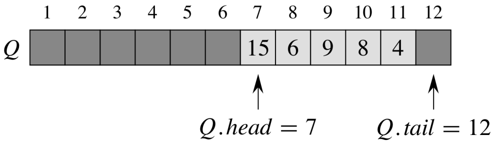
Pop(S)

```
if Stack-Empty(S)
    error "underflow"
else
    S.top ← S.top − 1
    return S[S.top + 1]
```

Queues (first-in, first-out)

- Insert operation called ENQUEUE(Q,x)
- Delete operation called DEQUEUE(Q)

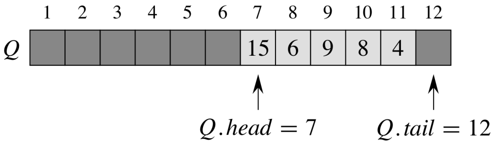
Queue Implementation



Implementation using arrays: Q consists of elements $S[Q.head, \ldots, Q.tail - 1]$

- $Q.head$ points at the first element
- $Q.tail$ points at the next location where a newly arrived element will be placed

Queue Implementation



What is the running time of these operations? $O(1)$

```
ENQUEUE(Q,x)
Q[Q.tail] = x
if Q.tail = Q.length
    Q.tail ← 1
else Q.tail ← Q.tail + 1
```

```
DEQUEUE(Q)
x = Q[Q.head]
if Q.head = Q.length
    Q.head ← 1
else Q.head ← Q.head + 1
return x
```

Stacks and Queues

Positives

- Very efficient
- Natural operations

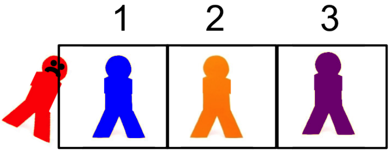
Negatives

- Limited support: for example, no search
- Implementations using arrays have a *fixed* capacity

Linked List

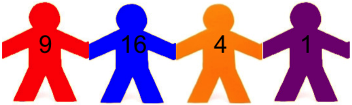
Objects are arranged in a linear order

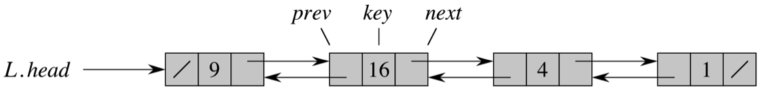
Not indexes in array



But pointers in each object

Linked List

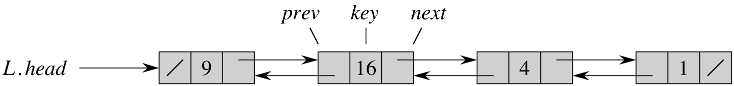




A list can be

- Single linked or double linked
- Sorted or unsorted
- etc.

Searching a Linked List



Task: Given k return pointer to first element with key k

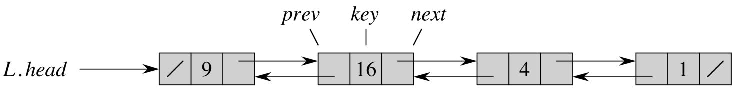
LIST-SEARCH(L,K)

```
x ← L.head
while x ≠ nil and x.key ≠ k
    x ← x.next
return x
```

Running time? $O(n)$

What if no element with key k exists? returns nil

Inserting into a Linked List



Task: Insert a new element x

LIST-INSERT(L,x)

$x.next \leftarrow L.head$

if $L.head \neq nil$

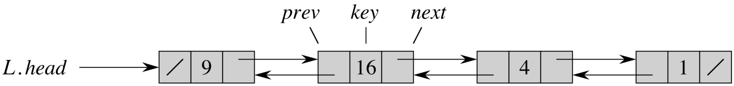
  $L.head.prev \leftarrow x$

$L.head \leftarrow x$

$x.prev = NIL$

Running time?

Inserting into a Linked List



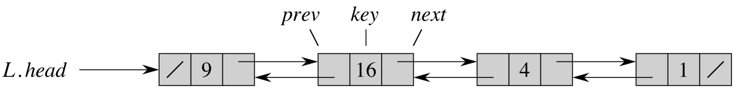
Task: Insert a new element x

LIST-INSERT(L,x)

1. $x.next \leftarrow L.head$
2. **if** $L.head \neq nil$
3. $\quad L.head.prev \leftarrow x$
4. $L.head \leftarrow x$
5. $x.prev = NIL$

Running time? $O(1)$

Deleting From a Linked List



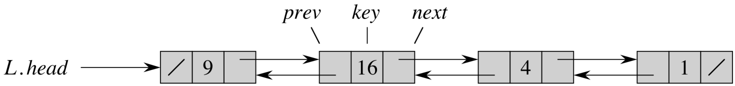
Task: Given a pointer to an element x remove it from L

List-Delete(L,x)

if $x.prev \neq nil$
    $x.prev.next \leftarrow x.next$
else $L.head \leftarrow x.next$
if $x.next \neq nil$
    $x.next.prev \leftarrow x.prev$

Running time? $O(1)$

Sentinels



Note: If x is in the middle of the list then

simplified

LIST-DELETE(L,x)

if $x.prev \neq nil$

  $x.prev.next \leftarrow x.next$

else $L.head \leftarrow x.next$

if $x.next \neq nil$

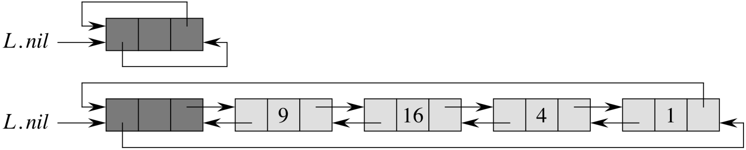
  $x.next.prev \leftarrow x.prev$

LIST-DELETE'(L,x)

$x.prev.next \leftarrow x.next$

$x.next.prev \leftarrow x.prev$

Sentinels



simplified

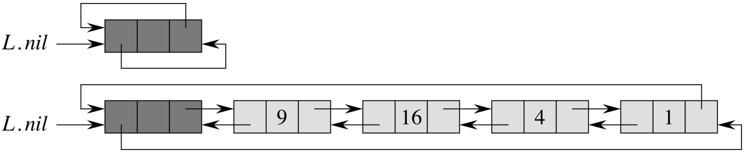
List-Delete(L,x)

1. **if** $x.prev \neq nil$
2. $x.prev.next \leftarrow x.next$
3. **else** $L.head \leftarrow x.next$
4. **if** $x.next \neq nil$
5. $x.next.prev \leftarrow x.prev$

List-Delete'(L,x)

1. $x.prev.next \leftarrow x.next$
2. $x.next.prev \leftarrow x.prev$

Sentinels



simplified

List-Insert(L,x)

$x.next \leftarrow L.head$
if $L.head \neq nil$
    $L.head.prev \leftarrow x$
$L.head \leftarrow x$
$x.prev = NIL$

List-Insert'(L,x)

$x.next \leftarrow L.nil.next$
$L.nil.next.prev \leftarrow x$
$L.nil.next \leftarrow x$
$x.prev \leftarrow L.nil$

Summary Linked List

- Dynamic data structure without predefined capacity
- Insertion: $O(1)$
- Deletion: $O(1)$ (if double linked)
  - Question in book: can you do it for single linked?
- Search: $O(n)$

Summary Linked List

- Dynamic data structure without predefined capacity
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  - Question in book: can you do it for single linked?
- Search: $O(n)$

Search $O(n)$ = no fun!

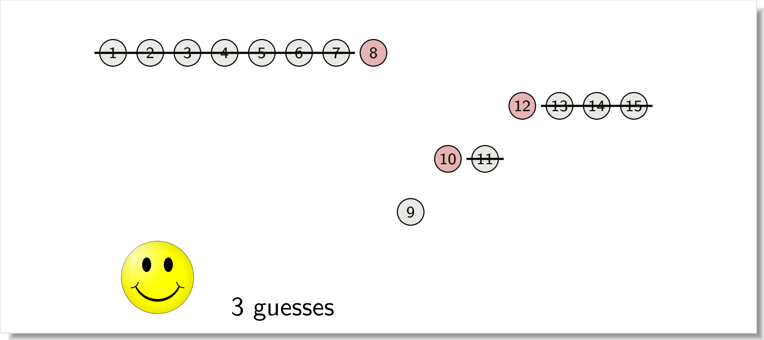
We will have fun: **Binary Search Trees**

BINARY SEARCH TREES

Idea

Guessing Game:

- Ola thinks of an integer between 1 and 15
- When you guess a number, answer either *correct*, *smaller*, or *larger*
  - For example: is it 5? Ola: *larger*
- What is your best strategy to minimize number of guesses?

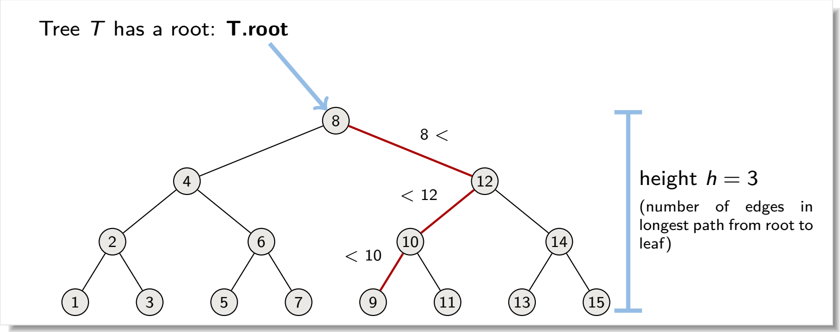


Binary Search Trees

Encodes a strategy whatever number we look for

Key property:

- If y is in the left subtree of x then $y.key < x.key$
- If y is in the right subtree of x then $y.key \geq x.key$



Binary Search Trees

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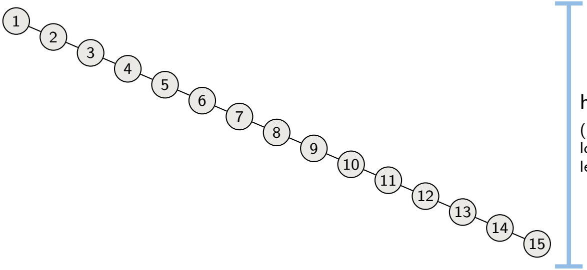


height $h = 14$

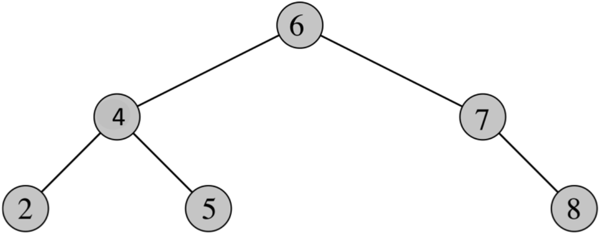
(number of edges in longest path from root to leaf)

Basic operations take time proportional to height: $O(h)$

QUERYING A BINARY SEARCH TREE

(Searching, Minimum, Maximum, Successor, Predecessor)

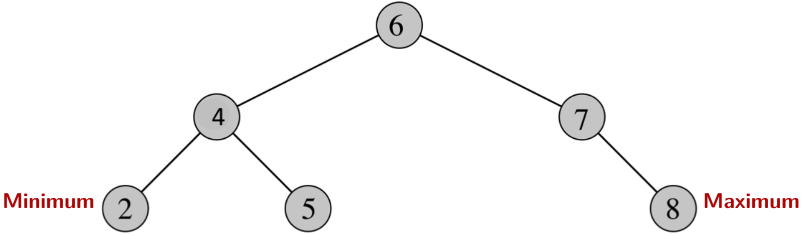
Searching



What is the running time? $O(h)$

```
TREE-SEARCH(x, k)
  if x == NIL or k == key[x]
      return x
  if k < x.key
      return TREE-SEARCH(x.left, k)
  else return TREE-SEARCH(x.right, k)
```

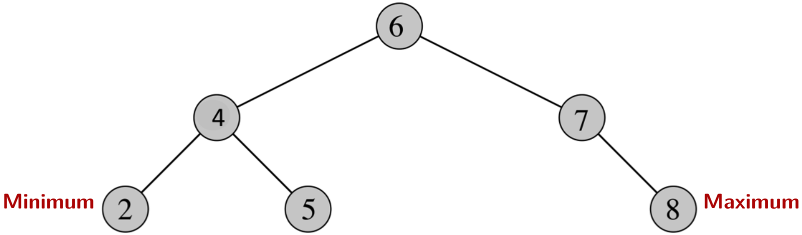
Minimum and Maximum



By key property:

- Minimum is located in leftmost node
- Maximum is located in rightmost node

Minimum and Maximum

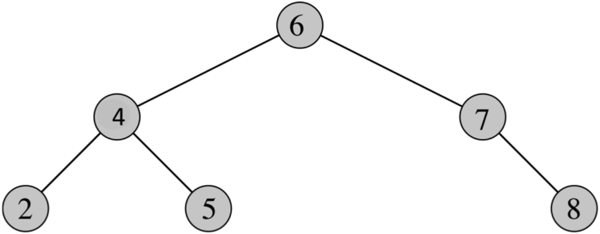


What is the running time? $O(h)$

TREE-MINIMUM(x)
  while $x.left \neq$ NIL
    $x = x.left$
  return x

TREE-MAXIMUM(x)
  while $x.right \neq$ NIL
    $x = x.right$
  return x

Successor



Successor of a noce x is the node y such that $y.key$ is the

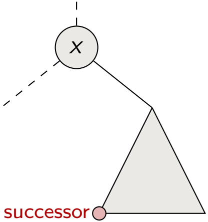
$$\text{“smallest key”} > x.key$$

- What is the successor of 6?
- What is the successor of 5?

Two cases when finding successor of x:

Case 1: x has a non-empty right subtree

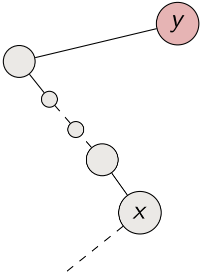
x's successor is the minimum in the right subtree



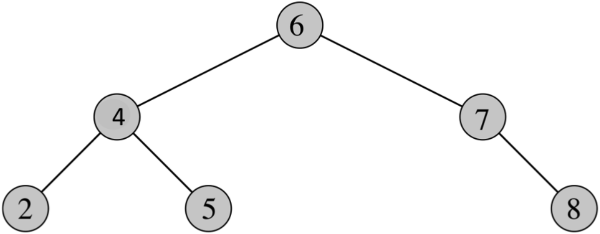
Case 2: x has an empty right subtree

As long as we to the left up the tree we're visiting smaller keys

x's successor is y is the node that x is the predecessor of (x is the maximum in y's left subtree)



Successor (Predecessor is symmetric)



What is the running time? $O(h)$

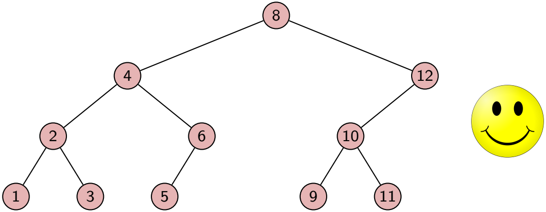
TREE-SUCCESSOR(x)
  if $x.right \neq$ NIL
    return TREE-MINIMUM$(x.right)$
  $y = x.p$
  while $y \neq$ NIL and $x == y.right$
    $x = y$
    $y = y.p$
  return y

PRINTING A BINARY SEARCH TREE

(Inorder, Preorder, Postorder)

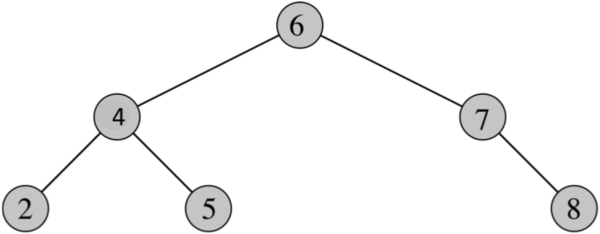
Printing Inorder (Idea)

- Print left subtree recursively
- Print root
- Print right subtree recursively



Output: 1,2,3,4,5,6,8,9,10,11,12

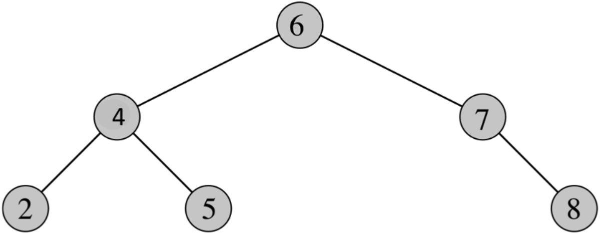
Inorder tree walk



What is the running time? $\Theta(n)$

```
INORDER-TREE-WALK(x)
  if x ≠ NIL
      INORDER-TREE-WALK(x.left)
      print key[x]
      INORDER-TREE-WALK(x.right)
```

Printing Preorder and Postorder



PREORDER-TREE-WALK(x)

1. **if** $x \neq NIL$
2. print $key[x]$
3. PREORDER-TREE-WALK($x.left$)
4. PREORDER-TREE-WALK($x.right$)

POSTORDER-TREE-WALK(x)

1. **if** $x \neq NIL$
2. POSTORDER-TREE-WALK($x.left$)
3. POSTORDER-TREE-WALK($x.right$)
4. print $key[x]$

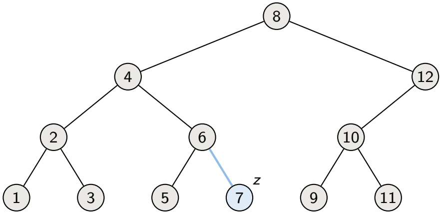
MODIFYING A BINARY SEARCH TREE

(Insertion and Deletion)

Idea of inserting z

- Search for $z.key$
- When arrived at *nil* insert z at that position

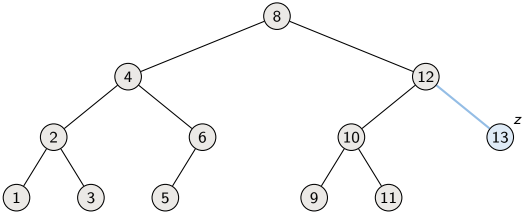
Ex: insert z with key 7



Idea of inserting z

- Search for $z.key$
- When arrived at *nil* insert z at that position

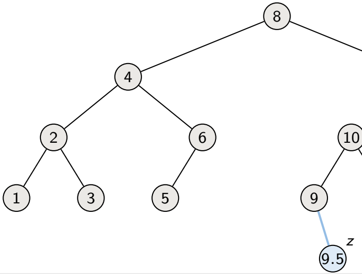
Ex: insert z with key 13



Idea of inserting z

- Search for $z.key$
- When arrived at *nil* insert z at that position

Ex: insert z with key 9.5



Insertion

"search" phase

"insert" phase

```
TREE-INSERT(T, z)
  y = NIL
  x = T.root
  while x ≠ NIL
      y = x
      if z.key < x.key
          x = x.left
      else x = x.right
  z.p = y
  if y == NIL
      T.root = z          // tree T was empty
  elseif z.key < y.key
      y.left = z
  else y.right = z
```

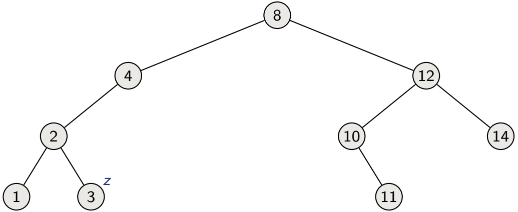
What is the running time? $O(h)$

Idea of deletion

Conceptually 3 cases:

- If z has no children, remove it

Ex: Delete z



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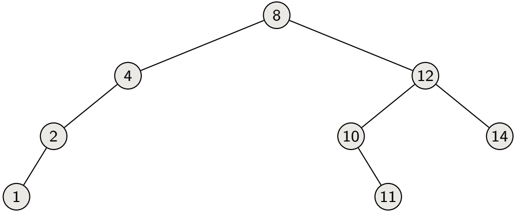
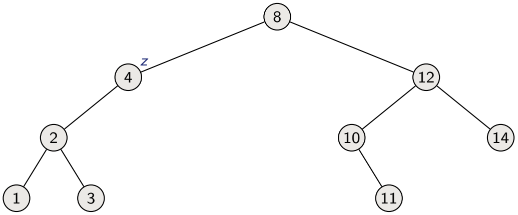


Idea of deletion

Conceptually 3 cases:

- If z has no children, remove it
- If z has one child, then make that child take z'z position in the tree

Ex: Delete z



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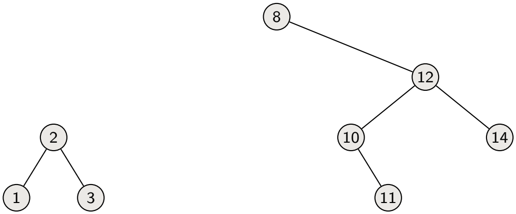

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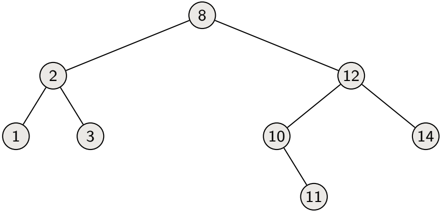
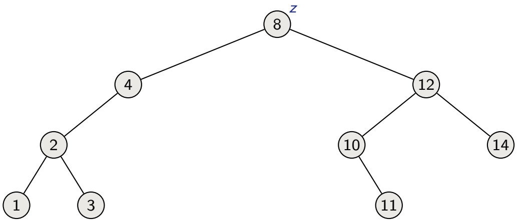


Idea of deletion

Conceptually 3 cases:

- If z has no children, remove it
- If z has one child, then make that child take z'z position in the tree
- If z has two children, then find its successor y and replace z by y

Ex: Delete z



Idea of deletion

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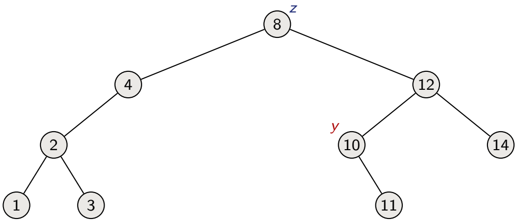

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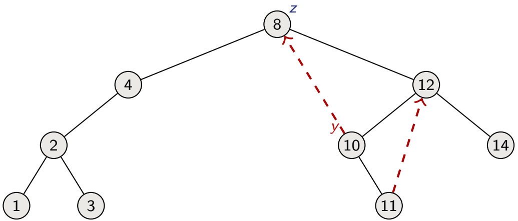

Idea of deletion

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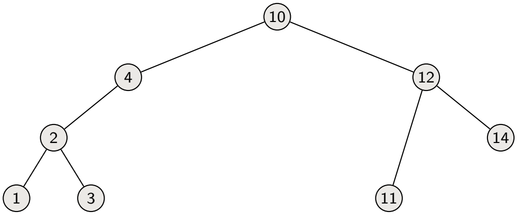
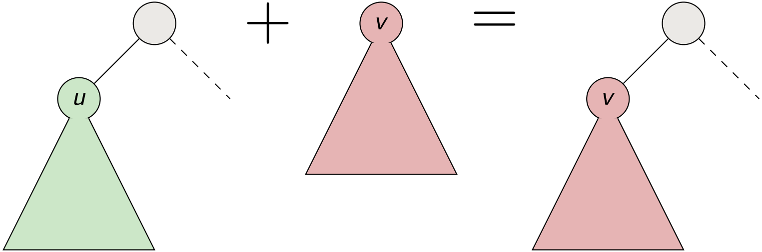
Ex: Delete z



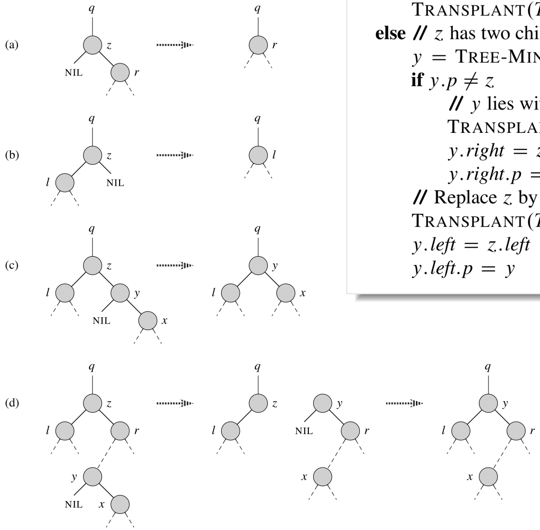
Deletion Implementation: Transplant

TRANSPLANT(T, u, v)
if $u.p$ == NIL
    $T.root = v$
elseif $u == u.p.left$
    $u.p.left = v$
else $u.p.right = v$
if $v \neq$ NIL
    $v.p = u.p$

TRANSPLANT(T, u, v) replaces subtree rooted at u with that rooted at v



Deletion Procedure




```
TREE-DELETE(T, z)
  if z.left == NIL
      TRANSPLANT(T, z, z.right)            // z has no left child
  elseif z.right == NIL
      TRANSPLANT(T, z, z.left)             // z has just a left child
  else // z has two children.
      y = TREE-MINIMUM(z.right)            // y is z's successor
      if y.p ≠ z
          // y lies within z's right subtree but is not the root of this
          TRANSPLANT(T, y, y.right)
          y.right = z.right
          y.right.p = y
      // Replace z by y.
      TRANSPLANT(T, z, y)
      y.left = z.left
      y.left.p = y
```

Summary

Query operations: Search, Max, Min, Predecessor, Successor: **O(h)** time

Modifying operations: Insertion, Deletion: **O(h)** time

Exist efficient procedures to keep tree balanced (AVL trees, red-black trees, etc.)