Algorithms: Strassen's Algorithm for Matrix Multiplication + Heaps and Heapsort

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Lecture 6, 05.03.2025

DIVIDE-AND-CONQUER

Merge Sort

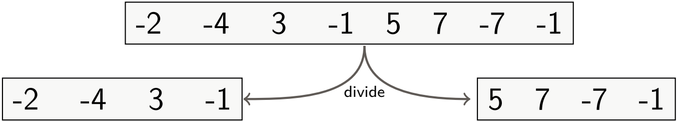
Maximum-Subarray Problem

Matrix Multiplication

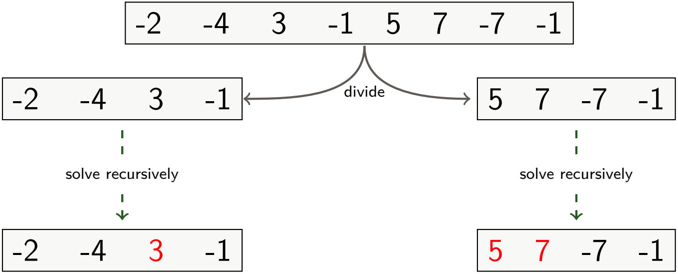
Recall Maximum-Subarray Problem

-2	-4	3	-1	5	7	-7	-1

Recall Maximum-Subarray Problem



Recall Maximum-Subarray Problem



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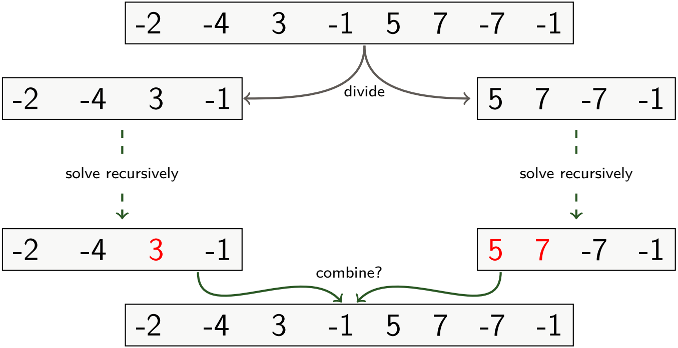

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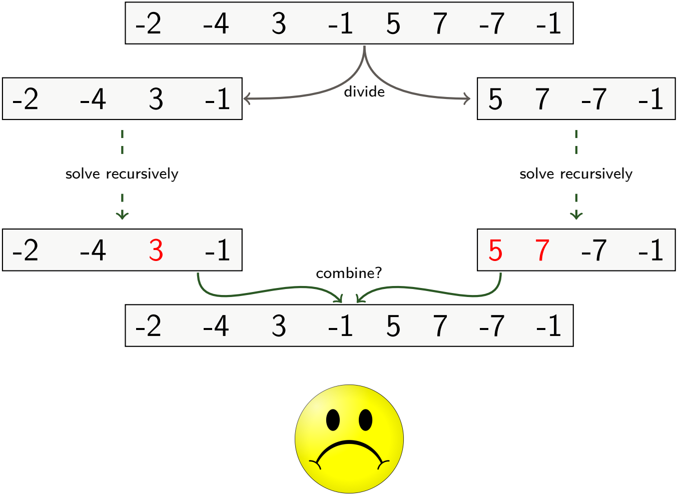

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Also find the maximum subarray that crosses the midpoint!

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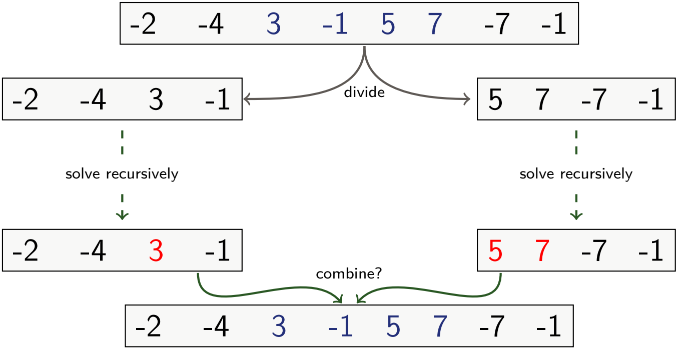

Finding maximum subarray crossing midpoint

Finding maximum subarray crossing midpoint

- Any subarray crossing the midpoint $A[mid]$ is made of two subarrays $A[i \ldots mid]$ and $A[mid + 1, \ldots, j]$ where $low \leq i \leq mid$ and $mid < j \leq high$
- Find maximum subarrays of the form $A[i \ldots mid]$ and $A[mid + 1 \ldots j]$ and then combine them.

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Crossing subarray

FIND-MAX-CROSSING-SUBARRAY$(A, low, mid, high)$

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// Find a maximum subarray of the form A[i..mid].
left-sum = −∞
sum = 0
for i = mid downto low
    sum = sum + A[i]
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// Find a maximum subarray of the form A[mid+1..j].
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for j = mid+1 to high
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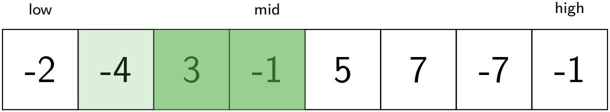

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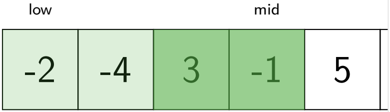


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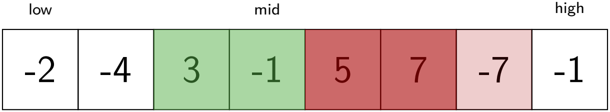

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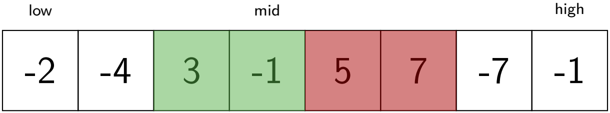

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Recursion for the running time is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

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Recursion for the running time is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

Hence, $T(n) = \Theta(n \log n)$

MATRIX MULTIPLICATION

Matrix Multiplication

Definition

Input: Two $n \times n$ (square) matrices, $A = (a_{ij})$ and $B = (b_{ij})$

Output: $n \times n$ matrix $C = (c_{ij})$, where $C = A \cdot B$

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Example ($n = 2$):

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21}, \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22}, \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21}, \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22}. \end{aligned}$$

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Example ($n = 2$):

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

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Example ($n = 2$):

$$\begin{pmatrix} 5 & 3 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

How to multiply two matrices?

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$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

$$c_{1,1} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} + \ldots, a_{1,n}b_{n,1} = \sum_{k=1}^{n} a_{1k}b_{k1}$$

How to multiply two matrices?

$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

$$c_{2,1} = a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} + \ldots, a_{2,n}b_{n,1} = \sum_{k=1}^{n} a_{2k}b_{k1}$$

How to multiply two matrices?

$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

$$c_{2,2} = a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} + \ldots, a_{2,n}b_{n,2} = \sum_{k=1}^{n} a_{2k}b_{k2}$$

How to multiply two matrices?

$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
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  for i = 1 to n
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          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} 3 & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} 5 & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} 5 & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} 5 & -1 \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} 5 & 3 \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} 5 & 3 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Running time?

Naive Algorithm

Well simply multiply the matrices...

SQUARE-MAT-MULT(A, B, n)

  let C be a new $n \times n$ matrix
  for $i = 1$ **to** n
    for $j = 1$ **to** n
      $c_{ij} = 0$
      for $k = 1$ **to** n
        $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
  return C

Running time? $\Theta(n^3)$

Naive Algorithm

Well simply multiply the matrices...

SQUARE-MAT-MULT(A, B, n)

let C be a new $n \times n$ matrix
for $i = 1$ **to** n
  for $j = 1$ **to** n
    $c_{ij} = 0$
    for $k = 1$ **to** n
      $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C

Running time? $\Theta(n^3)$

Space?

Naive Algorithm

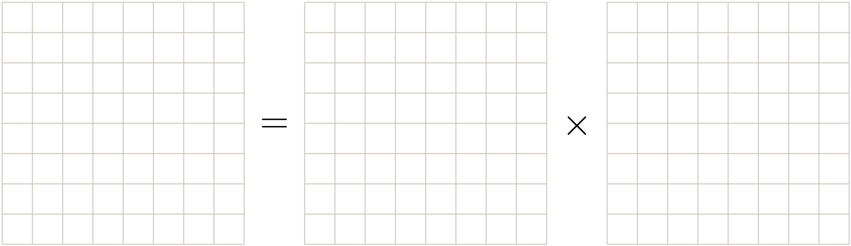
Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Running time? $\Theta(n^3)$ 

Space? $\Theta(n^2)$

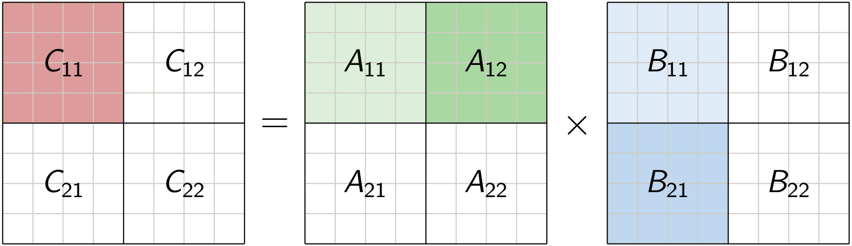
Smart Algorithm: Divide-and-Conquer



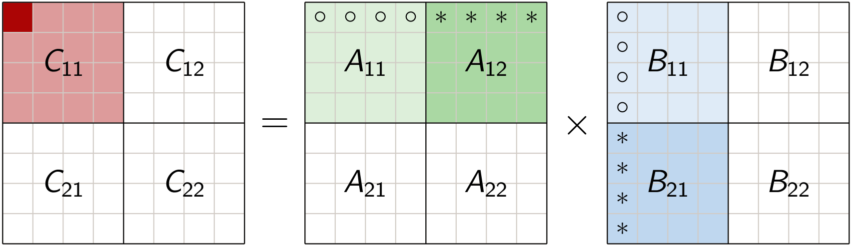
Smart Algorithm: Divide-and-Conquer

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

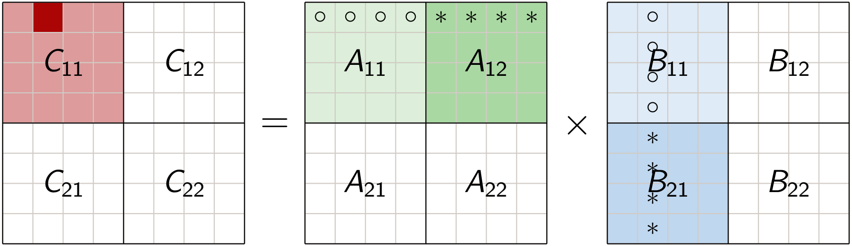
Smart Algorithm: Divide-and-Conquer



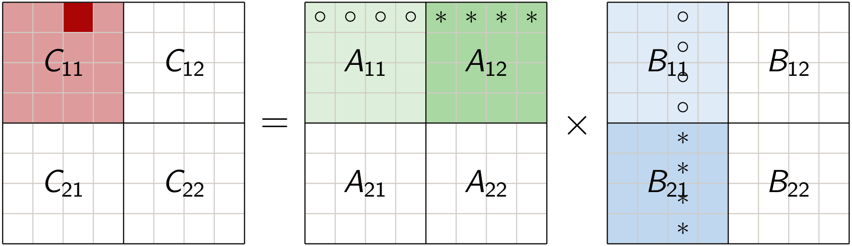
Smart Algorithm: Divide-and-Conquer



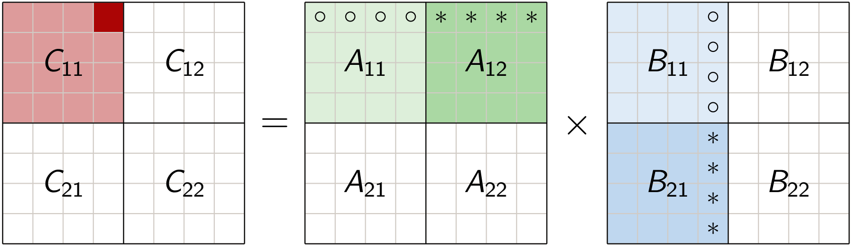
Smart Algorithm: Divide-and-Conquer



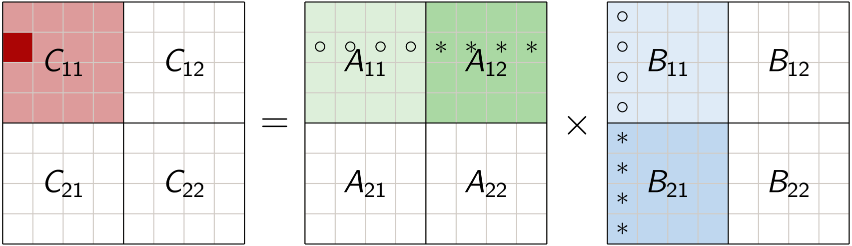
Smart Algorithm: Divide-and-Conquer



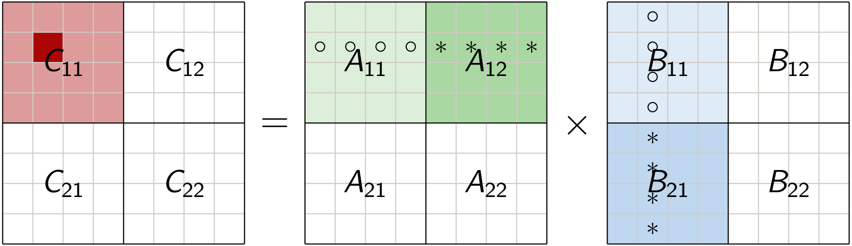
Smart Algorithm: Divide-and-Conquer



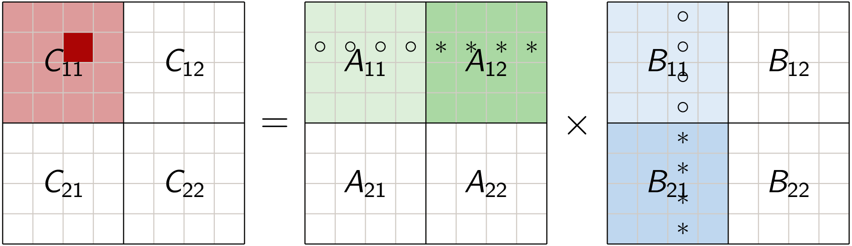
Smart Algorithm: Divide-and-Conquer



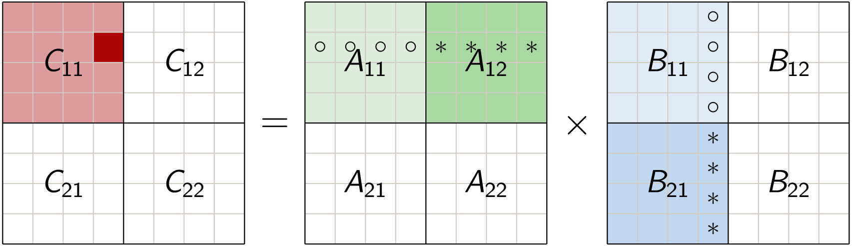
Smart Algorithm: Divide-and-Conquer



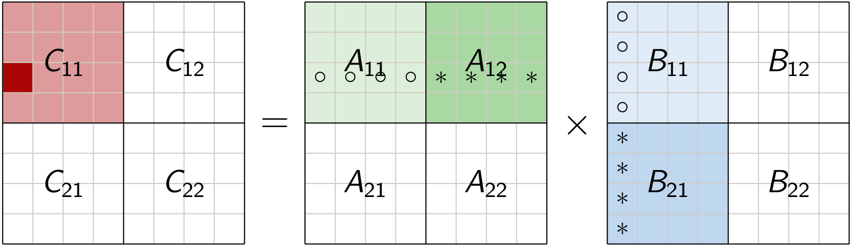
Smart Algorithm: Divide-and-Conquer



Smart Algorithm: Divide-and-Conquer



Smart Algorithm: Divide-and-Conquer



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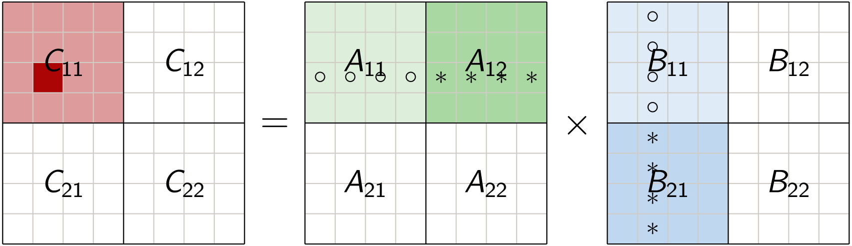
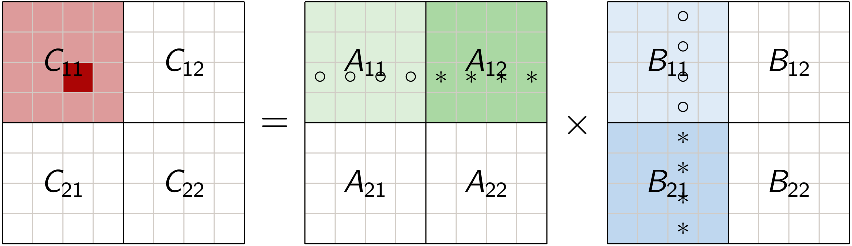
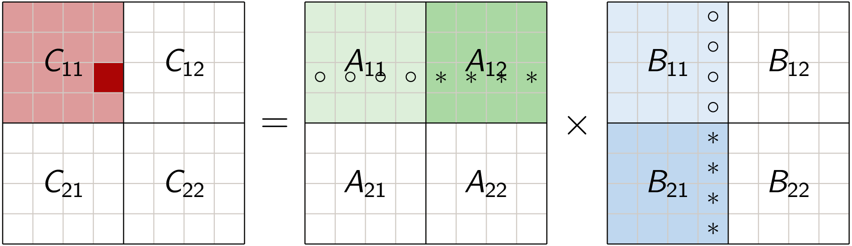


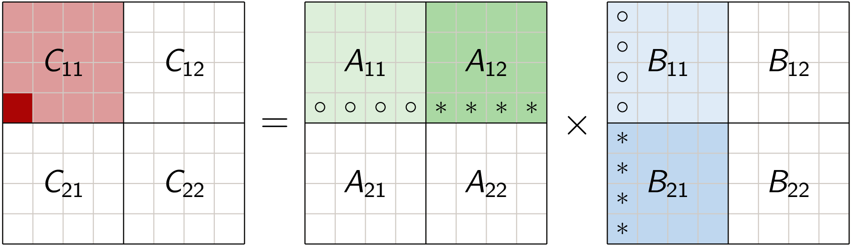
Smart Algorithm: Divide-and-Conquer



Smart Algorithm: Divide-and-Conquer



Smart Algorithm: Divide-and-Conquer



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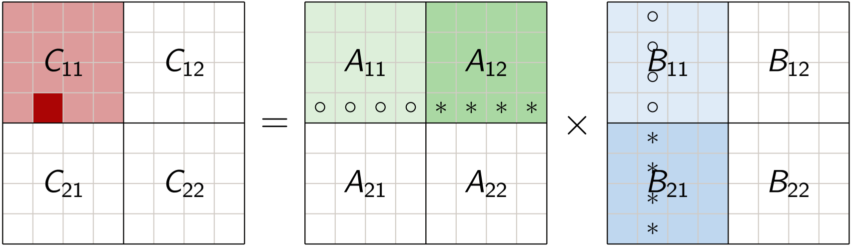

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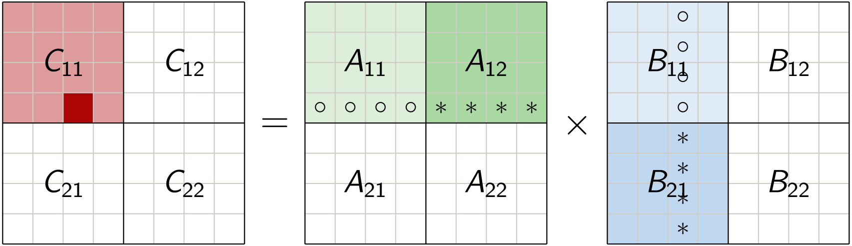
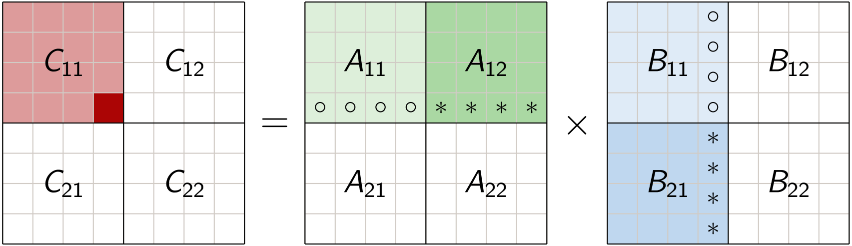


Smart Algorithm: Divide-and-Conquer



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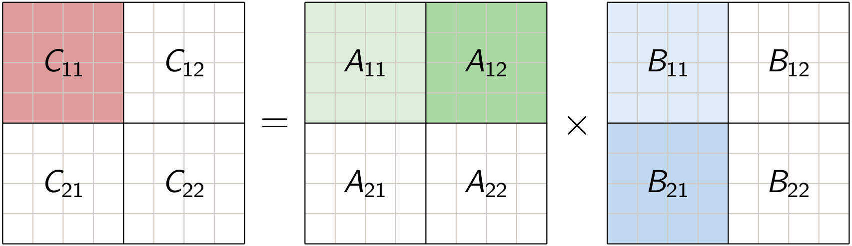
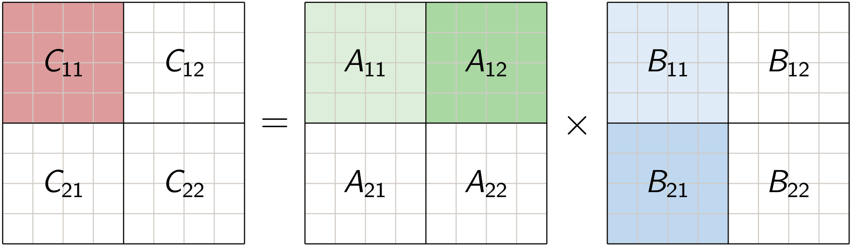


Smart Algorithm: Divide-and-Conquer



$$C_{11} = A_{11}B_{11} + A_{12}B_{21} \qquad C_{12} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21} \qquad C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

Divide-and-Conquer Algorithm

Divide each of A, B, C into four $n/2 \times n/2$ matrices: so that

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Divide-and-Conquer Algorithm

Divide each of A, B, C into four $n/2 \times n/2$ matrices: so that

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Conquer: Since

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
$$C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$$

we recursively solve 8 matrix multiplications that each multiply two $n/2 \times n/2$ matrices.

Divide-and-Conquer Algorithm

Divide each of A, B, C into four $n/2 \times n/2$ matrices: so that

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Conquer: Since

$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$
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$C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$

we recursively solve 8 matrix multiplications that each multiply two $n/2 \times n/2$ matrices.

Combine: Make the additions to get C

Pseudocode and Analysis

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

```
REC-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  if n == 1
      c_11 = a_11 · b_11
  else partition A, B, and C into n/2 × n/2 submatrices
      C_11 = REC-MAT-MULT(A_11, B_11, n/2) + REC-MAT-MULT(A_12, B_21, n/2)
      C_12 = REC-MAT-MULT(A_11, B_12, n/2) + REC-MAT-MULT(A_12, B_22, n/2)
      C_21 = REC-MAT-MULT(A_21, B_11, n/2) + REC-MAT-MULT(A_22, B_21, n/2)
      C_22 = REC-MAT-MULT(A_21, B_12, n/2) + REC-MAT-MULT(A_22, B_22, n/2)
  return C
```

Pseudocode and Analysis

```
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  if n == 1
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      C_21 = REC-MAT-MULT(A_21, B_11, n/2) + REC-MAT-MULT(A_22, B_21, n/2)
      C_22 = REC-MAT-MULT(A_21, B_12, n/2) + REC-MAT-MULT(A_22, B_22, n/2)
  return C
```

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

Base case: $n = 1$. Perform one scalar multiplication: $\Theta(1)$

Pseudocode and Analysis

```
REC-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  if n == 1
      c11 = a11 · b11
  else partition A, B, and C into n/2 × n/2 submatrices
      C11 = REC-MAT-MULT(A11, B11, n/2) + REC-MAT-MULT(A12, B21, n/2)
      C12 = REC-MAT-MULT(A11, B12, n/2) + REC-MAT-MULT(A12, B22, n/2)
      C21 = REC-MAT-MULT(A21, B11, n/2) + REC-MAT-MULT(A22, B21, n/2)
      C22 = REC-MAT-MULT(A21, B12, n/2) + REC-MAT-MULT(A22, B22, n/2)
  return C
```

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

Base case: $n = 1$. Perform one scalar multiplication: $\Theta(1)$

Recursive case: $n > 1$.

- Dividing takes $\Theta(1)$ time if careful and $\Theta(n^2)$ if simply copying

Pseudocode and Analysis

```
REC-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  if n == 1
      c11 = a11 · b11
  else partition A, B, and C into n/2 × n/2 submatrices
      C11 = REC-MAT-MULT(A11, B11, n/2) + REC-MAT-MULT(A12, B21, n/2)
      C12 = REC-MAT-MULT(A11, B12, n/2) + REC-MAT-MULT(A12, B22, n/2)
      C21 = REC-MAT-MULT(A21, B11, n/2) + REC-MAT-MULT(A22, B21, n/2)
      C22 = REC-MAT-MULT(A21, B12, n/2) + REC-MAT-MULT(A22, B22, n/2)
  return C
```

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

Base case: $n = 1$. Perform one scalar multiplication: $\Theta(1)$

Recursive case: $n > 1$.

- Dividing takes $\Theta(1)$ time if careful and $\Theta(n^2)$ if simply copying
- Conquering makes 8 recursive calls, each multiplying $n/2 \times n/2$ matrices $\Rightarrow 8T(n/2)$

Pseudocode and Analysis

```
REC-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  if n == 1
      c11 = a11 · b11
  else partition A, B, and C into n/2 × n/2 submatrices
      C11 = REC-MAT-MULT(A11, B11, n/2) + REC-MAT-MULT(A12, B21, n/2)
      C12 = REC-MAT-MULT(A11, B12, n/2) + REC-MAT-MULT(A12, B22, n/2)
      C21 = REC-MAT-MULT(A21, B11, n/2) + REC-MAT-MULT(A22, B21, n/2)
      C22 = REC-MAT-MULT(A21, B12, n/2) + REC-MAT-MULT(A22, B22, n/2)
  return C
```

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

Base case: $n = 1$. Perform one scalar multiplication: $\Theta(1)$

Recursive case: $n > 1$.

- Dividing takes $\Theta(1)$ time if careful and $\Theta(n^2)$ if simply copying
- Conquering makes 8 recursive calls, each multiplying $n/2 \times n/2$ matrices $\Rightarrow 8T(n/2)$
- Combining takes time $\Theta(n^2)$ time to add $n/2 \times n/2$ matrices.

Pseudocode and Analysis

```
REC-MAT-MULT(A, B, n)
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Base case: $n = 1$. Perform one scalar multiplication: $\Theta(1)$

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Recurrence is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 8T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

Pseudocode and Analysis

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Master method $\Rightarrow T(n) = \Theta(n^3)$

Volker Strassen

STRASSEN'S ALGORITHM FOR MATRIX MULTIPLICATION

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Strassen's method

Divide each of A, B, C into four $n/2 \times n/2$ matrices: so that

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

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Combine: Let

$C_{11} = M_1 + M_4 - M_5 + M_7$ $\quad$ $C_{12} = M_3 + M_5$

$C_{21} = M_2 + M_4$ $\quad$ $C_{22} = M_1 - M_2 + M_3 + M_6$

Analysis of Strassen's Method

Base case: $n = 1 \Rightarrow$ it takes time $\Theta(1)$

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Notes about Strassen's method

- First to beat $\Theta(n^3)$ time
- Faster known today, method by Coppersmith and Winograd runs in time $O(n^{2.376})$ recently improved by Vassilevska Williams to $O(n^{2.3727})$.
- Big open problem how to multiply matrices in best way
- Naive method better for small instances because of hidden constants

Karatsuba's algorithm

Problem: given two n-digit long integers x and y base b, find $x \cdot y$

Grade school algorithm: runtime

Karatsuba's algorithm

Problem: given two n-digit long integers x and y base b, find $x \cdot y$

Grade school algorithm: runtime $O(n^2)$

Can we do better than that?

Multiplying integers via divide and conquer

Divide: each number into two halves:

$$x = x_H \cdot b^{n/2} + x_L$$
$$y = y_H \cdot b^{n/2} + y_L$$

Then we have

$$\begin{aligned} x \cdot y &= (x_H \cdot b^{n/2} + x_L) \cdot (y_H \cdot b^{n/2} + y_L) \\ &= x_H \cdot y_H \cdot b^n + (x_H y_L + x_L y_H) \cdot b^{n/2} + x_L y_L \end{aligned}$$

Runtime?

Naive approach: compute $x_H \cdot y_H, x_H \cdot y_L, x_L \cdot y_H, x_L \cdot y_L$

All additions take $\Theta(n)$

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Suffices to compute $x_H y_H$, $x_L y_L$, $(x_H + x_L)(y_H + y_L)$

All additions take $\Theta(n)$

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$T(n) = \Theta(n^{\log_2 3})$

Summary of divide-and-conquer

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- Divide-and-conquer simple but powerful algorithmic paradigm
- Merge-sort and maximum subarray both run in time $\Theta(n \log n)$
- Strassen's algorithm for matrix multiplication in time $\Theta(n^{\log_2 7})$ where $\log_2 7 \approx 2.8$.

HEAPS AND HEAPSORT

Heapsort

Algorithm	worst-case running time	in-place
Insertion Sort	$\Theta(n^2)$	YES
Merge Sort	$\Theta(n \log n)$	NO

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Uses a cool datastructure: heaps

Data Structures = “Building Blocks”

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Algorithm →

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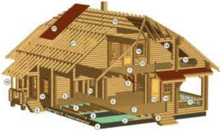
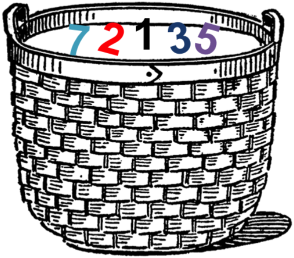
Algorithm →



Algorithm →

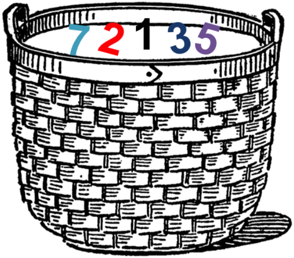
Data structures = dynamic sets of items



Data structure containing numbers

Data structures = dynamic sets of items

What kind of operations do we want to do?

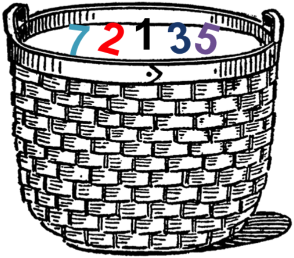


Data structure containing numbers

Data structures = dynamic sets of items

What kind of operations do we want to do?

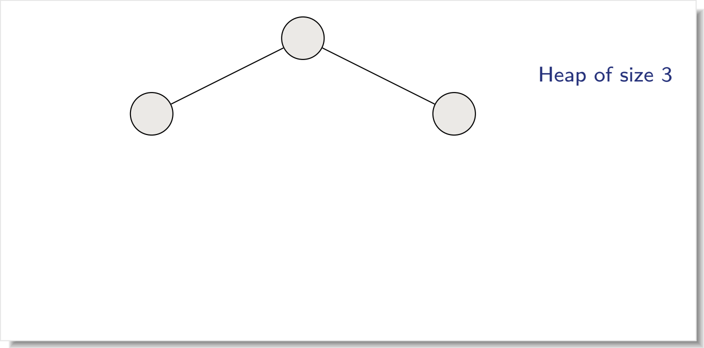
- Modifying operations: insertion, deletion, ...
- Query operations: search, maximum, minimum, ...



Data structure containing numbers

(Binary) heap data structure

Heap A (not garbage-collected storage) is a nearly complete binary tree



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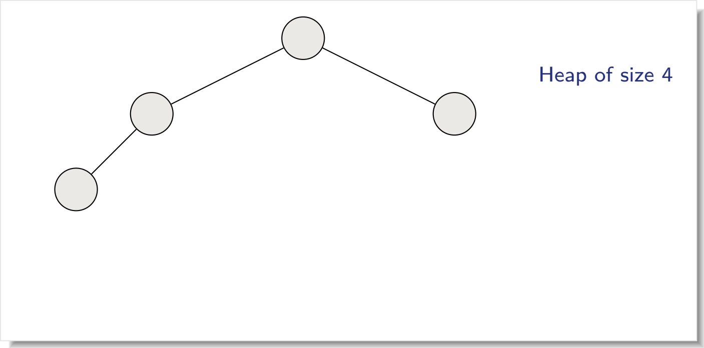

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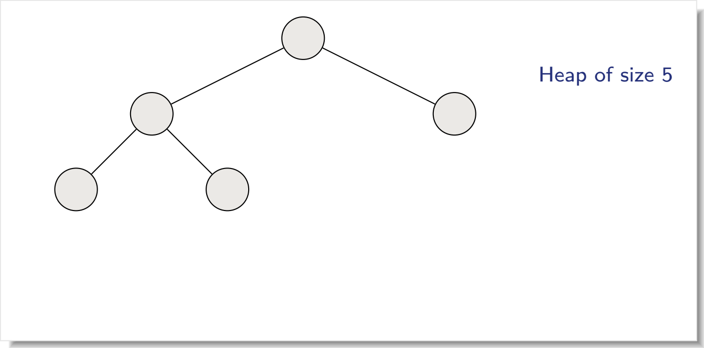

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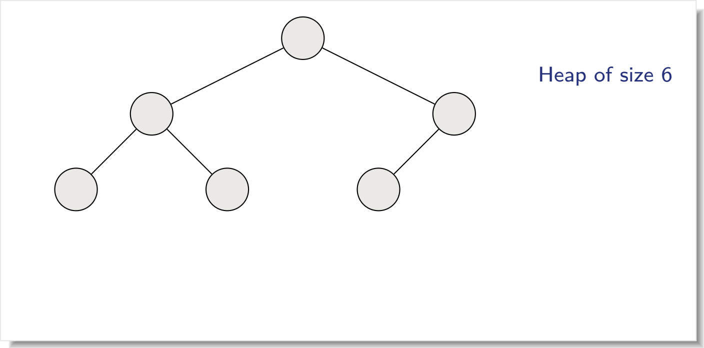

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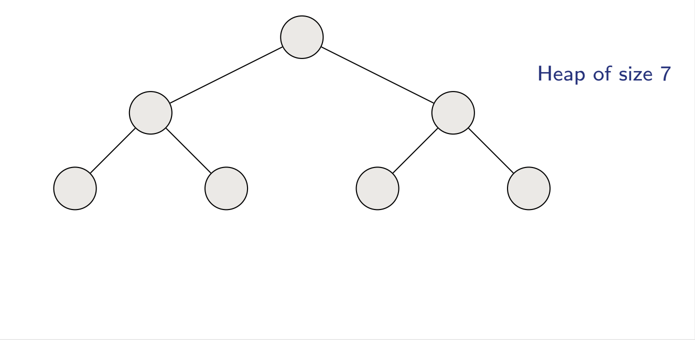

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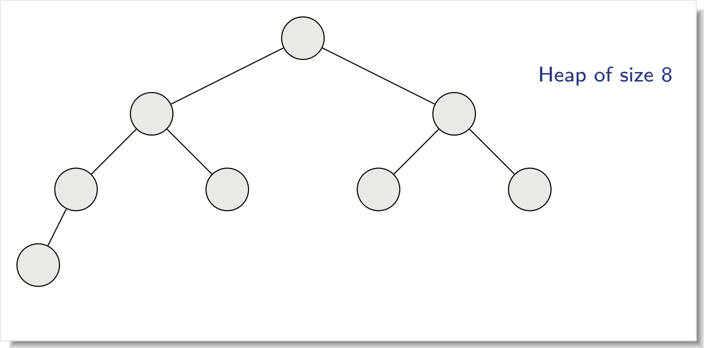

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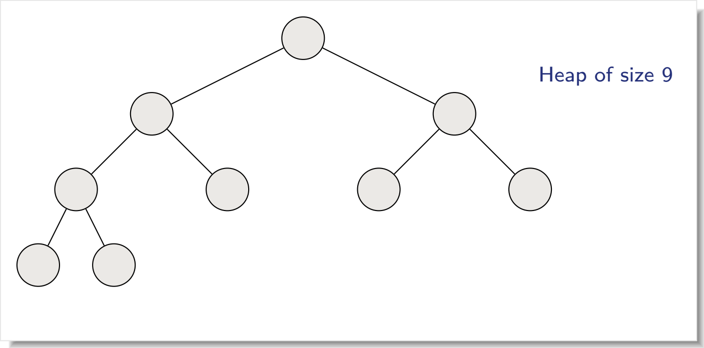

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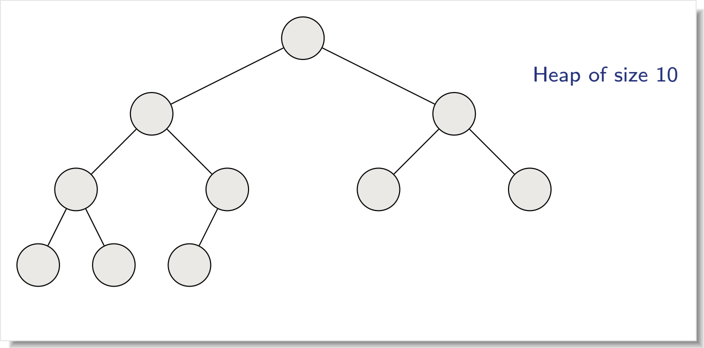

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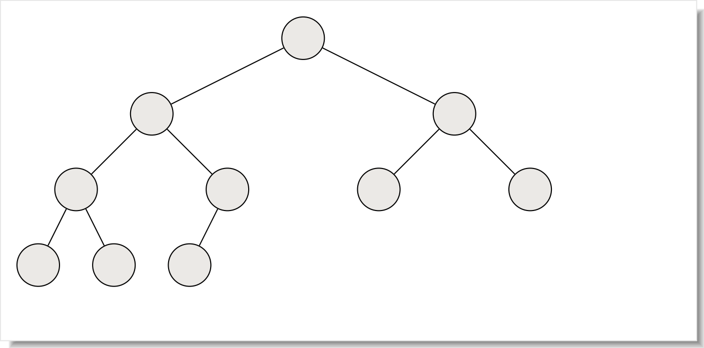

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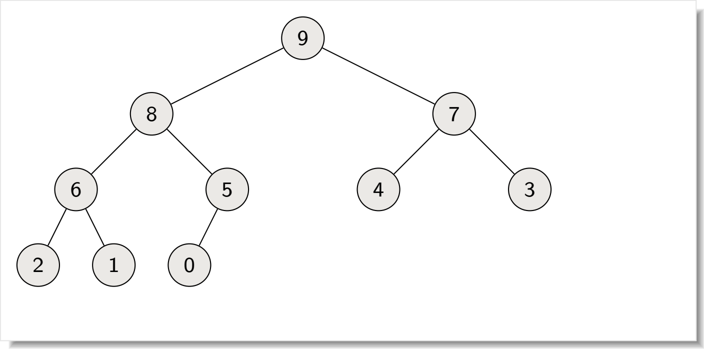

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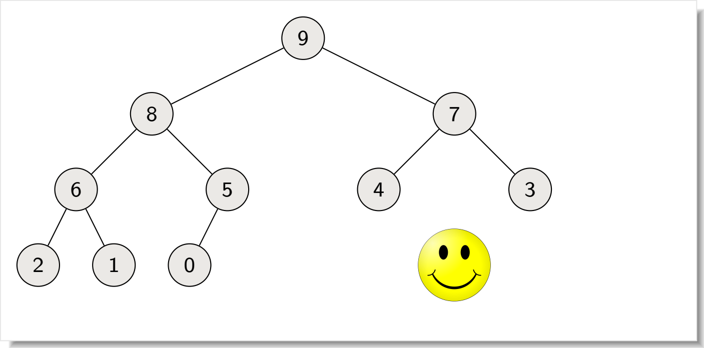

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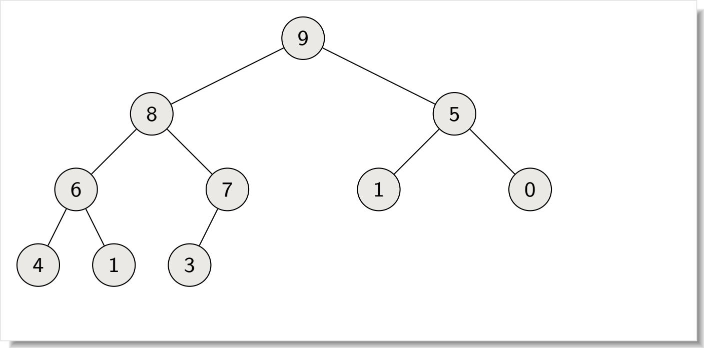

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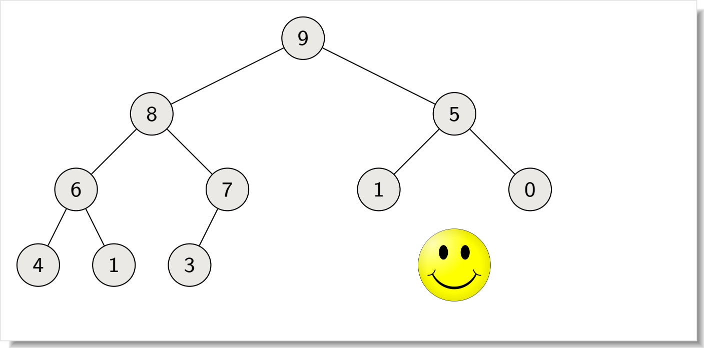

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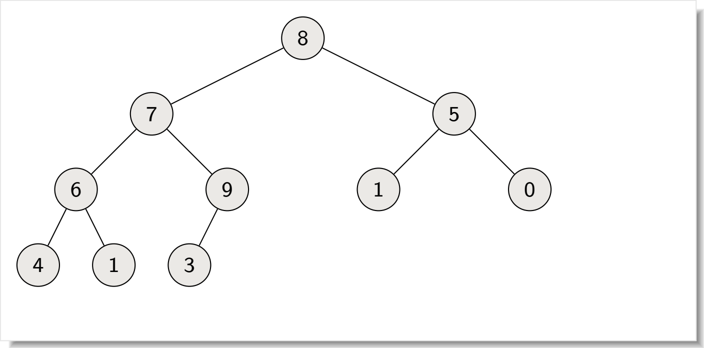

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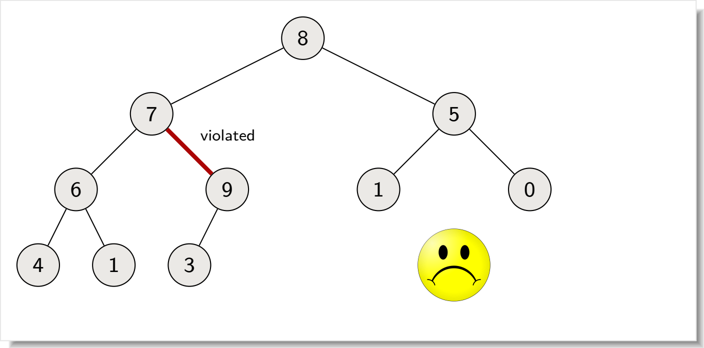

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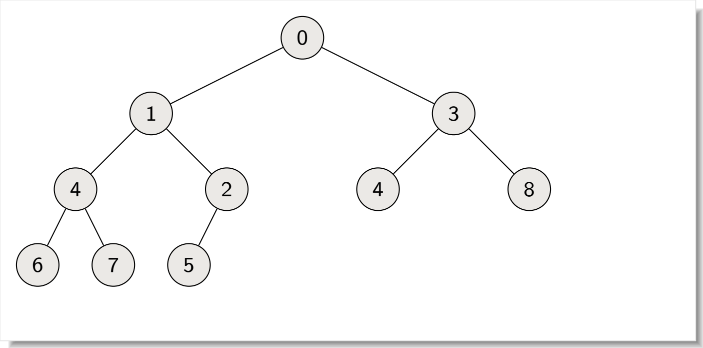

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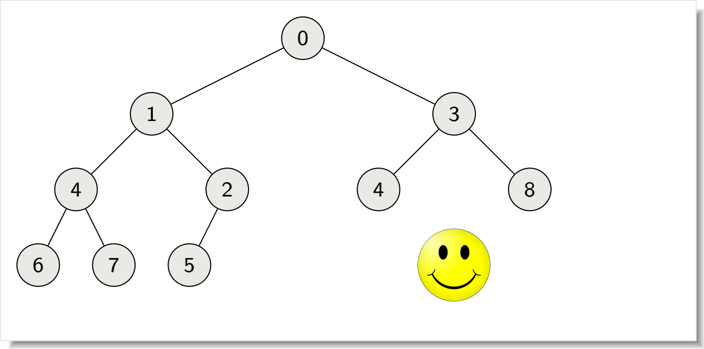
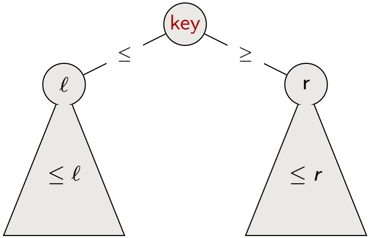
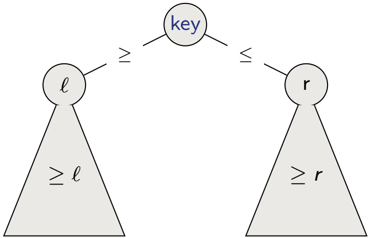


Max-Heap $\Rightarrow$ maximum element is the root

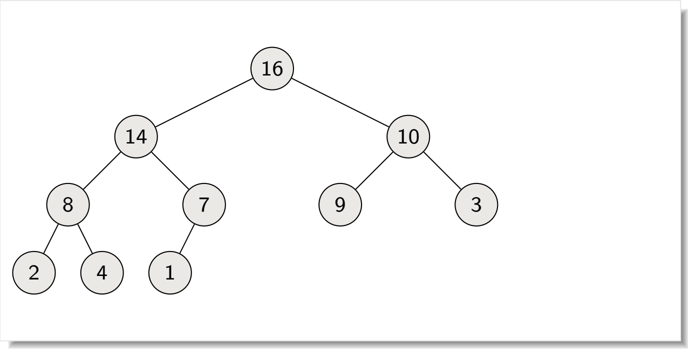


Min-Heap $\Rightarrow$ minimum element is the root



Height of a heap

Height of node = # of edges on a longest simple path from the node down to a leaf



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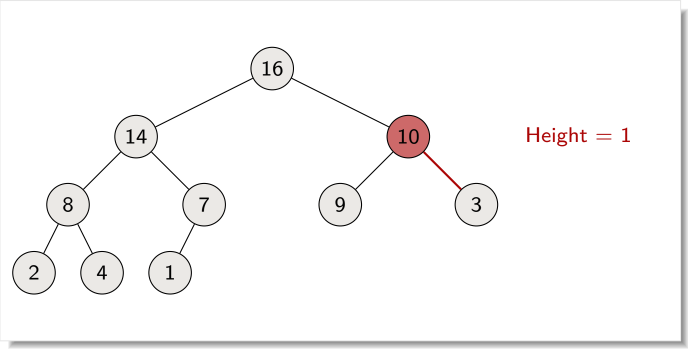

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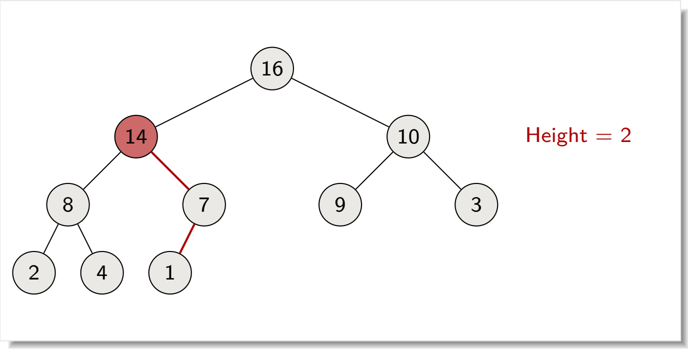

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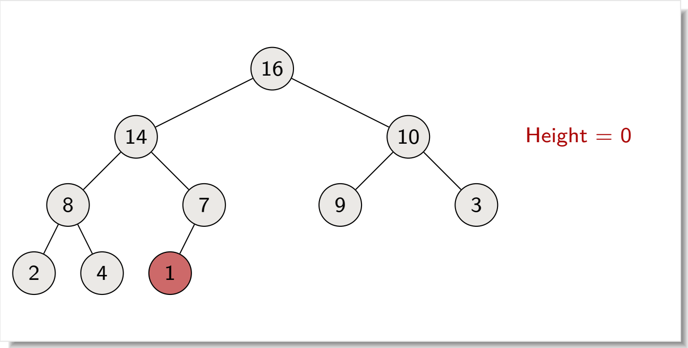

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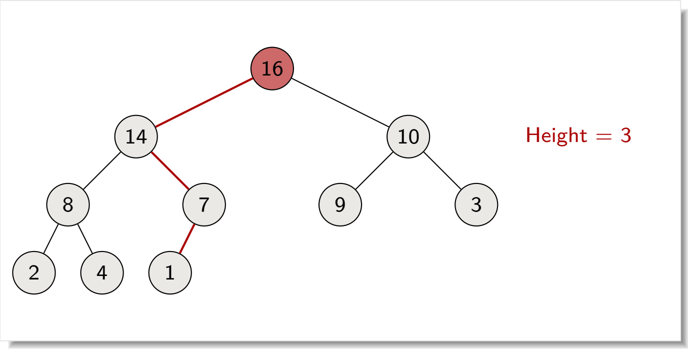

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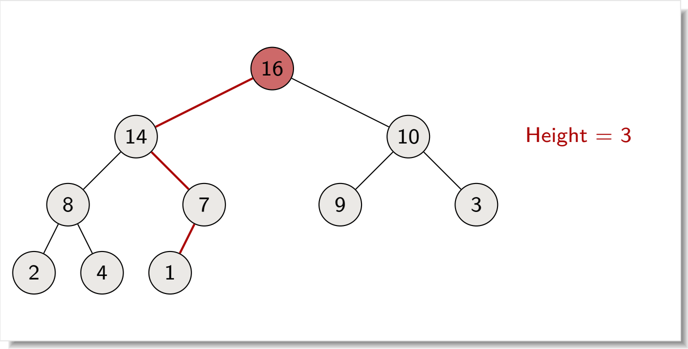
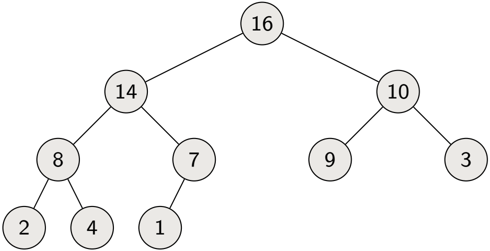
Height of node = # of edges on a longest simple path from the node down to a leaf

Height of heap = height of root = $\Theta(\log n)$

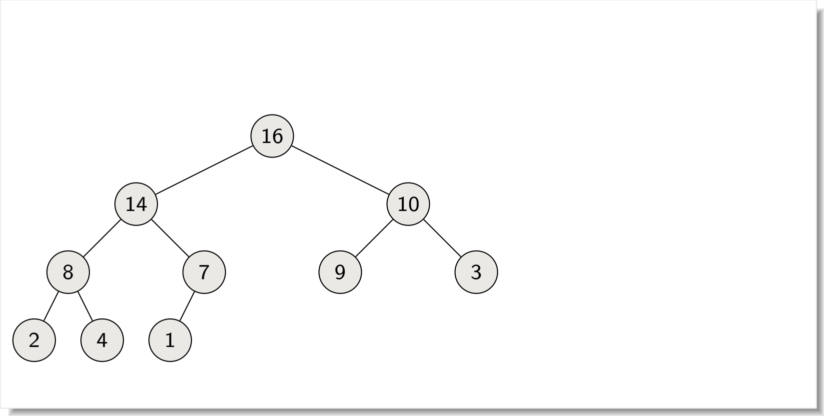


How to store a heap/tree?



How to store a heap/tree?

pointer to left and right children



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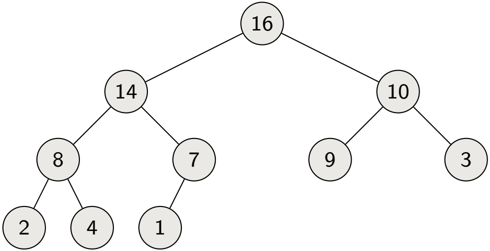
Use that tree is almost complete to store it in array



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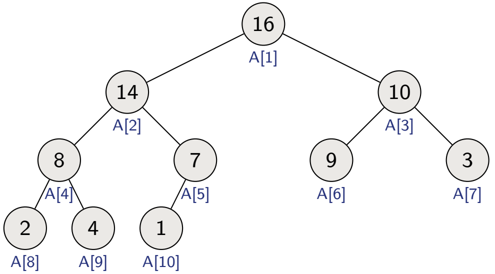
A =

16	14	10	8	7	9	3	2	4	1



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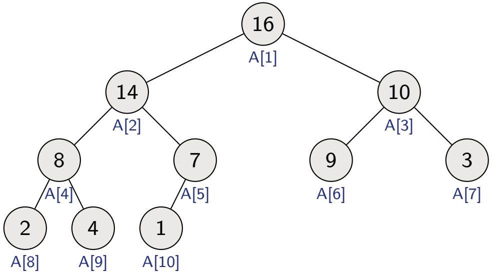

In this representation:

ROOT is $A[1]$

LEFT(i) =???

RIGHT(i) =???

PARENT(i) =???

How to store a heap/tree?

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Use that tree is almost complete to store it in array

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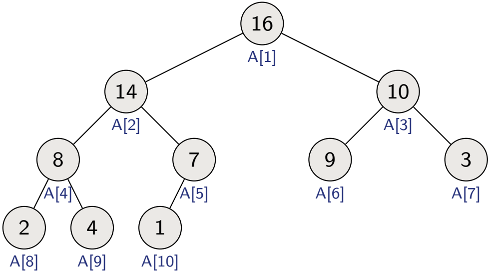

In this representation:

Root is $A[1]$

$\text{Left}(i) = 2i$

$\text{Right}(i) = ???$

$\text{Parent}(i) = ???$

How to store a heap/tree?

~~pointer to left and right children~~

Use that tree is almost complete to store it in array

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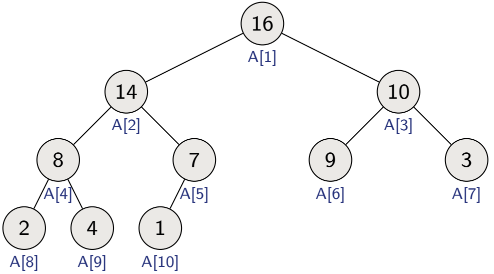

In this representation:

$\textsc{Root}$ is $A[1]$

$\textsc{Left}(i) = 2i$

$\textsc{Right}(i) = 2i + 1$

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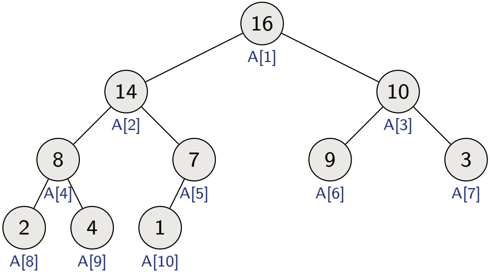


In this representation:

Root is $A[1]$

$\textsc{Left}(i) = 2i$

$\textsc{Right}(i) = 2i + 1$

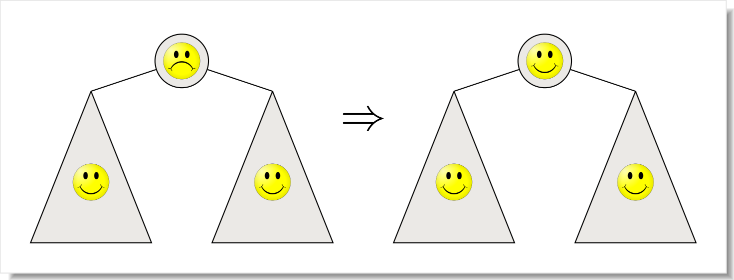
$\textsc{Parent}(i) = \lfloor i/2 \rfloor$

BUILDING AND MANIPULATING HEAPS

Maintaining the heap property

MAX-HEAPIFY is important for manipulating heaps:

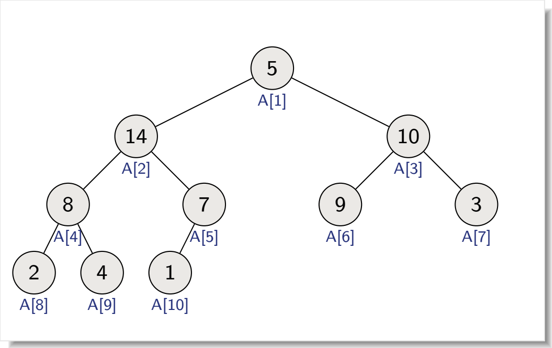
Given an i such that the subtrees of i are heaps, it ensures that the subtree rooted at i is a heap satisfy the heap property



MAX-HEAPIFY(A, i, n)

Algorithm:

- Compare $A[i], A[\text{LEFT}(i)], A[\text{RIGHT}(i)]$
- If necessary, swap $A[i]$ with the largest of the two children to preserve heap property
- Continue this process of comparing and swapping down the heap, until subtree rooted at i is max-heap



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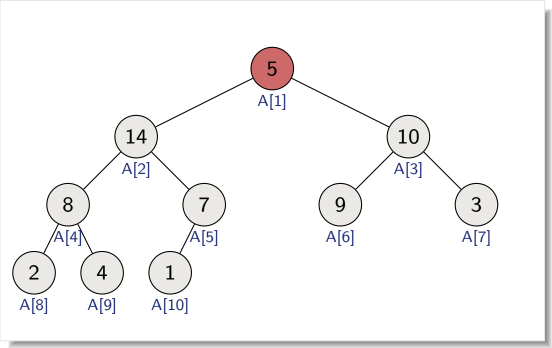

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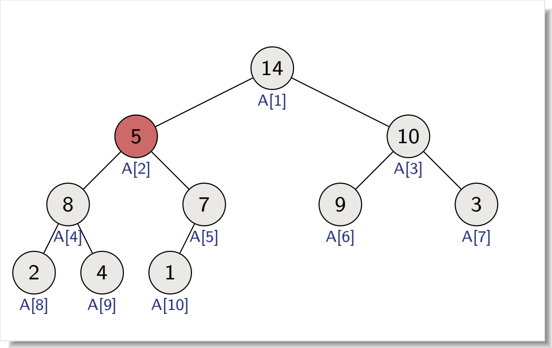

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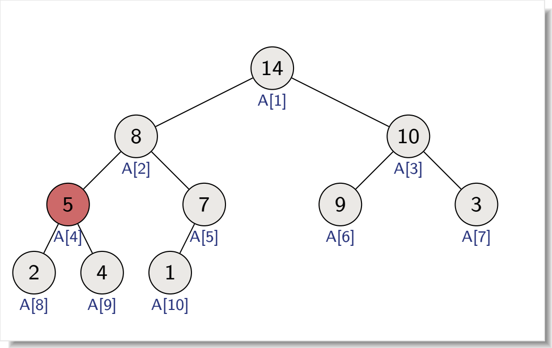

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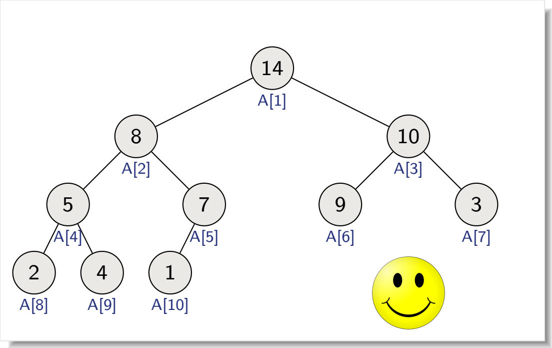

Pseudo-code and analysis

```
MAX-HEAPIFY(A, i, n)
  l = LEFT(i)
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  if l ≤ n and A[l] > A[i]
      largest = l
  else largest = i
  if r ≤ n and A[r] > A[largest]
      largest = r
  if largest ≠ i
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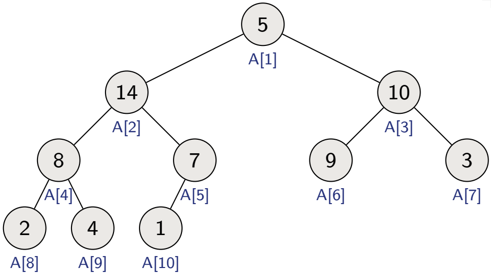
```



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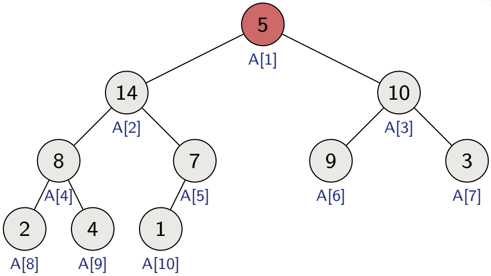

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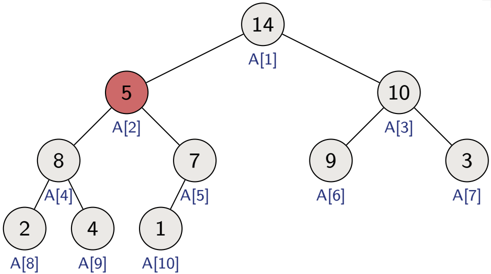

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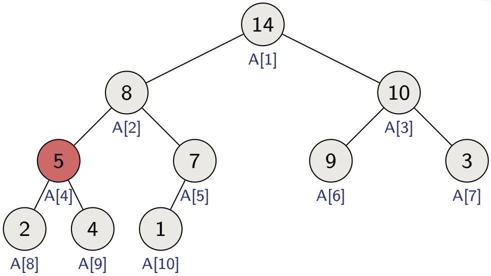

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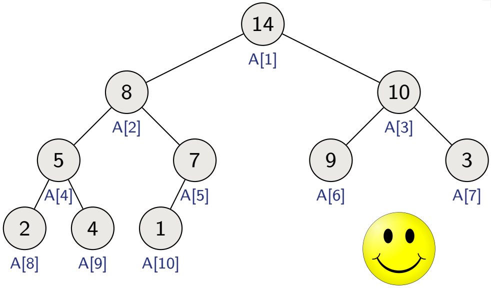

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Space?

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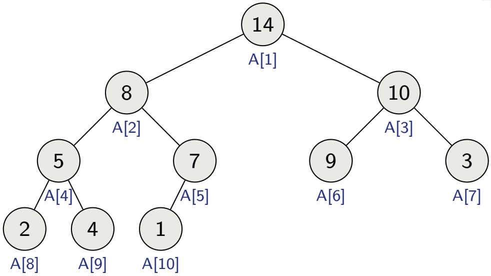

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$\Theta(\text{height of } i) = O(\log n)$

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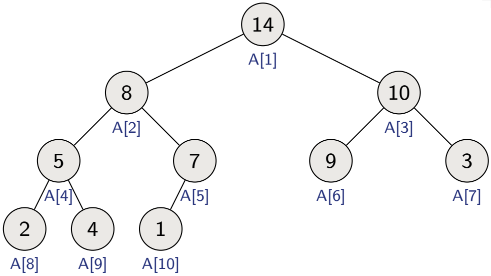


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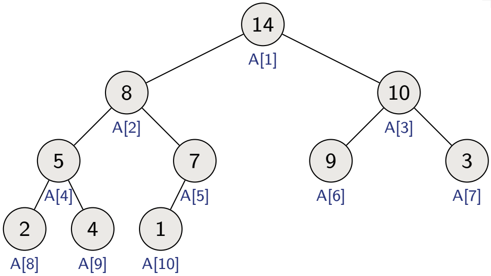


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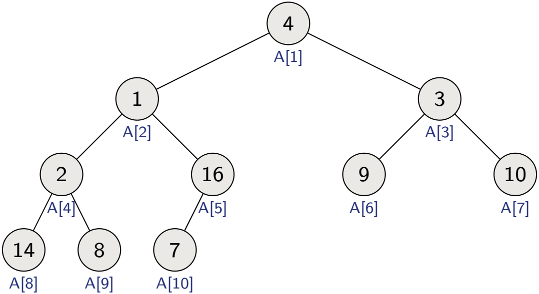
Building a heap

$\textsc{Build-Max-Heap}(A, n)$

for $i = \lfloor n/2 \rfloor$ **downto** 1
$\quad \textsc{Max-Heapify}(A, i, n)$

Given unordered array A of length n, $\textsc{Build-Max-Heap}$ outputs a heap

4	1	3	2	16	9	10	14	8	7



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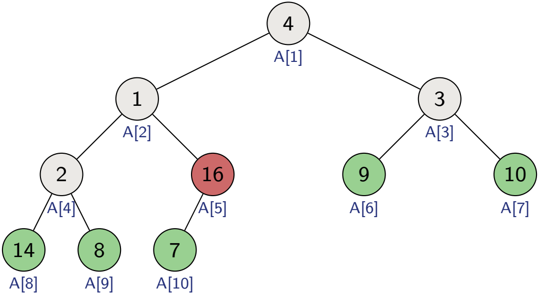

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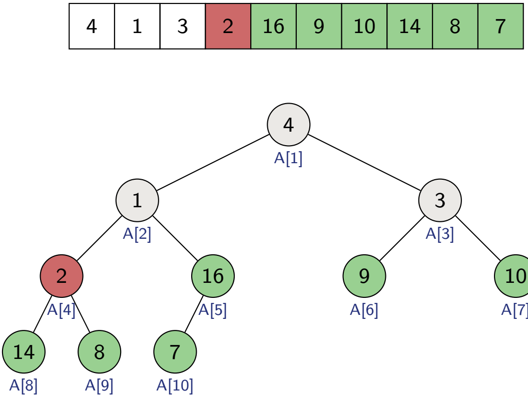

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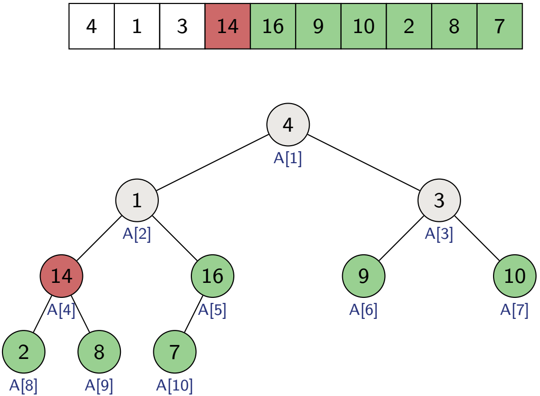

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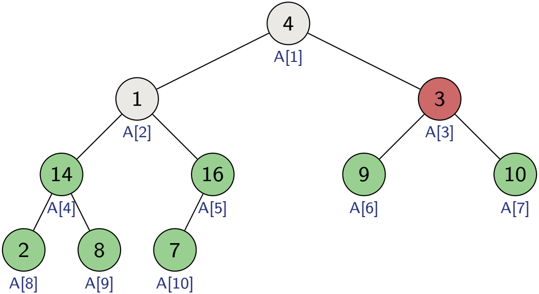

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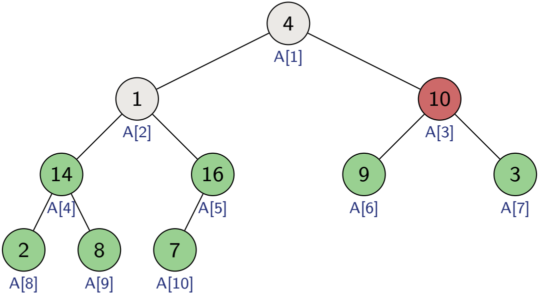

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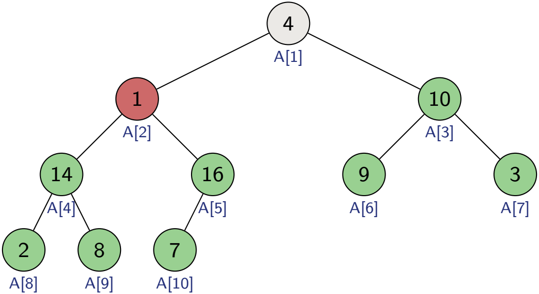

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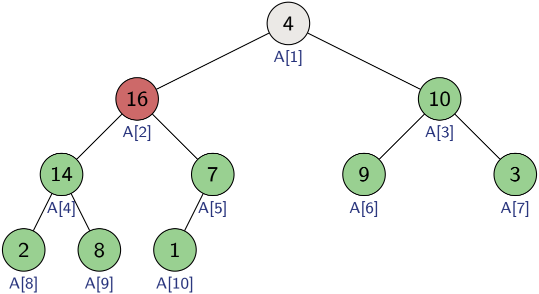

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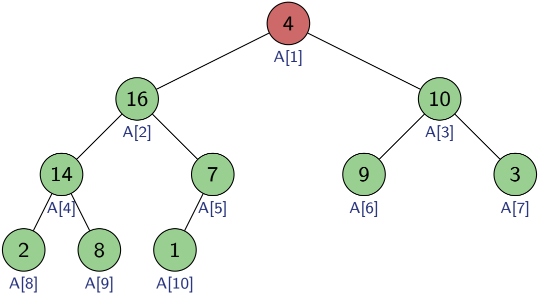

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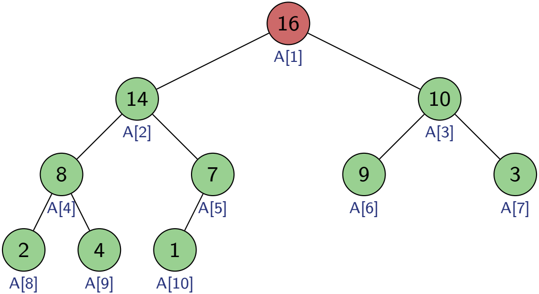

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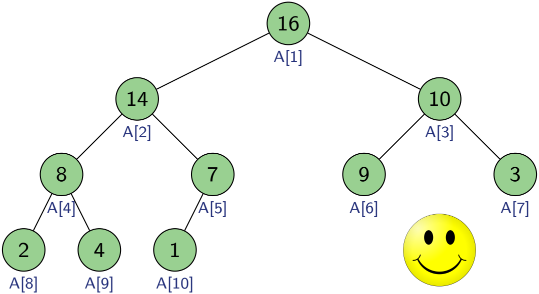


Analysis

```
BUILD-MAX-HEAP(A, n)
  for i = ⌊n/2⌋ downto 1
      MAX-HEAPIFY(A, i, n)
```

What is the worst-case running time of BUILD-MAX-HEAP?

Analysis

BUILD-MAX-HEAP(A, n)

for $i = \lfloor n/2 \rfloor$ **downto** 1

MAX-HEAPIFY(A, i, n)

What is the worst-case running time of BUILD-MAX-HEAP?

Simple bound: $O(n)$ calls to MAX-HEAPIFY, each of which takes $O(\lg n)$ time $\Rightarrow O(n \lg n)$ in total

Analysis

BUILD-MAX-HEAP(A, n)
for $i = \lfloor n/2 \rfloor$ **downto** 1
    MAX-HEAPIFY(A, i, n)

What is the worst-case running time of BUILD-MAX-HEAP?

Simple bound: $O(n)$ calls to MAX-HEAPIFY, each of which takes $O(\lg n)$ time $\Rightarrow O(n \lg n)$ in total

Tighter analysis: Time to run MAX-HEAPIFY is linear in the height of the node it's run on. Hence, the time is bounded by

$$\sum_{h=0}^{\lg n} \{\text{\# nodes of height } h\} O(h) = O\left(n \sum_{h=0}^{\lg n} \frac{h}{2^h}\right),$$

which is $O(n)$ since $\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$.

HEAPSORT

The heapsort algorithm

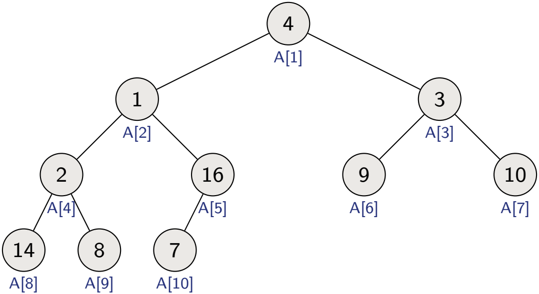
- Builds a max-heap from the array
- Starting with the root (the maximum element), the algorithm places the maximum element into the correct place in the array by swapping it with the element in the last position in the array
- "Discard" this last node (knowing that it is in its correct place) by decreasing the heap size, and calling MAX-HEAPIFY on the new (possibly incorrectly-placed) root
- Repeat this "discarding" process until only one node (the smallest element) remains, and therefore is in the correct place in the array

Example

HEAPSORT(A, n)

```
BUILD-MAX-HEAP(A, n)
for i = n downto 2
    exchange A[1] with A[i]
    MAX-HEAPIFY(A, 1, i − 1)
```

4	1	3	2	16	9	10	14	8	7



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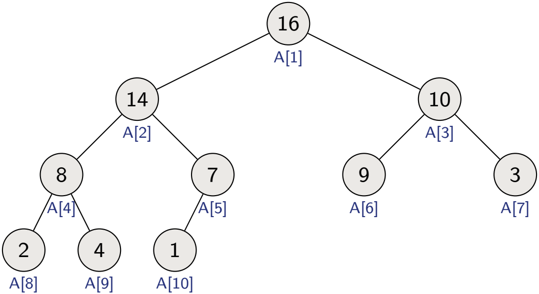
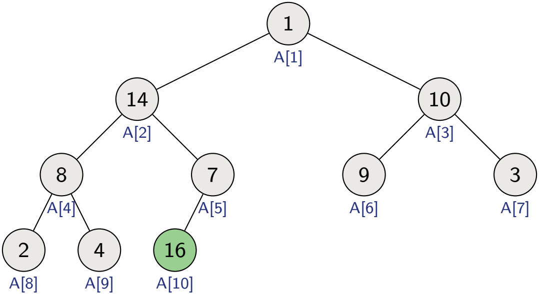


Example

Heapsort(A, n)
  Build-Max-Heap(A, n)
  for $i = n$ **downto** 2
    exchange $A[1]$ with $A[i]$
    Max-Heapify$(A, 1, i - 1)$

1	14	10	8	7	9	3	2	4	16



Example

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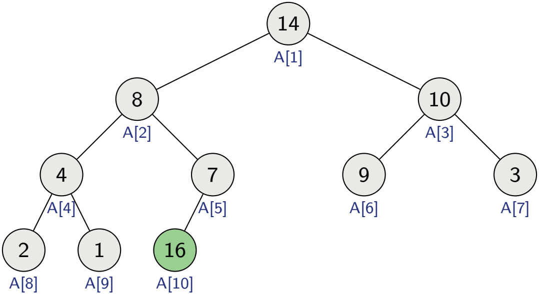
BUILD-MAX-HEAP(A, n)
for $i = n$ **downto** 2
exchange $A[1]$ with $A[i]$
MAX-HEAPIFY$(A, 1, i-1)$

14	8	10	4	7	9	3	2	1	16



Example

HEAPSORT(A, n)

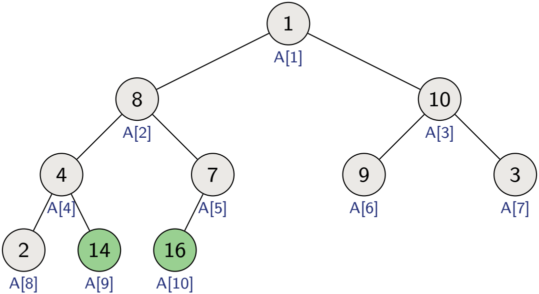
BUILD-MAX-HEAP(A, n)

for $i = n$ **downto** 2

exchange $A[1]$ with $A[i]$

MAX-HEAPIFY$(A, 1, i - 1)$

1	8	10	4	7	9	3	2	14	16



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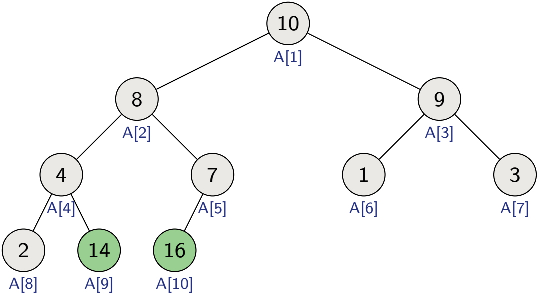
10	8	9	4	7	1	3	2	14	16



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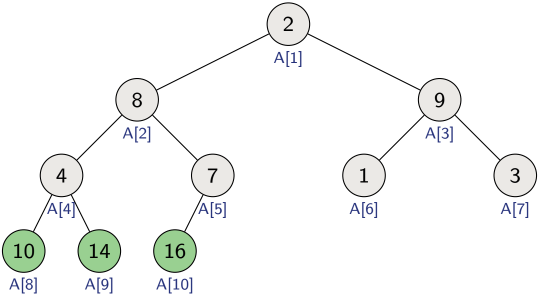

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BUILD-MAX-HEAP(A, n)
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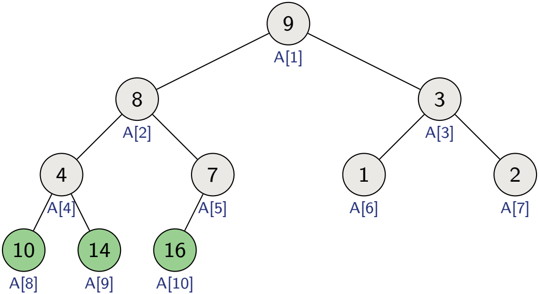
exchange $A[1]$ with $A[i]$
MAX-HEAPIFY$(A, 1, i - 1)$

9	8	3	4	7	1	2	10	14	16



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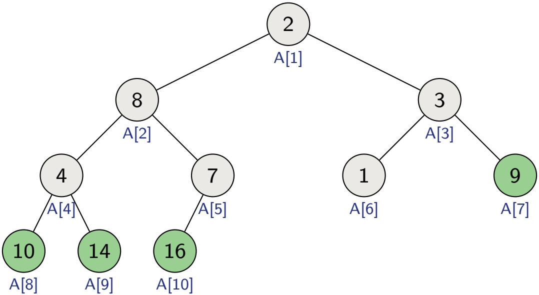

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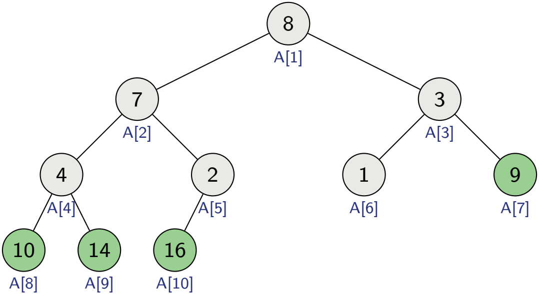

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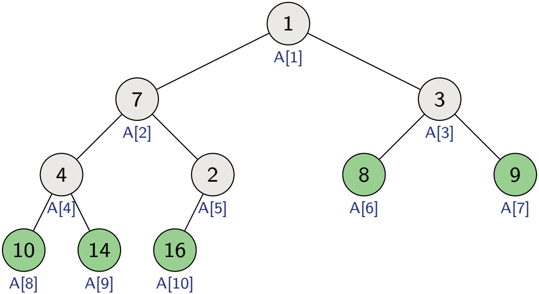

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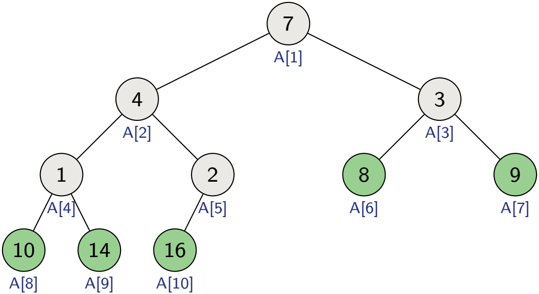

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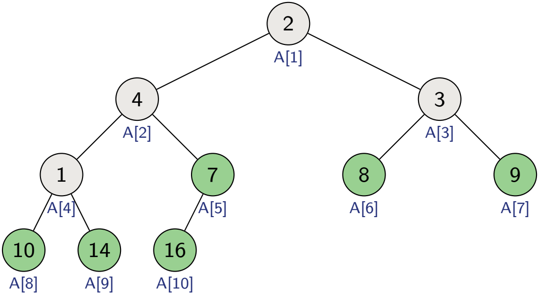
2	4	3	1	7	8	9	10	14	16



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HEAPSORT(A, n)
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```
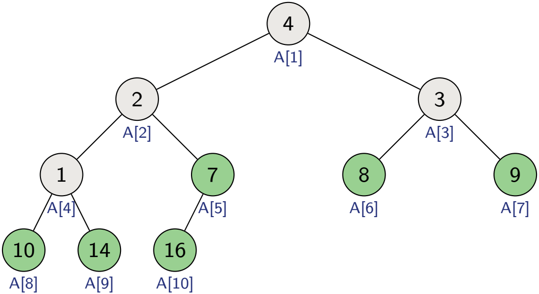

4	2	3	1	7	8	9	10	14	16



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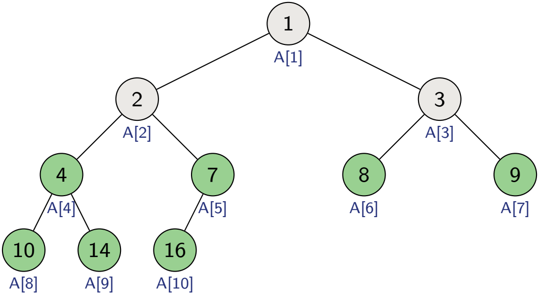

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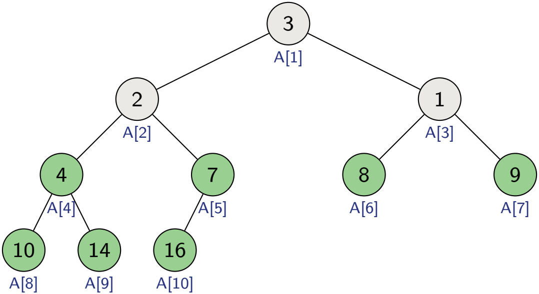
3	2	1	4	7	8	9	10	14	16



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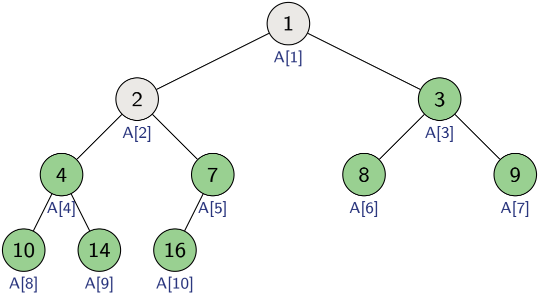

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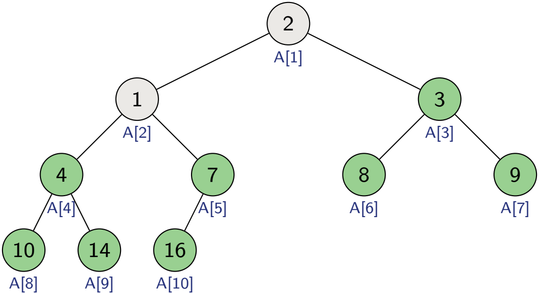
2	1	3	4	7	8	9	10	14	16



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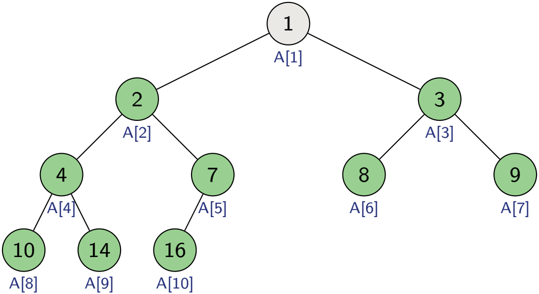

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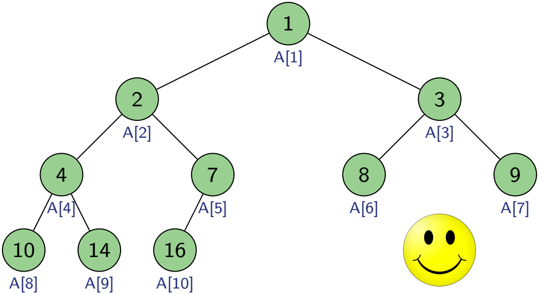

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- BUILD-MAX-HEAP: $O(n)$

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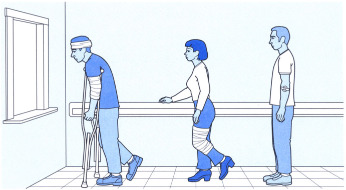
- BUILD-MAX-HEAP: $O(n)$
- **for** loop: $n - 1$ times
- exchange elements: $O(1)$
- MAX-HEAPIFY: $O(\lg n)$

Total time: $O(n \lg n)$

Summary

- Heapsort runs in time $O(n \log n)$ and is in-place
- Great algorithm but a well-implemented quicksort usually beats it in practice
- Heaps are nice, next lecture we will see how to use them for priority queues and we will also start with other data structures

HEAP IMPLEMENTATION OF PRIORITY QUEUE



Priority Queue

- Maintains a dynamic set S of elements
- Each set element has a key — an associated value that regulates its importance

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INCREASE-KEY(S,x,k): increases value of element x's key to k;
assume $k \geq x$'s current key value

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Example max-priority queue application: schedule jobs on shared computer

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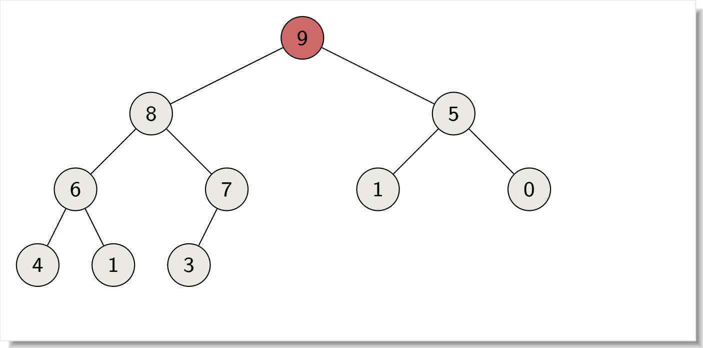
Example max-priority queue application: schedule jobs on shared computer

Heaps efficiently implement priority queues

Finding maximum element

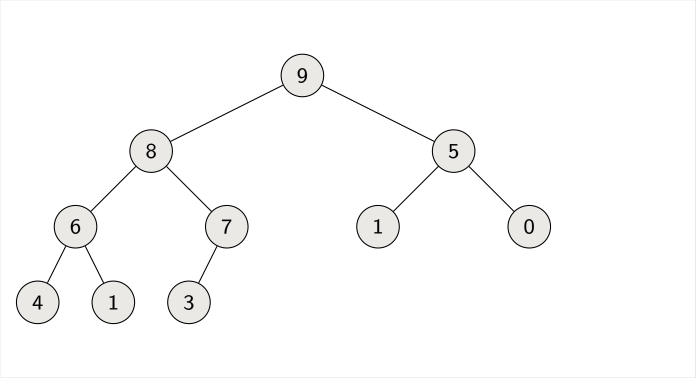
HEAP-MAXIMUM(A)
    return $A[1]$

Simply return the root in time $\Theta(1)$



Extracting maximum element

HEAP-EXTRACT-MAX(A, n)
if $n < 1$
  error "heap underflow"
$max = A[1]$
$A[1] = A[n]$
$n = n - 1$
MAX-HEAPIFY$(A, 1, n)$
return max



Extracting maximum element

1. Make sure heap is not empty

HEAP-EXTRACT-MAX(A, n)

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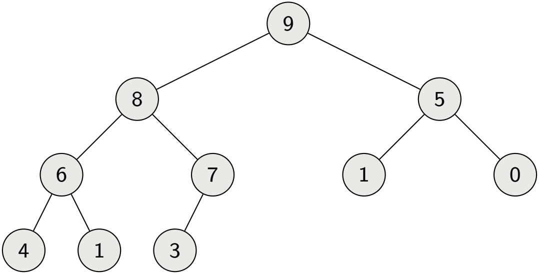

Extracting maximum element

1. Make sure heap is not empty
2. Make a copy of the maximum element (the root)

HEAP-EXTRACT-MAX(A, n)
- **if** $n < 1$
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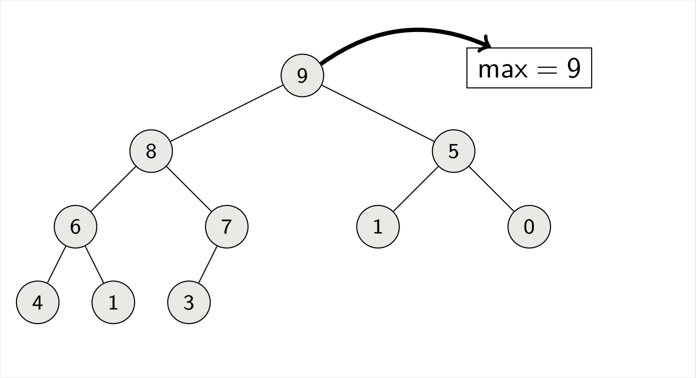
- **return** max



Extracting maximum element

1. Make sure heap is not empty
2. Make a copy of the maximum element (the root)
3. Make the last node in the tree the new root

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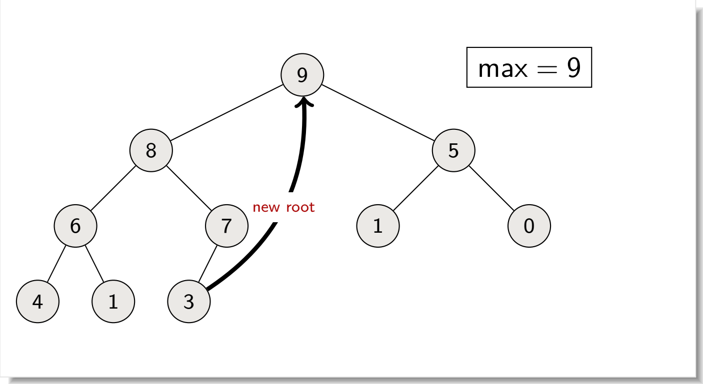

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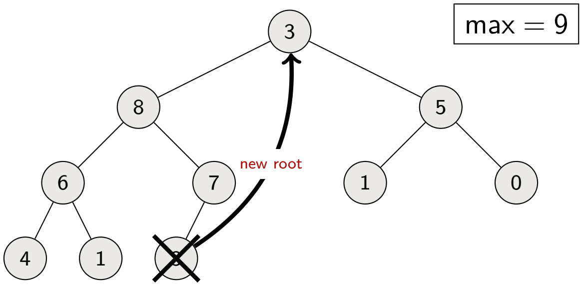

Extracting maximum element

1. Make sure heap is not empty
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4. Re-heapify the heap, with one fewer node

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  n = n - 1
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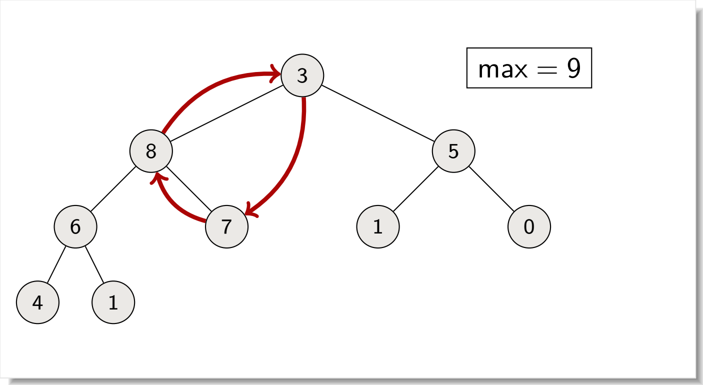
```



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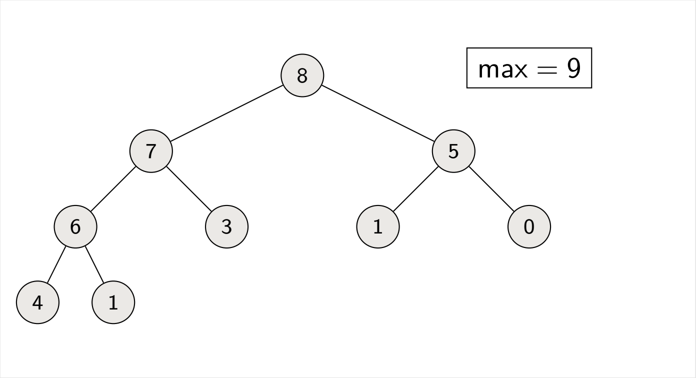

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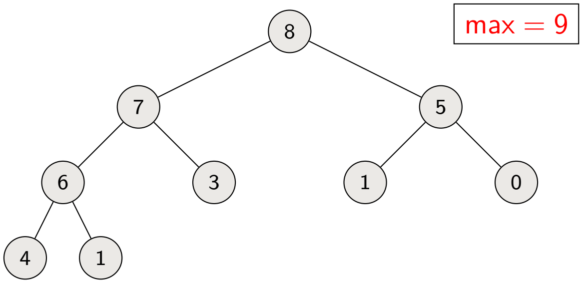

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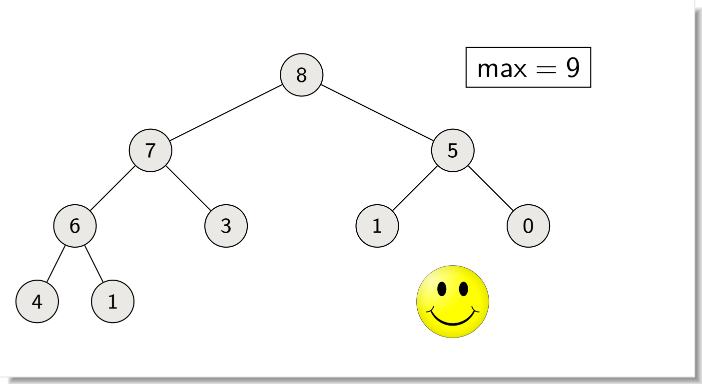

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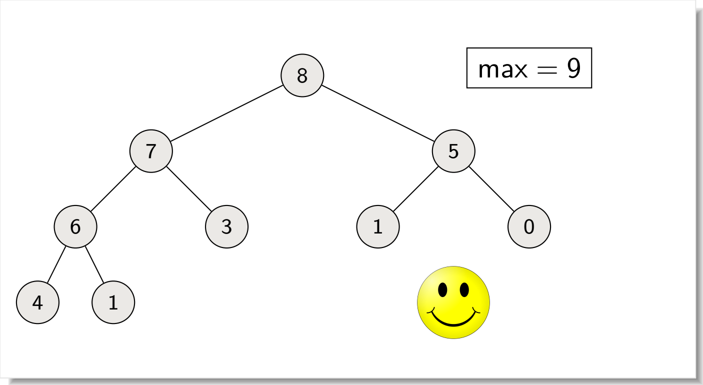
MAX-HEAPIFY$(A, 1, n)$
return *max*

Analysis:



Extracting maximum element

Analysis: Constant-time assignments plus time for MAX-HEAPIFY

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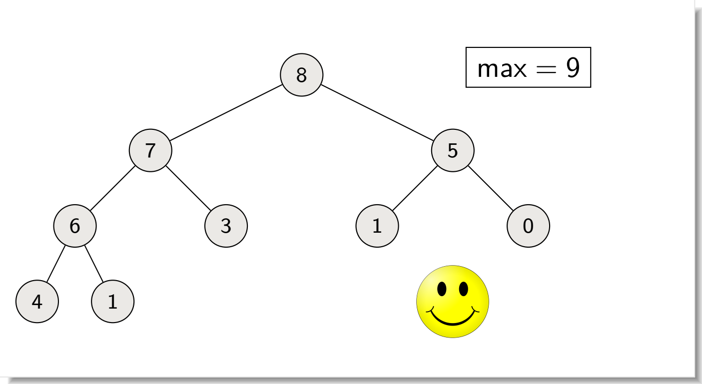

Extracting maximum element

Analysis: Constant-time assignments plus time for MAX-HEAPIFY

Hence, it runs in time $O(\lg n)$

HEAP-EXTRACT-MAX(A, n)
- **if** $n < 1$
  - **error** "heap underflow"
- $max = A[1]$
- $A[1] = A[n]$
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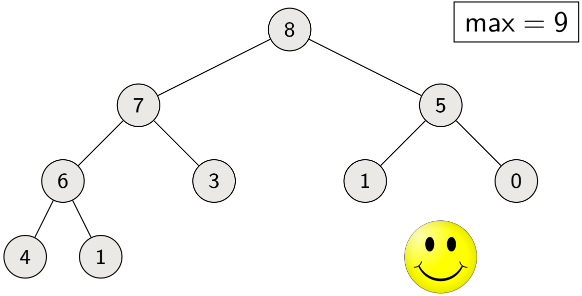
- **return** max

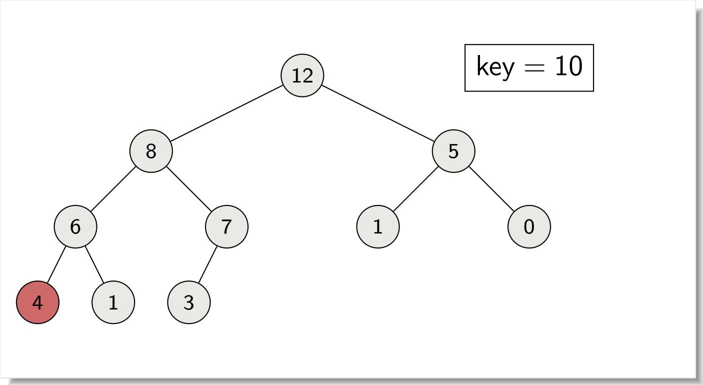


Increasing key value

Given a heap A, index i, and new value *key*

HEAP-INCREASE-KEY(A, i, key)

```
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```



Increasing key value

Given a heap A, index i, and new value key

1. Make sure $key \geq A[i]$

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
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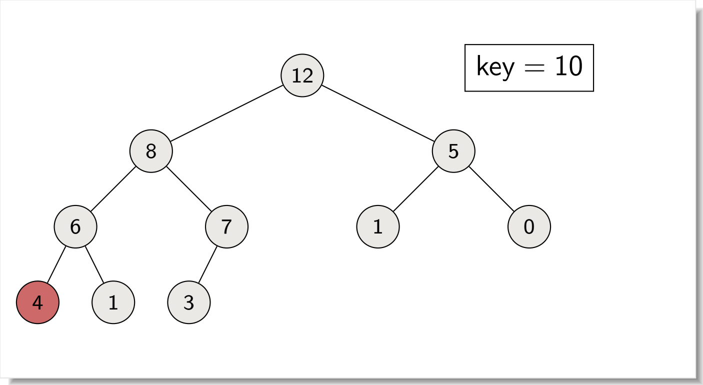

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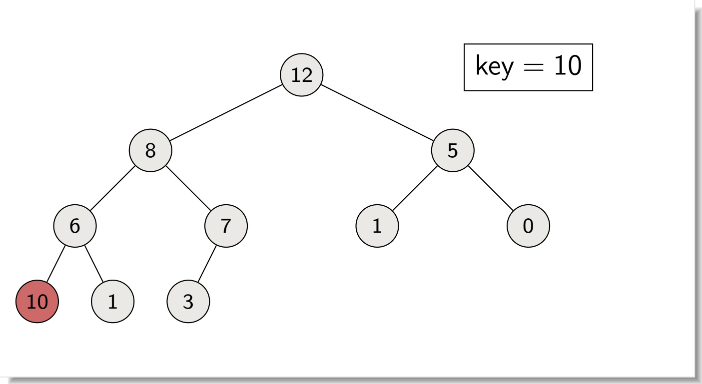

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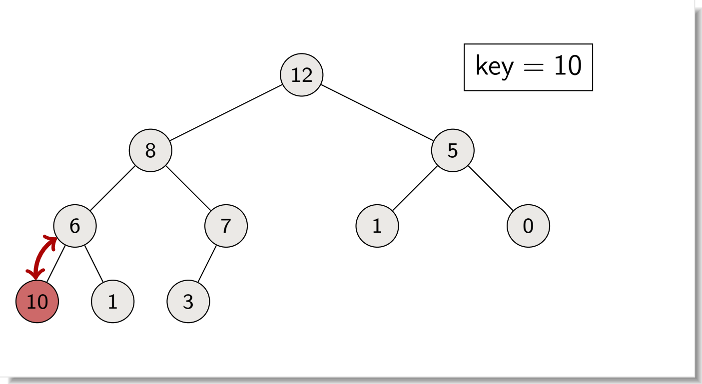

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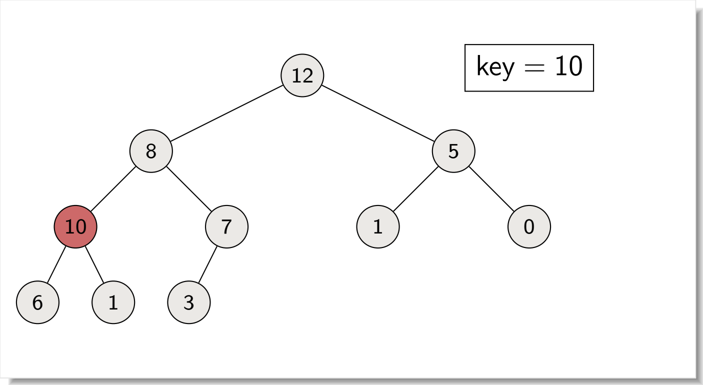

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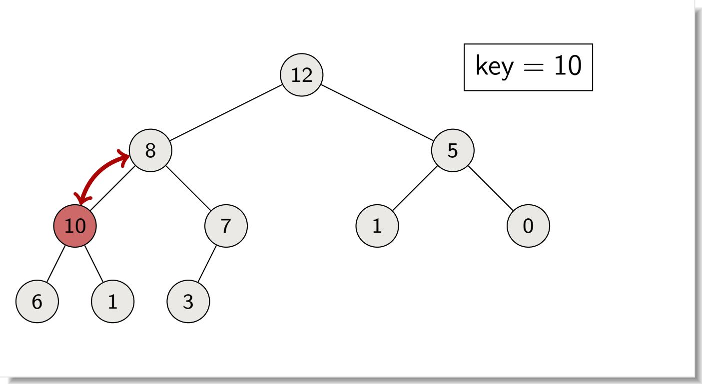

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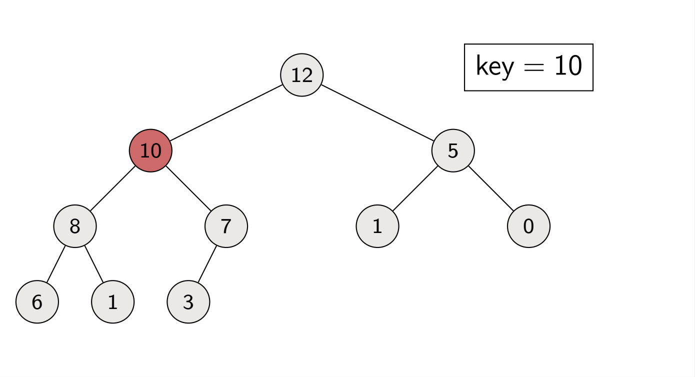

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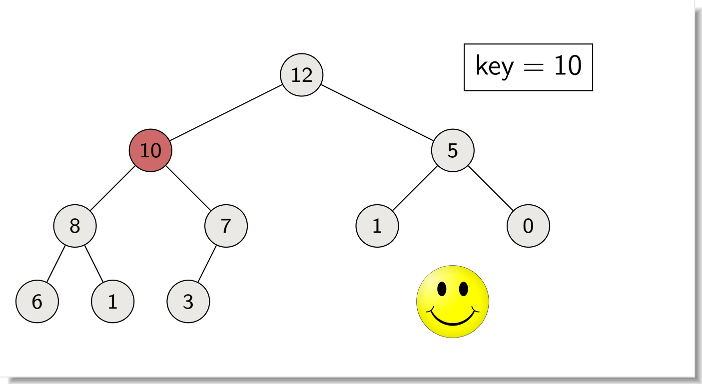

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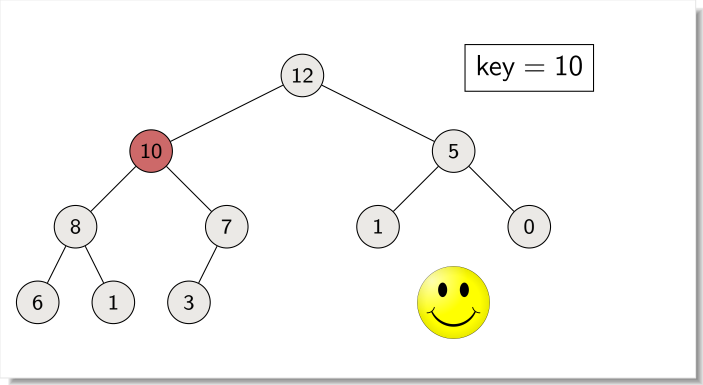

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Upward path from node i has length $O(\lg n)$ in an n-element heap

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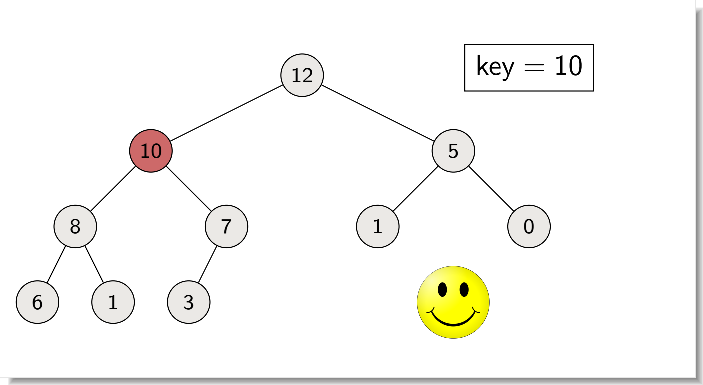

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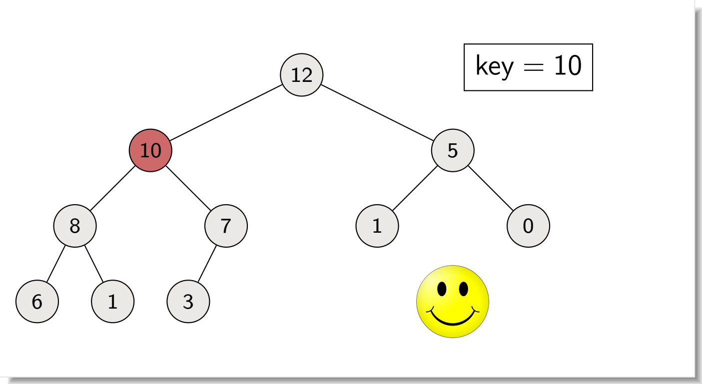

Inserting into the heap

Given a new *key* to insert into heap

MAX-HEAP-INSERT(A, key, n)

$n = n + 1$
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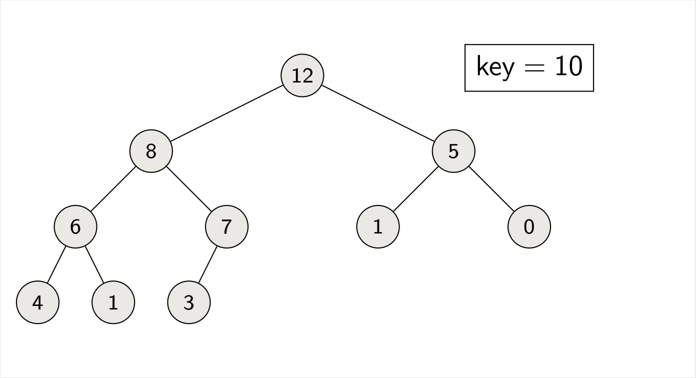

Inserting into the heap

Given a new *key* to insert into heap

1. Increment the heap size

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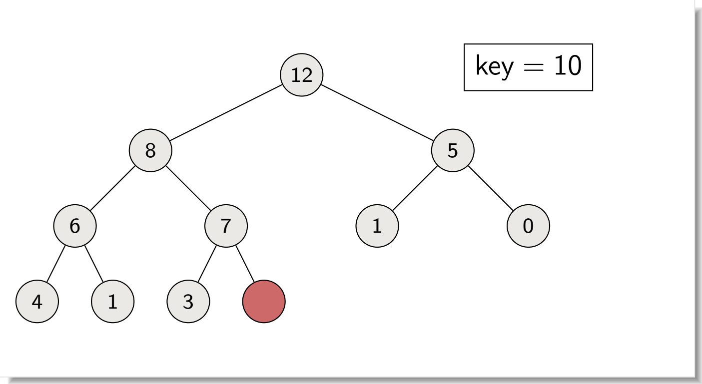

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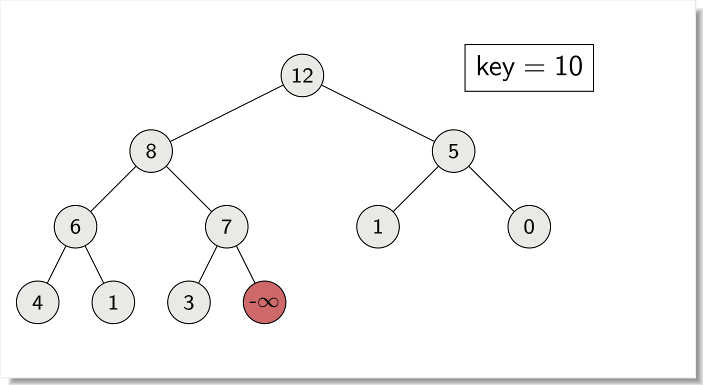

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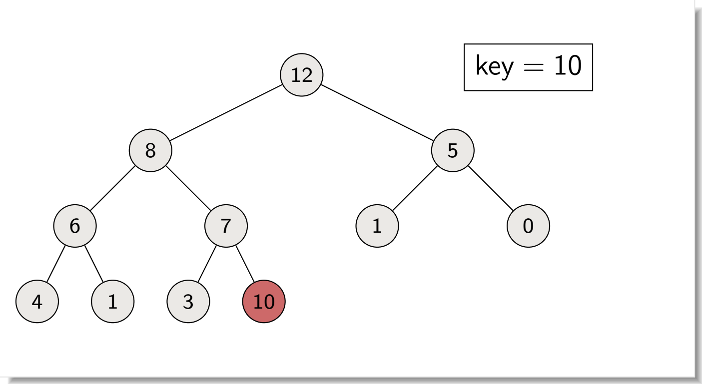

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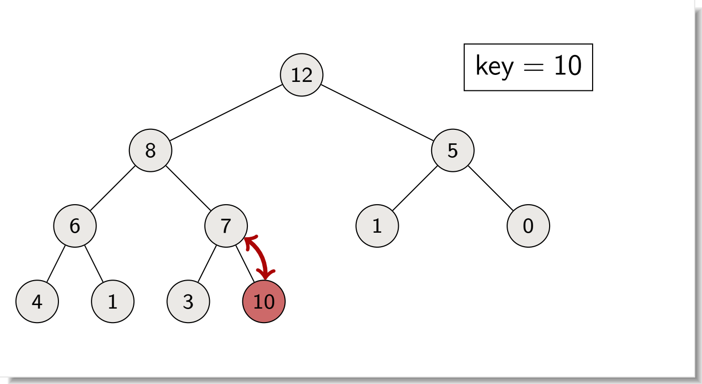

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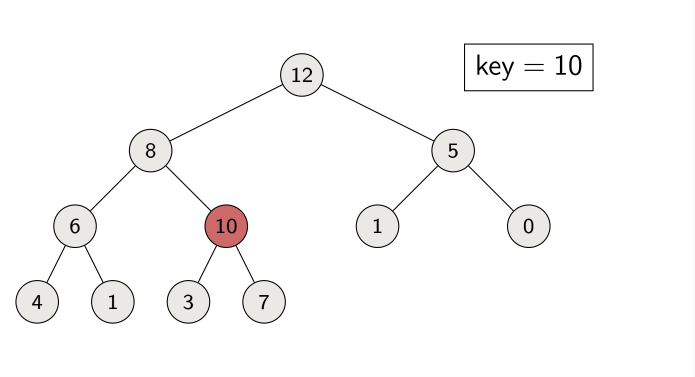

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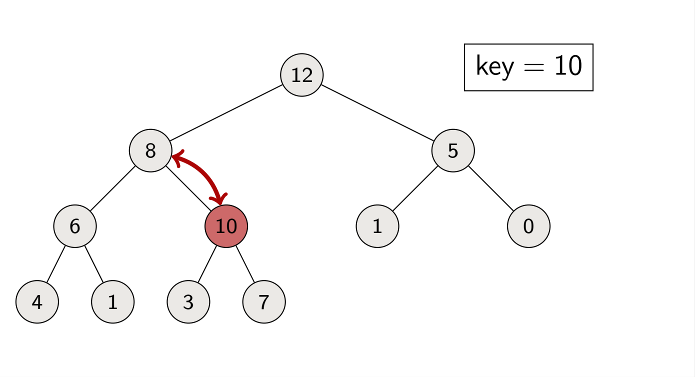

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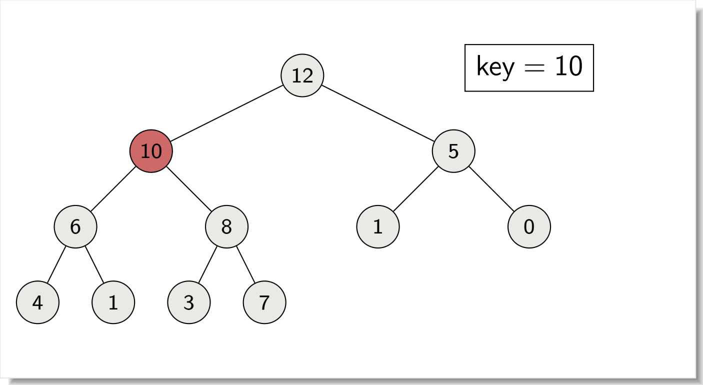

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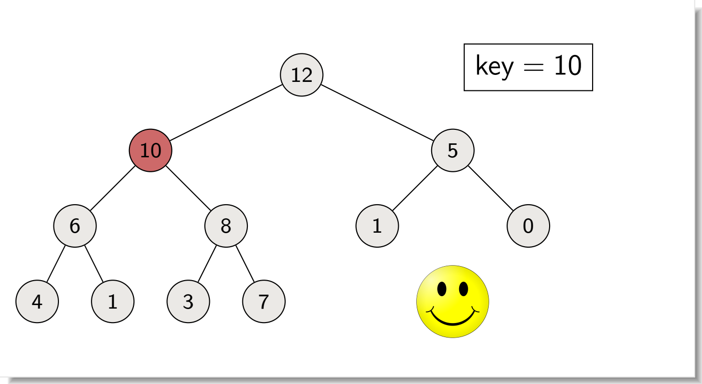

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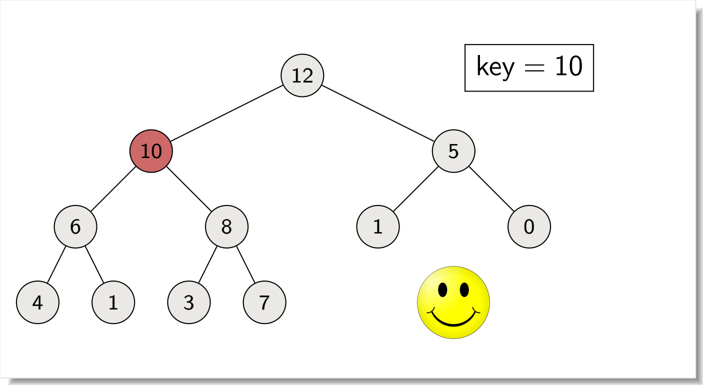

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Analysis: Constant time assignments

$+$

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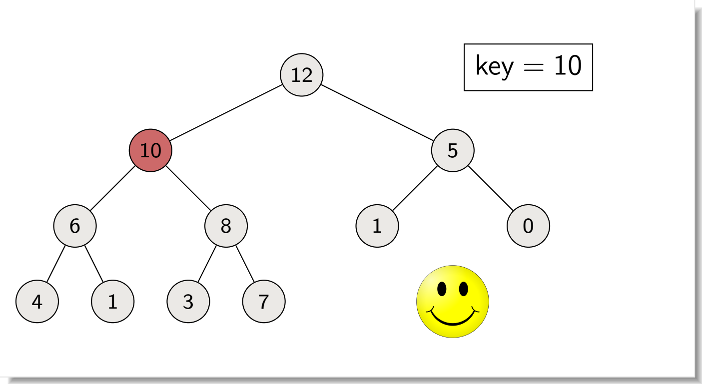

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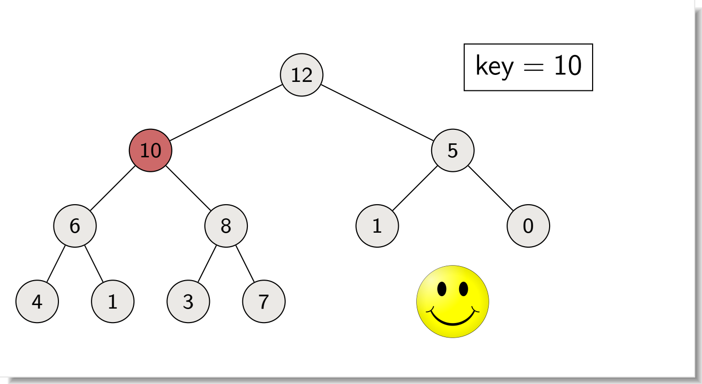

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- Min-priority queues are implemented with min-heaps similarly