Algorithms: Divide-and-Conquer (Matrix multiplication)

Alessandro Chiesa, Ola Svensson

EPFL School of Computer and Communication Sciences

Lecture 5, 04.03.2025

RECALL LAST LECTURE

Solving Recurrences:

- Substitution method
- Recursion Trees
- (Master Method)

Maximum-Subarray Problem

Recall Recursion trees

Example: $T(1) = c$ and $T(n) = 2T(n/2) + cn^2$

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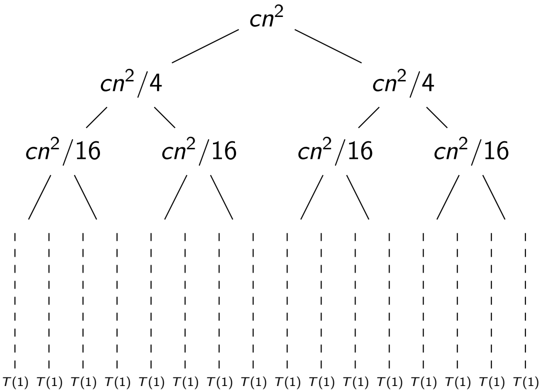

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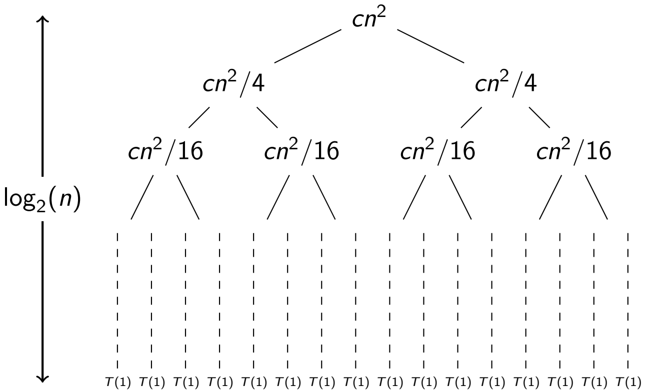

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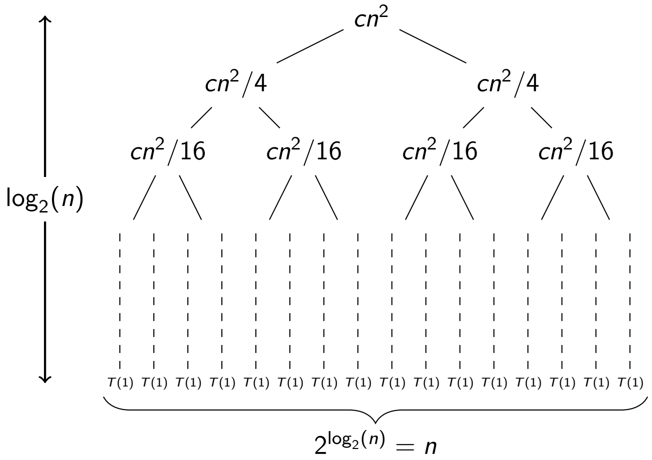

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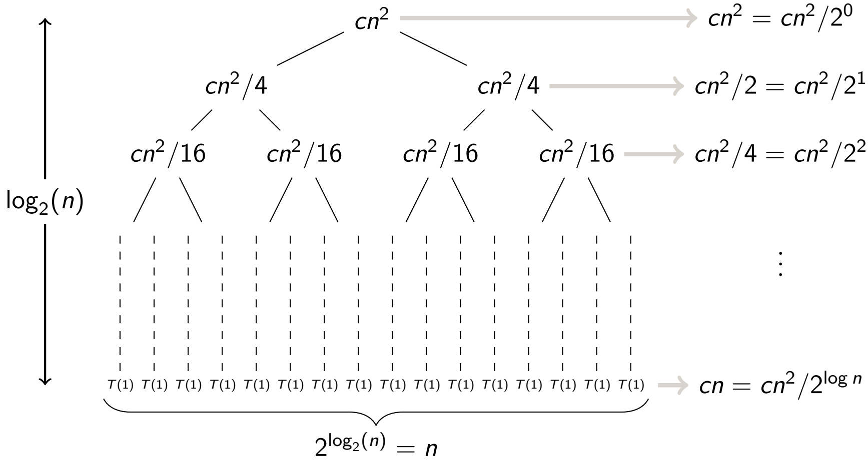

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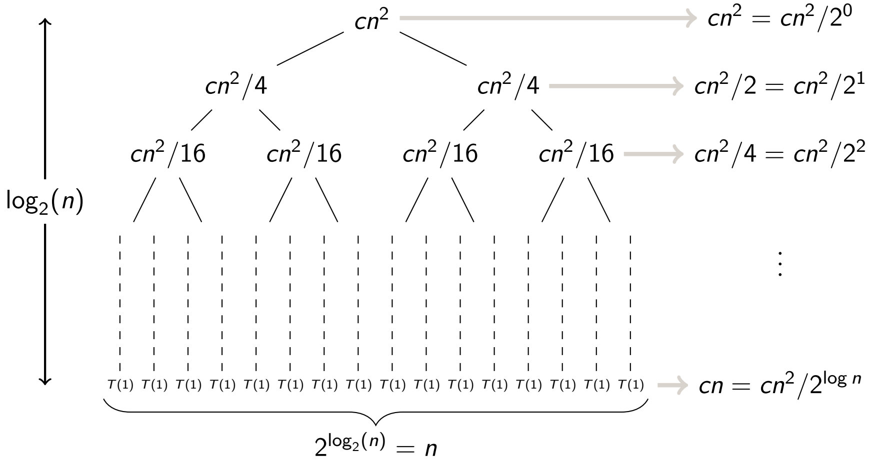

Qualified guess: $T(n) = cn^2 \sum_{i=0}^{\log n} \frac{1}{2^i} \leq cn^2 = \Theta(n^2)$

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$$\vdots$$

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$$cn \longrightarrow cn$$

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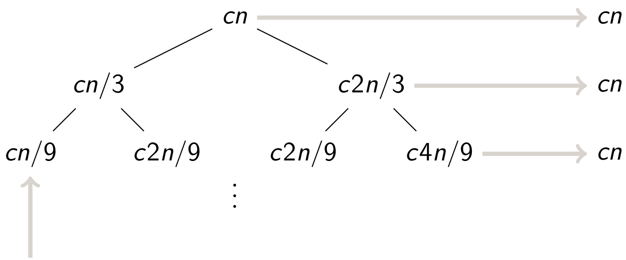

leftmost branch peters out after $\log_3 n$ levels

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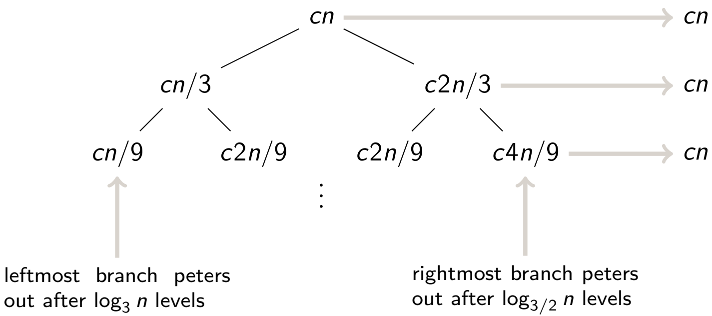

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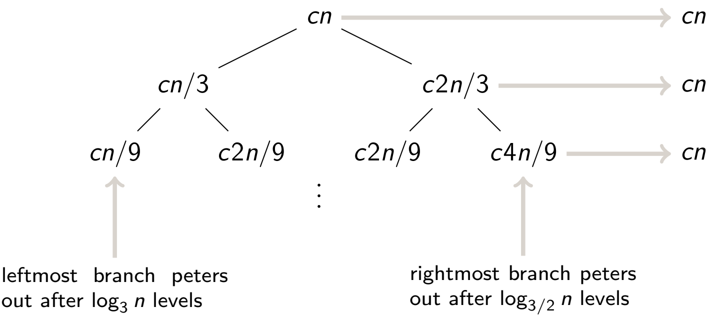

- There are $\log_3 n$ full levels and after $\log_{3/2} n$ levels the problem size is down to 1.

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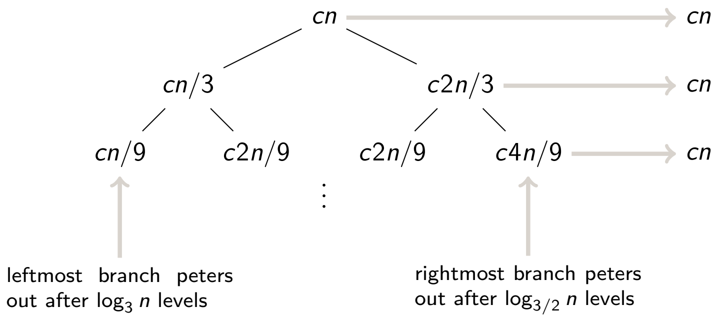

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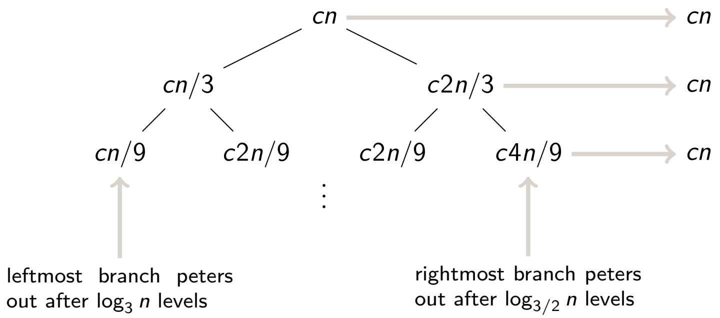

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- Each level contributes $\approx cn$

Qualified guess: exist positive constants a, b so that

$$a \cdot n \log_3(n) \leq T(n) \leq b \cdot n \log_{3/2} n \Rightarrow T(n) = \Theta(n \log n)$$

Master method

Used to black-box solve recurrences of the form $T(n) = aT(n/b) + f(n)$

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Let $a \geq 1$ and $b > 1$ be constants, let $T(n)$ be defined on the nonnegative integers by the recurrence

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- *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$*

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Our favorite example: $T(1) = c$ and $T(n) = 2T(n/2) + cn$

- $f(n) = O(n)$ and $a = b = 2$ so $\log_b(a) = 1$ and $f(n) = \Theta(n^{\log_b(a)})$.

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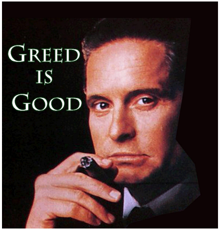
- $f(n) = O(n)$ and $a = b = 2$ so $\log_b(a) = 1$ and $f(n) = \Theta(n^{\log_b(a)})$.
- By Master theorem, we have $T(n) = \Theta(n \log n)$:)



MAXIMUM-SUBARRAY PROBLEM

Maximum-subarray problem

"If we let $A[i] =$ (price after day i) $-$ (price after day $i-1$) then if the maximum subarray is $A[i \ldots j]$ then we should have bought just before day i and sold just after day j."

Definition

INPUT: An array $A[1 \ldots n]$ of numbers

OUTPUT: Indices i and j such that $A[i \ldots j]$ has the greatest sum of any nonempty, contiguous subarray of A, along with the sum of the values in $A[i \ldots j]$

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-2	-4	3	-1	5	7	-7	-2	4	-3	2

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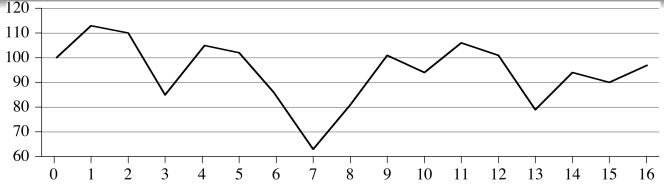
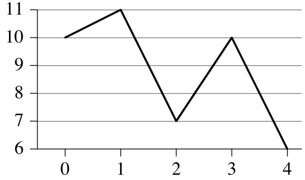
output is $i = 3$ and $j = 6$ and the sum 14

Maximum-subarray problem

More examples



Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	−3	−25	20	−3	−16	−23	18	20	−7	12	−5	−22	15	−4	7

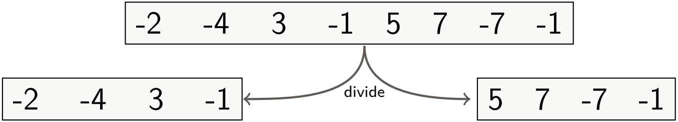


Day	0	1	2	3	4
Price	10	11	7	10	6
Change		1	−4	3	−4

Divide-and-Conquer

-2	-4	3	-1	5	7	-7	-1

Divide-and-Conquer



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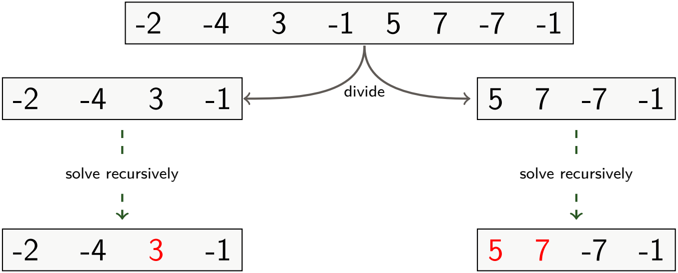

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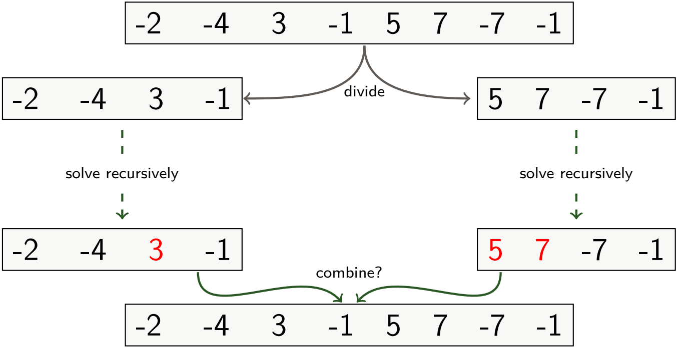

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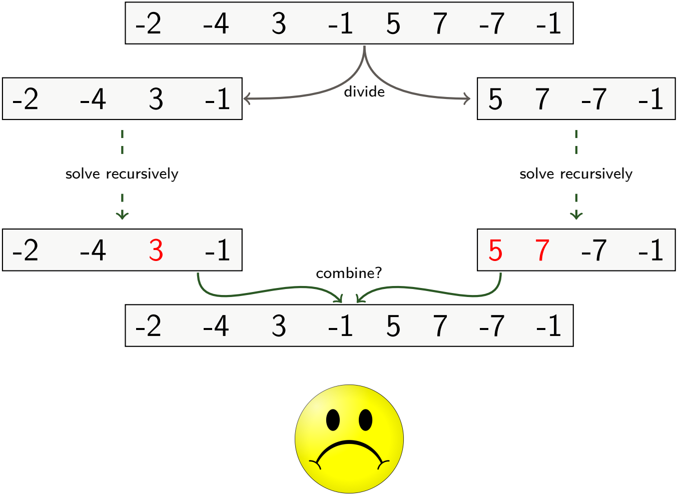

Solution

Also find the maximum subarray that crosses the midpoint!

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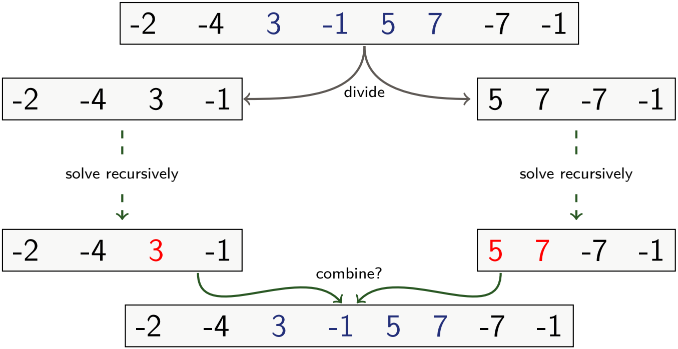

Divide-and-Conquer approach

Divide the subarray into two subarrays of as equal size as possible. Find the midpoint *mid* of the subarrays, and consider the subarrays $A[low \dots mid]$ and $A[mid + 1 \dots high]$.

Conquer by finding maximum subarrays of $A[low \dots mid]$ and $A[mid + 1 \dots high]$.

Combine by finding a maximum subarray that crosses the midpoint, and using the best solution out of the three

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This strategy works because any subarray must either lie entirely on one side of the midpoint or cross the midpoint

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FIND-MAXIMUM-SUBARRAY(A, low, high)
 if high == low
     return (low, high, A[low])            // base case: only one element
 else mid = ⌊(low + high)/2⌋
     (left-low, left-high, left-sum) =
         FIND-MAXIMUM-SUBARRAY(A, low, mid)
     (right-low, right-high, right-sum) =
         FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
     (cross-low, cross-high, cross-sum) =
         FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
     if left-sum ≥ right-sum and left-sum ≥ cross-sum
         return (left-low, left-high, left-sum)
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Analysis

Assume that we can find max-crossing-subarray in time $\Theta(n)$

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Divide takes constant time,i.e., $\Theta(1)$

Conquer recursively solve two subproblems, each of size $n/2 \Rightarrow T(n/2)$

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Merge time dominated by FIND-MAX-CROSSING-SUBARRAY $\Rightarrow \Theta(n)$

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Conquer recursively solve two subproblems, each of size $n/2 \Rightarrow T(n/2)$

Merge time dominated by FIND-MAX-CROSSING-SUBARRAY $\Rightarrow \Theta(n)$

Recursion for the running time is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

Analysis

Assume that we can find max-crossing-subarray in time $\Theta(n)$

```
FIND-MAXIMUM-SUBARRAY(A, low, high)
  if high == low
      return (low, high, A[low])            // base case: only one element
  else mid = ⌊(low + high)/2⌋
      (left-low, left-high, left-sum) =
          FIND-MAXIMUM-SUBARRAY(A, low, mid)
      (right-low, right-high, right-sum) =
          FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
      (cross-low, cross-high, cross-sum) =
          FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
      if left-sum ≥ right-sum and left-sum ≥ cross-sum
          return (left-low, left-high, left-sum)
      elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
          return (right-low, right-high, right-sum)
      else return (cross-low, cross-high, cross-sum)
```

Divide takes constant time,i.e., $\Theta(1)$

Conquer recursively solve two subproblems, each of size $n/2 \Rightarrow T(n/2)$

Merge time dominated by FIND-MAX-CROSSING-SUBARRAY $\Rightarrow \Theta(n)$

Recursion for the running time is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

Hence, $T(n) = \Theta(n \log n)$

Finding maximum subarray crossing midpoint

Finding maximum subarray crossing midpoint

- Any subarray crossing the midpoint $A[mid]$ is made of two subarrays $A[i \ldots mid]$ and $A[mid+1, \ldots, j]$ where $low \leq i \leq mid$ and $mid < j \leq high$
- Find maximum subarrays of the form $A[i \ldots mid]$ and $A[mid+1 \ldots j]$ and then combine them.

-2	-4	3	-1	5	7	-7	-1

-2	-4	3	-1

5	7	-7	-1

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Crossing subarray

```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
  // Find a maximum subarray of the form A[i .. mid].
  left-sum = −∞
  sum = 0
  for i = mid downto low
      sum = sum + A[i]
      if sum > left-sum
          left-sum = sum
          max-left = i
  // Find a maximum subarray of the form A[mid + 1 .. j].
  right-sum = −∞
  sum = 0
  for j = mid + 1 to high
      sum = sum + A[j]
      if sum > right-sum
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  // Return the indices and the sum of the two subarrays.
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```

low			mid				high
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Crossing subarray

low			mid	
-2	-4	3	-1	5

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Crossing subarray

Running time?

Space?

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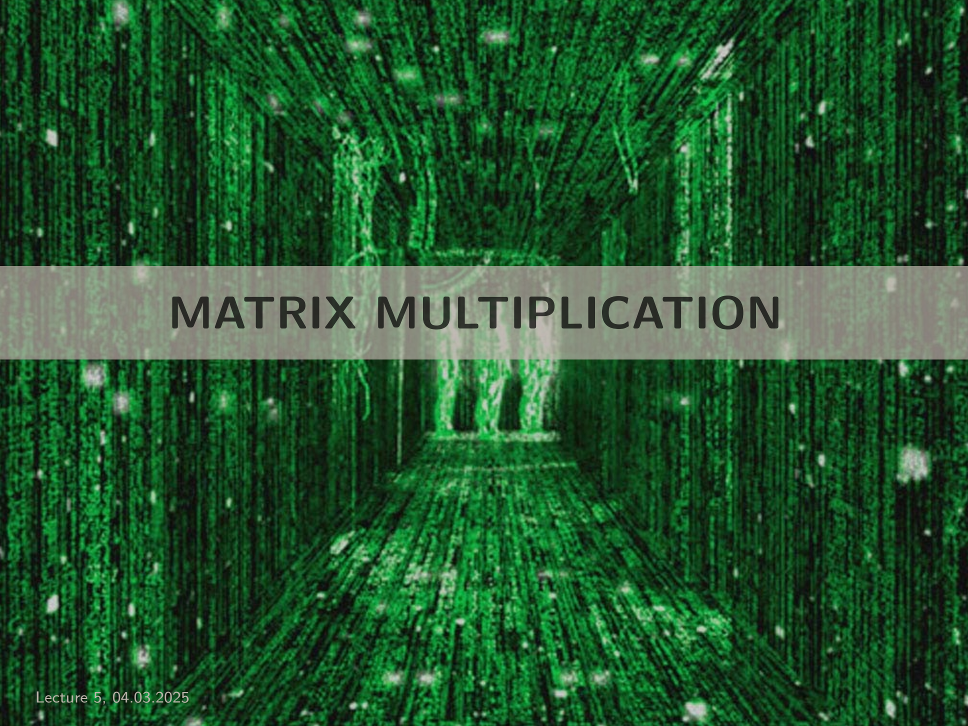

MATRIX MULTIPLICATION

Lecture 5, 04.03.2025

Matrix Multiplication

Definition

Input: Two $n \times n$ (square) matrices, $A = (a_{ij})$ and $B = (b_{ij})$

Output: $n \times n$ matrix $C = (c_{ij})$, where $C = A \cdot B$

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Example ($n = 2$):

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21}, \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22}, \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21}, \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22}. \end{aligned}$$

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$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

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Example ($n = 2$):

$$\begin{pmatrix} 5 & 3 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

How to multiply two matrices?

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$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

$$c_{1,1} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1} + \ldots, a_{1,n}b_{n,1} = \sum_{k=1}^{n} a_{1k}b_{k1}$$

How to multiply two matrices?

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$$c_{2,1} = a_{2,1}b_{1,1} + a_{2,2}b_{2,1} + a_{2,3}b_{3,1} + \ldots, a_{2,n}b_{n,1} = \sum_{k=1}^{n} a_{2k}b_{k1}$$

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$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix}$$

$$c_{2,2} = a_{2,1}b_{1,2} + a_{2,2}b_{2,2} + a_{2,3}b_{3,2} + \dots, a_{2,n}b_{n,2} = \sum_{k=1}^{n} a_{2k}b_{k2}$$

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Naive Algorithm

Well simply multiply the matrices...

```
SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
          c_ij = 0
          for k = 1 to n
              c_ij = c_ij + a_ik · b_kj
  return C
```

Example:

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

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Example:

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$$\begin{pmatrix} 5 & -1 \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

Naive Algorithm

Well simply multiply the matrices...

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SQUARE-MAT-MULT(A, B, n)
  let C be a new n × n matrix
  for i = 1 to n
      for j = 1 to n
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```

Example:

$$\begin{pmatrix} 5 & 3 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

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Running time?

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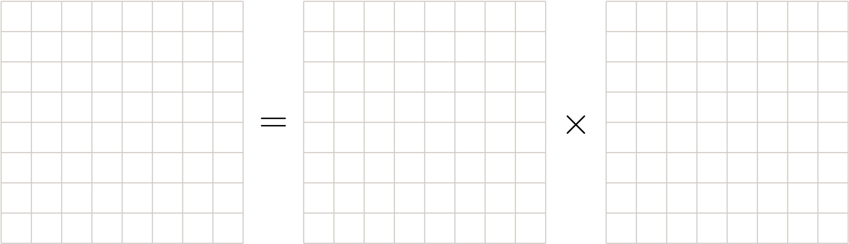
Running time? $\Theta(n^3)$

Space? $\Theta(n^2)$

Smart Algorithm: Divide-and-Conquer



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$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

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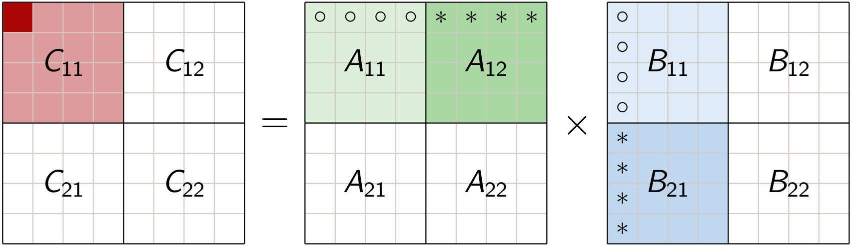

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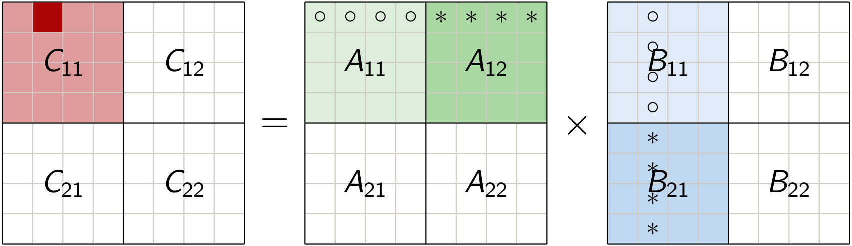

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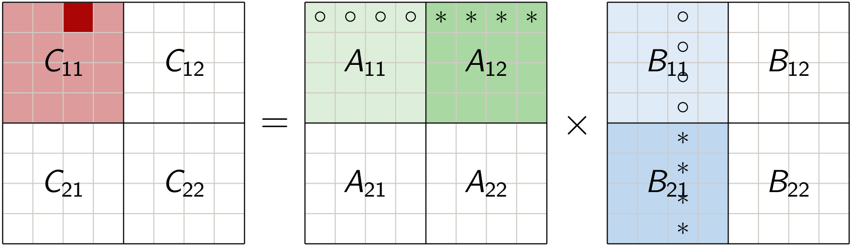

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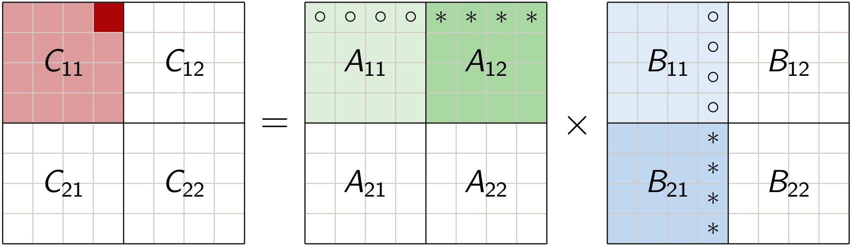

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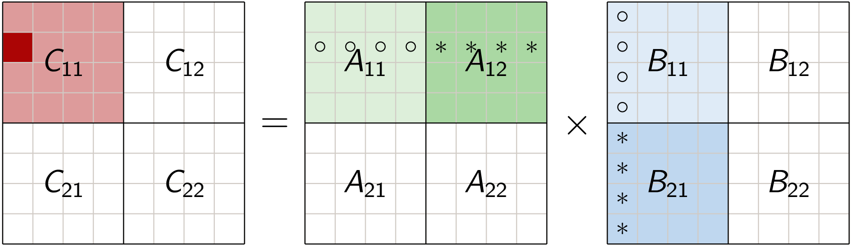

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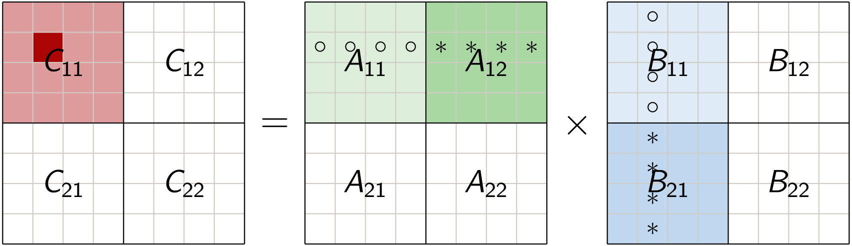

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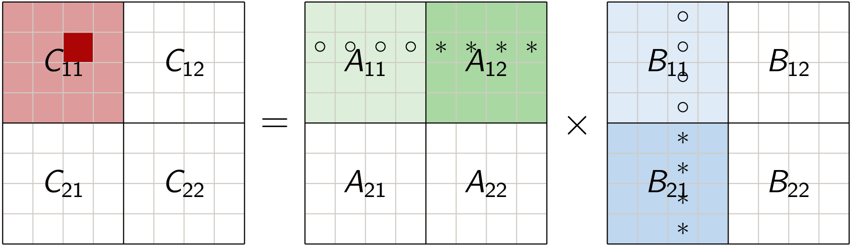

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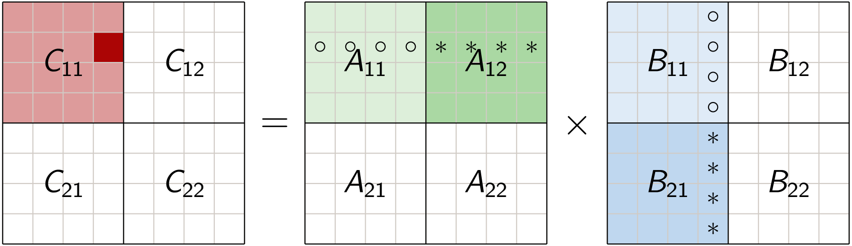

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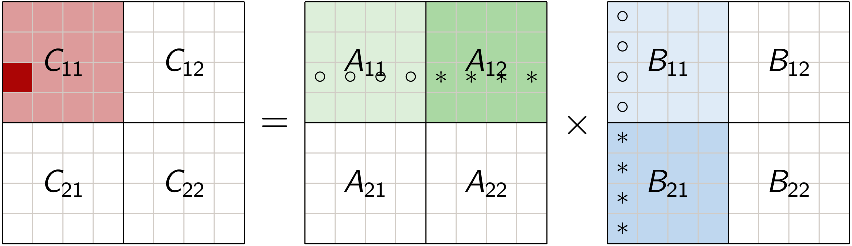

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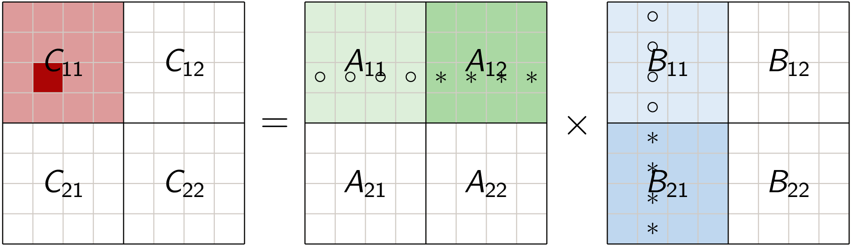

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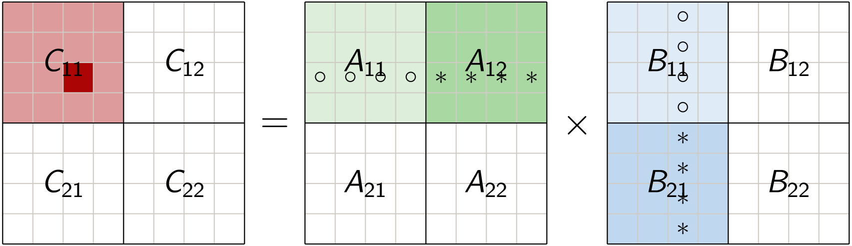

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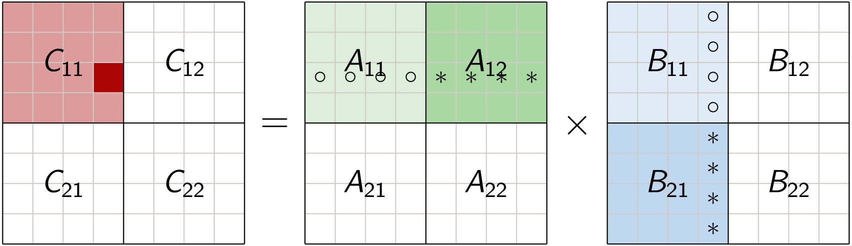

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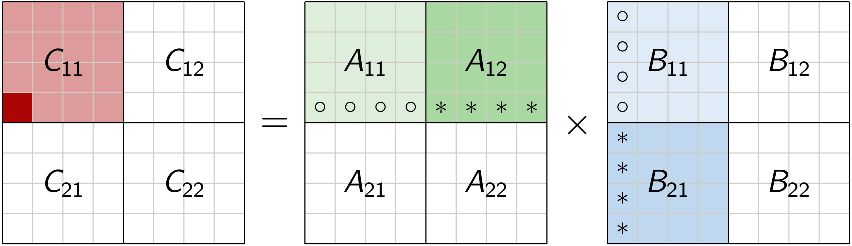

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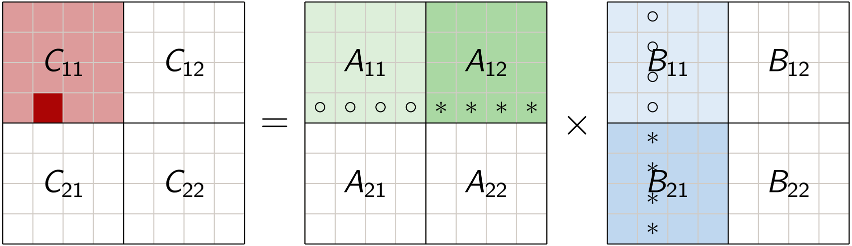

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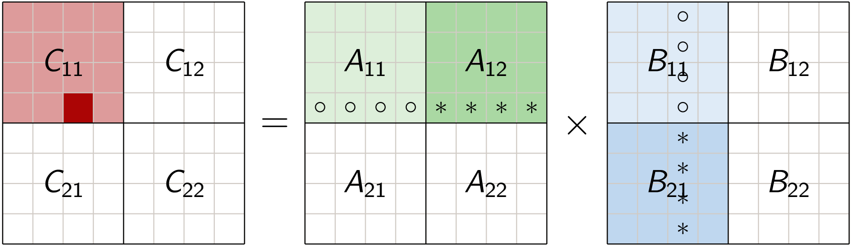

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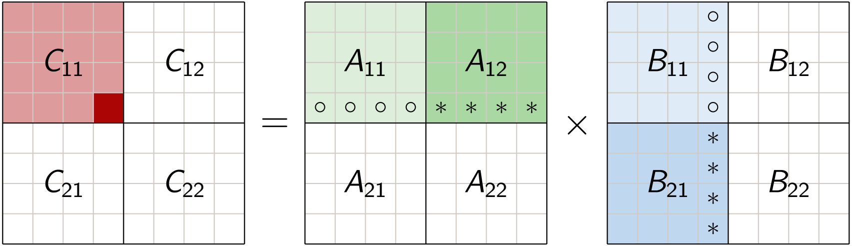

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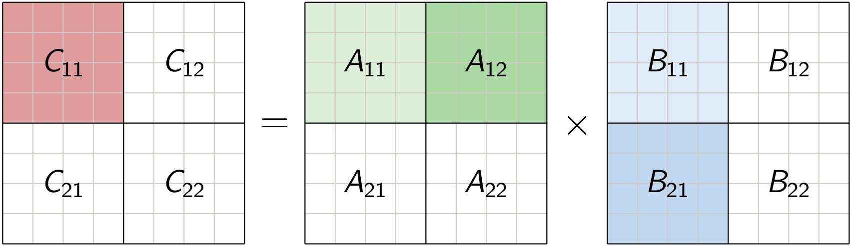

$$C_{11} = A_{11}B_{11} + A_{12}B_{21} \qquad C_{12} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21} \qquad C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

Divide-and-Conquer Algorithm

Divide each of A, B, C into four $n/2 \times n/2$ matrices: so that

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we recursively solve 8 matrix multiplications that each multiply two $n/2 \times n/2$ matrices.

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Combine: Make the additions to get C

Pseudocode and Analysis

Let $T(n)$ be the time to multiply two $n \times n$ matrices.

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REC-MAT-MULT(A, B, n)
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  if n == 1
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Master method $\Rightarrow T(n) = \Theta(n^3)$

Volker Strassen

STRASSEN'S ALGORITHM FOR MATRIX MULTIPLICATION

The Idea

Make less recursive calls

- Perform only 7 recursive multiplications of $n/2 \times n/2$ matrices, rather than 8

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Master method $\Rightarrow$ $T(n) = \Theta(n^{\log_2 7})$

Strassen's method

Divide each of A, B, C into four $n/2 \times n/2$ matrices: so that

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Conquer: Calculate recursively 7 matrix multiplications, each of two $n/2 \times n/2$ matrices:

$$\begin{aligned} M_1 &:= (A_{11} + A_{22})(B_{11} + B_{22}) & M_5 &:= (A_{11} + A_{12})B_{22} \\ M_2 &:= (A_{21} + A_{22})B_{11} & M_6 &:= (A_{21} - A_{11})(B_{11} + B_{12}) \\ M_3 &:= A_{11}(B_{12} - B_{22}) & M_7 &:= (A_{12} - A_{22})(B_{21} + B_{22}) \\ M_4 &:= A_{22}(B_{21} - B_{11}) \end{aligned}$$

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Combine: Let

$$\begin{aligned} C_{11} &= M_1 + M_4 - M_5 + M_7 & C_{12} &= M_3 + M_5 \\ C_{21} &= M_2 + M_4 & C_{22} &= M_1 - M_2 + M_3 + M_6 \end{aligned}$$

Analysis of Strassen's Method

Base case: $n = 1 \Rightarrow$ it takes time $\Theta(1)$

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- Dividing takes time $\Theta(n^2)$
- Conquering makes 7 recursive calls, each multiplying $n/2 \times n/2$ matrices $\Rightarrow 7T(n/2)$
- Combining takes time $\Theta(n^2)$ time to add $n/2 \times n/2$ matrices.

Recurrence is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 7T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

Analysis of Strassen's Method

Base case: $n = 1 \Rightarrow$ it takes time $\Theta(1)$

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Master method $\Rightarrow T(n) = \Theta(n^{\log_2 7})$

Notes about Strassen's method

- First to beat $\Theta(n^3)$ time
- Faster known today, method by Coppersmith and Winograd runs in time $O(n^{2.376})$ recently improved by Vassilevska Williams to $O(n^{2.3727})$.
- Big open problem how to multiply matrices in best way
- Naive method better for small instances because of hidden constants

Summary

- Divide-and-conquer simple but powerful algorithmic paradigm

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- Merge-sort and maximum subarray both run in time $\Theta(n \log n)$

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- Divide-and-conquer simple but powerful algorithmic paradigm
- Merge-sort and maximum subarray both run in time $\Theta(n \log n)$
- Strassen's algorithm for matrix multiplication in time $\Theta(n^{\log_2 7})$ where $\log_2 7 \approx 2.8$.

HEAPS AND HEAPSORT

Heapsort

Algorithm	worst-case running time	in-place
Insertion Sort	$\Theta(n^2)$	YES
Merge Sort	$\Theta(n \log n)$	NO

Heapsort:

Heapsort

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Heapsort:

- $O(n \lg n)$ worst case – like merge sort

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Uses a cool datastructure: heaps

Data Structures = “Building Blocks”

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Algorithm

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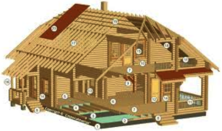
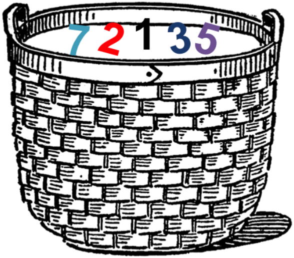
Algorithm



Algorithm

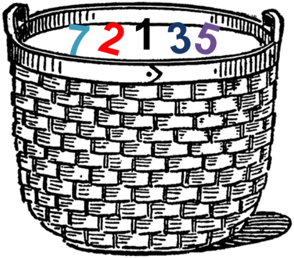
Data structures = dynamic sets of items



Data structure containing numbers

Data structures = dynamic sets of items

What kind of operations do we want to do?

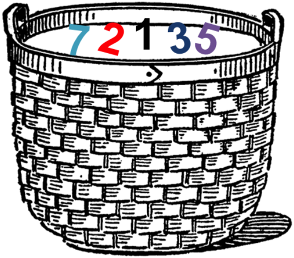


Data structure containing numbers

Data structures = dynamic sets of items

What kind of operations do we want to do?

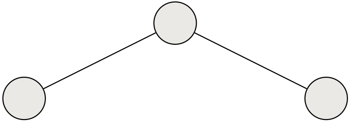
- Modifying operations: insertion, deletion, ...
- Query operations: search, maximum, minimum, ...



Data structure containing numbers

(Binary) heap data structure

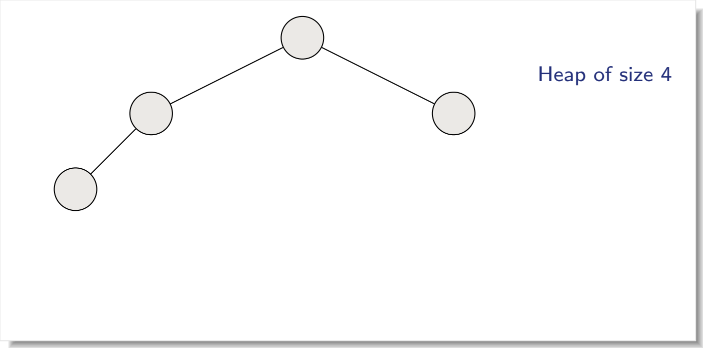
Heap A (not garbage-collected storage) is a nearly complete binary tree



Heap of size 3

(Binary) heap data structure

Heap A (not garbage-collected storage) is a nearly complete binary tree



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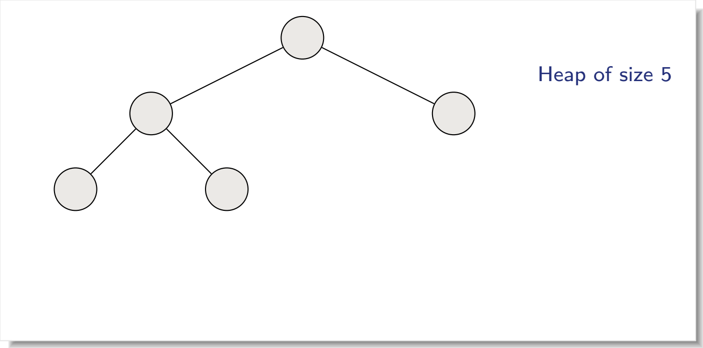

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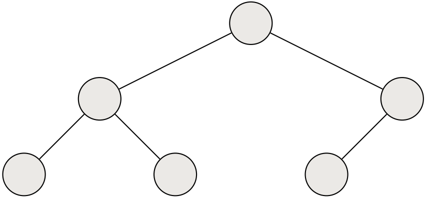
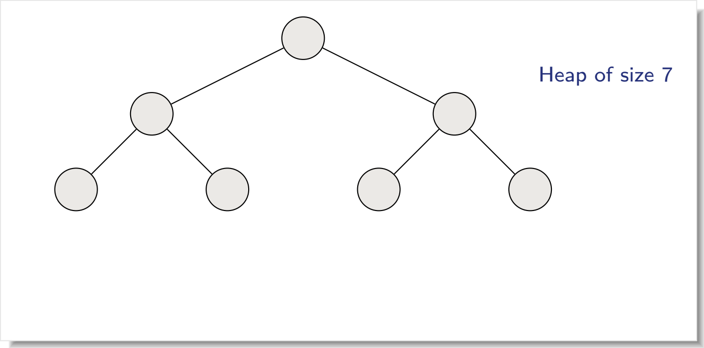


Heap of size 6

(Binary) heap data structure

Heap A (not garbage-collected storage) is a nearly complete binary tree



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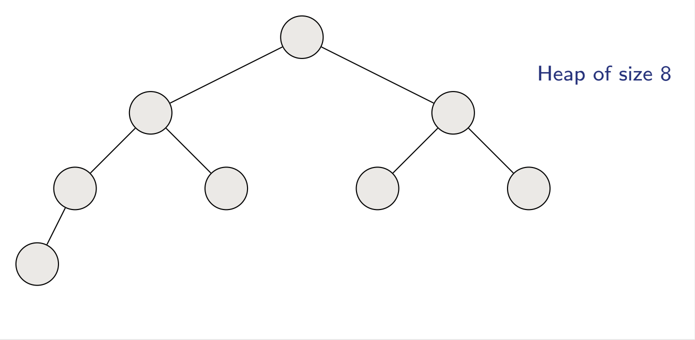

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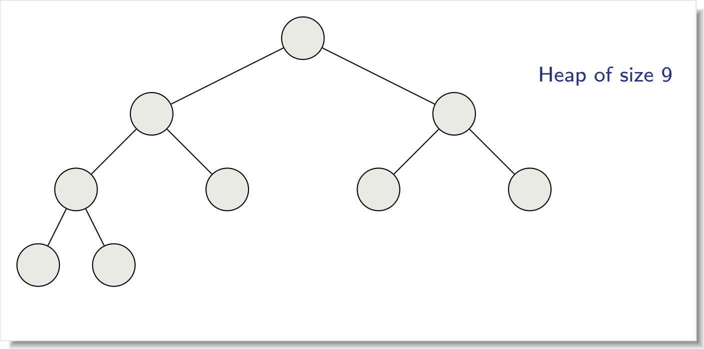

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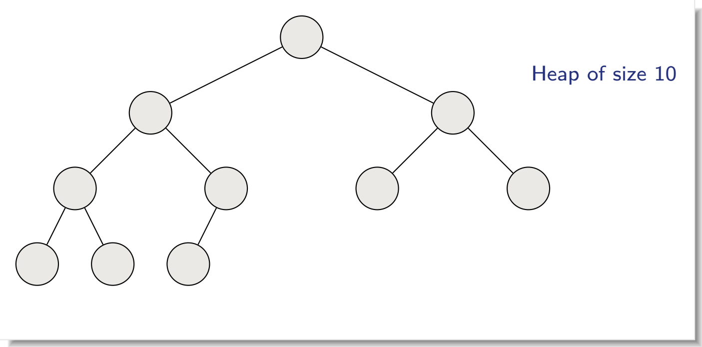
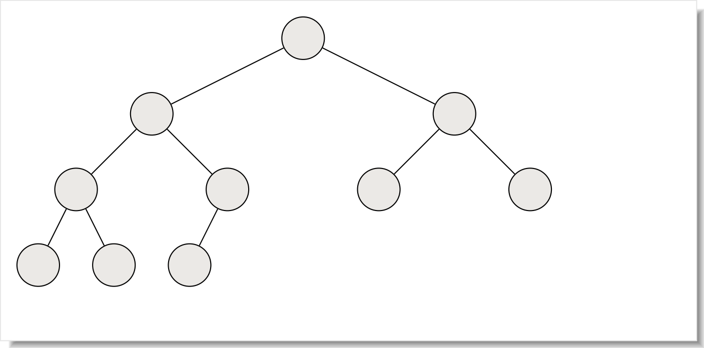


(Binary) heap data structure

Heap A (not garbage-collected storage) is a nearly complete binary tree

(Max)-Heap property: key of i's children is smaller or equal to i's key



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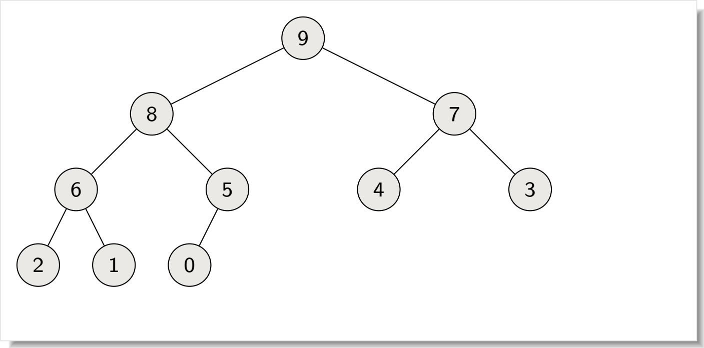

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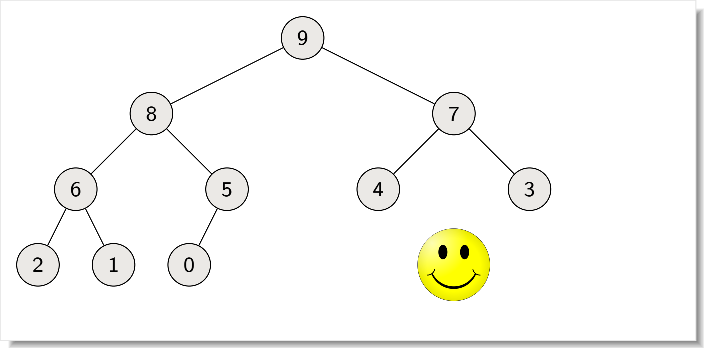

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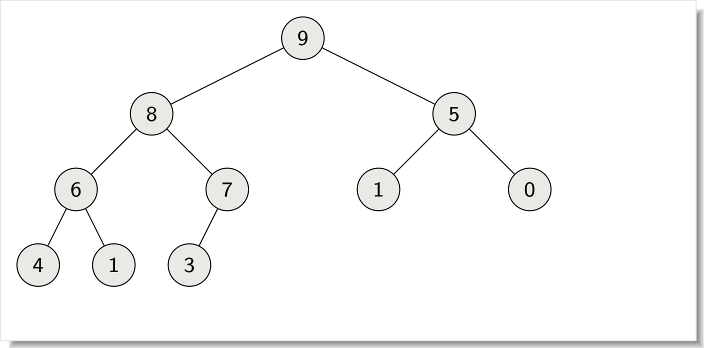

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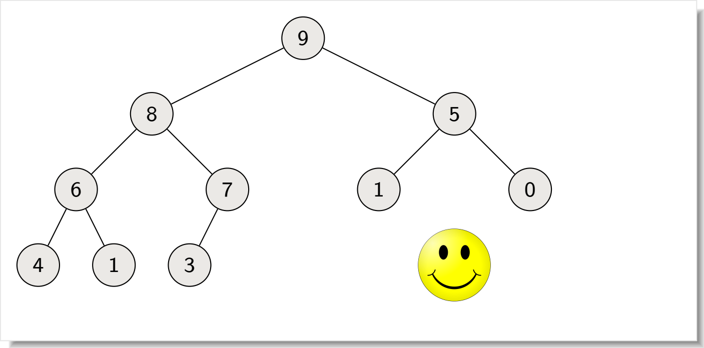

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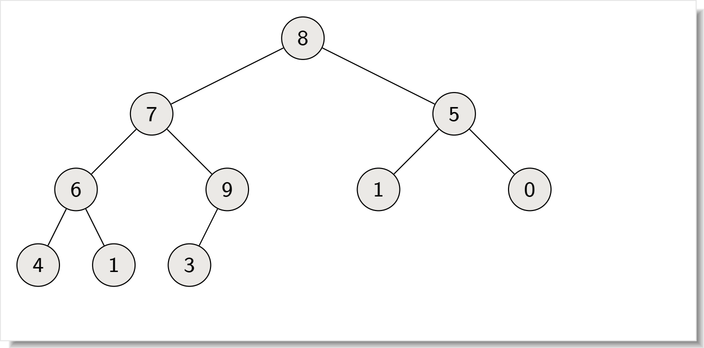

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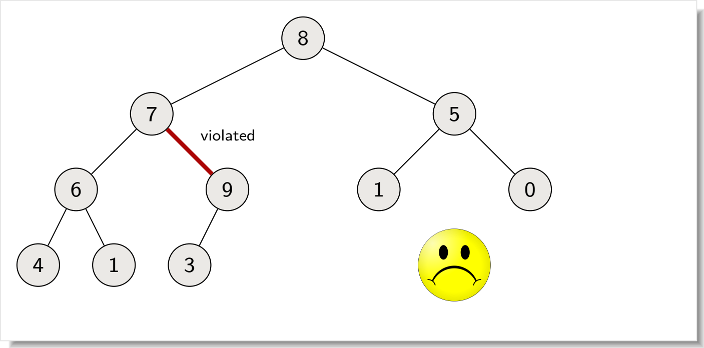

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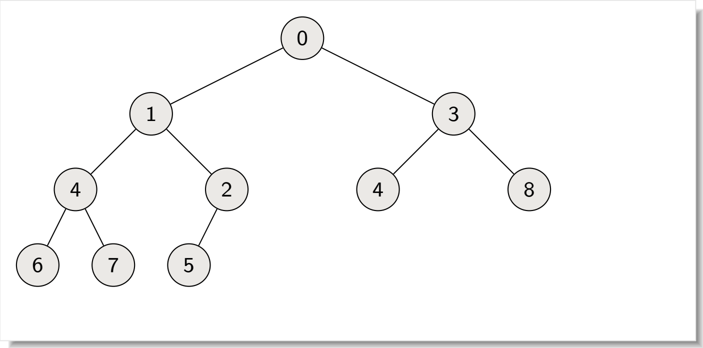
(Min)-Heap property: key of i's children is greater or equal to i's key



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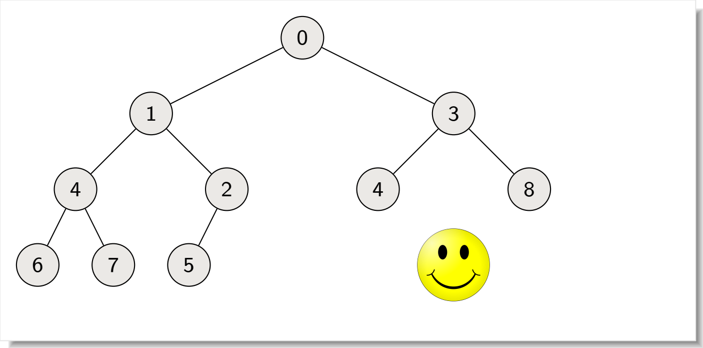
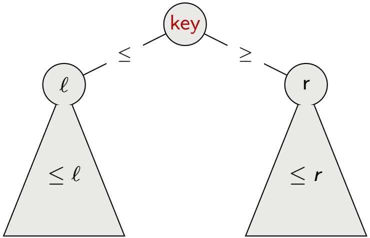
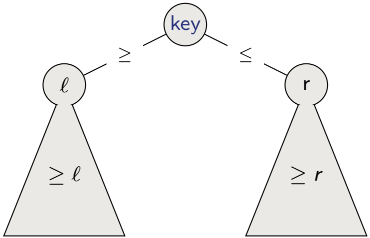


Max-Heap $\Rightarrow$ maximum element is the root

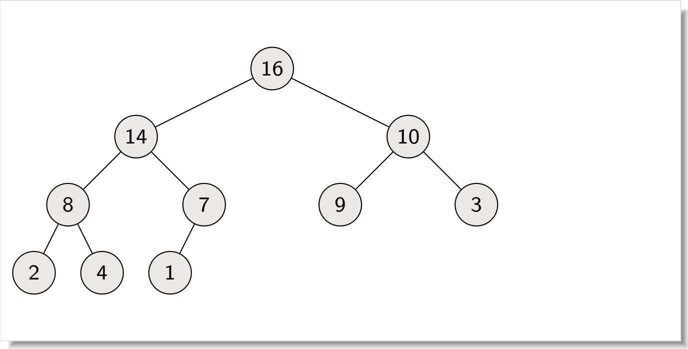


Min-Heap $\Rightarrow$ minimum element is the root



Height of a heap

Height of node = # of edges on a longest simple path from the node down to a leaf



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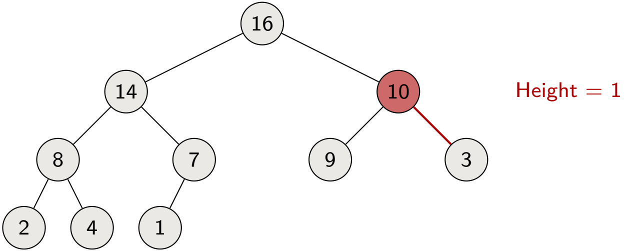

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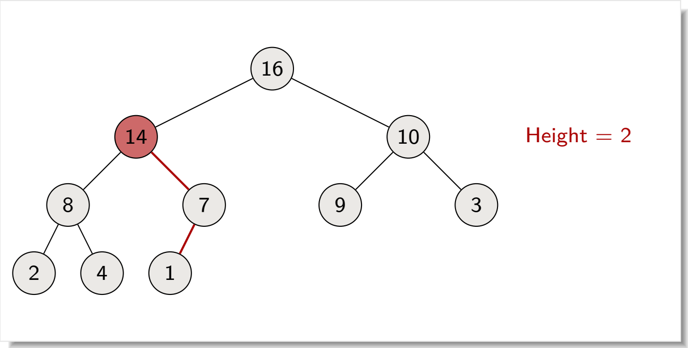

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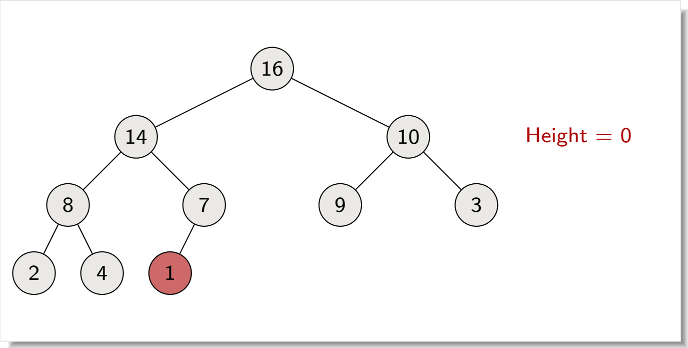
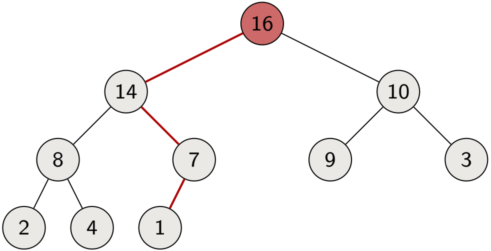


Height of a heap

Height of node = # of edges on a longest simple path from the node down to a leaf

Height of heap = height of root

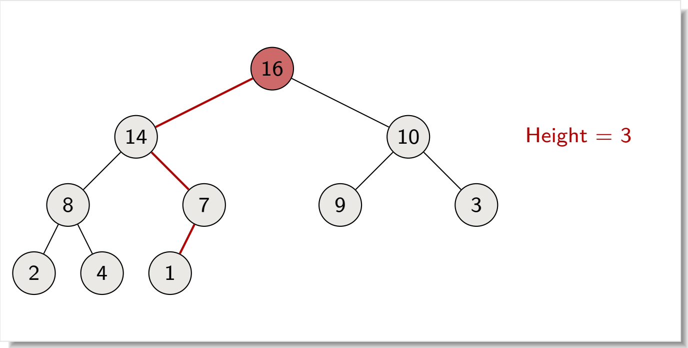


Height = 3

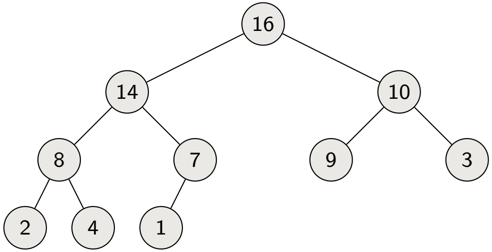
Height of a heap

Height of node = # of edges on a longest simple path from the node down to a leaf

Height of heap = height of root = $\Theta(\log n)$

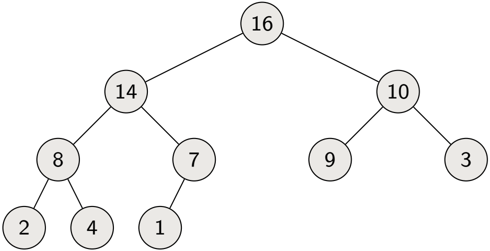


How to store a heap/tree?



How to store a heap/tree?

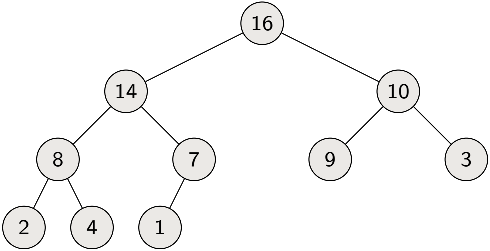
pointer to left and right children



How to store a heap/tree?

~~pointer to left and right children~~

Use that tree is almost complete to store it in array



How to store a heap/tree?

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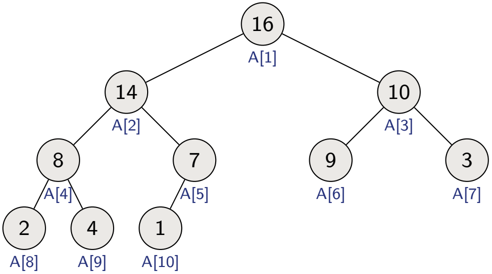
Use that tree is almost complete to store it in array

A =

16	14	10	8	7	9	3	2	4	1



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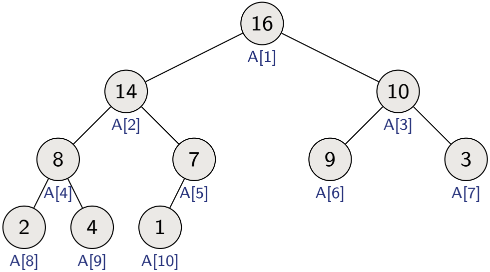

In this representation:

Root is $A[1]$

$\textsc{Left}(i) = ???$

$\textsc{Right}(i) = ???$

$\textsc{Parent}(i) = ???$

How to store a heap/tree?

~~pointer to left and right children~~

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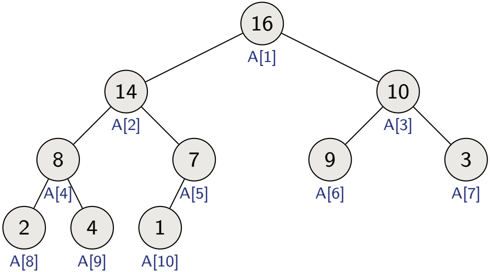


In this representation:

Root is $A[1]$

$\textsc{Left}(i) = 2i$

$\textsc{Right}(i) = ???$

$\textsc{Parent}(i) = ???$

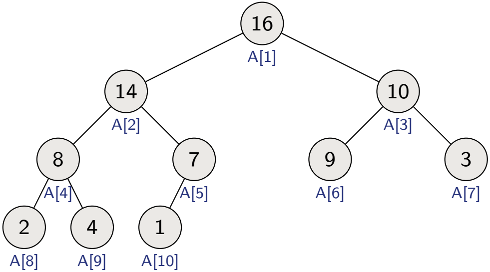
How to store a heap/tree?

~~pointer to left and right children~~

Use that tree is almost complete to store it in array

A =

16	14	10	8	7	9	3	2	4	1



In this representation:

Root is $A[1]$

$\textsc{Left}(i) = 2i$

$\textsc{Right}(i) = 2i + 1$

$\textsc{Parent}(i) = ???$

How to store a heap/tree?

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Use that tree is almost complete to store it in array

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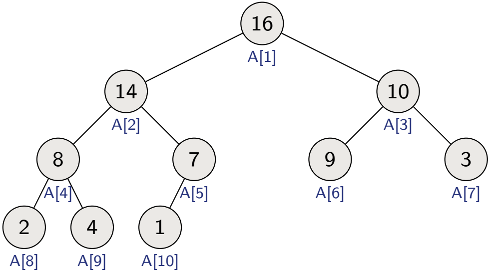
16	14	10	8	7	9	3	2	4	1



In this representation:

Root is $A[1]$

$\textsc{Left}(i) = 2i$

$\textsc{Right}(i) = 2i + 1$

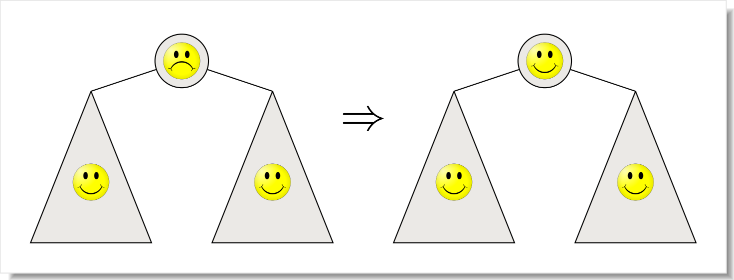
$\textsc{Parent}(i) = \lfloor i/2 \rfloor$

BUILDING AND MANIPULATING HEAPS

Maintaining the heap property

MAX-HEAPIFY is important for manipulating heaps:

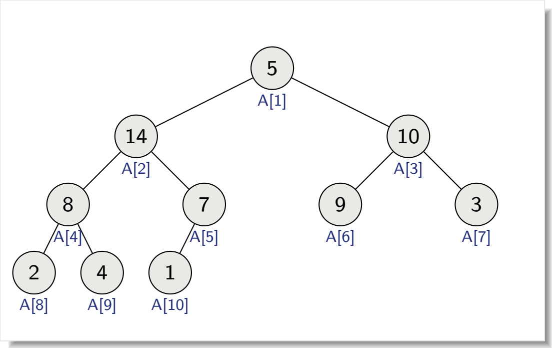
> Given an i such that the subtrees of i are heaps, it ensures that the subtree rooted at i is a heap satisfy the heap property



MAX-HEAPIFY(A, i, n)

Algorithm:

- Compare $A[i], A[\text{LEFT}(i)], A[\text{RIGHT}(i)]$
- If necessary, swap $A[i]$ with the largest of the two children to preserve heap property
- Continue this process of comparing and swapping down the heap, until subtree rooted at i is max-heap



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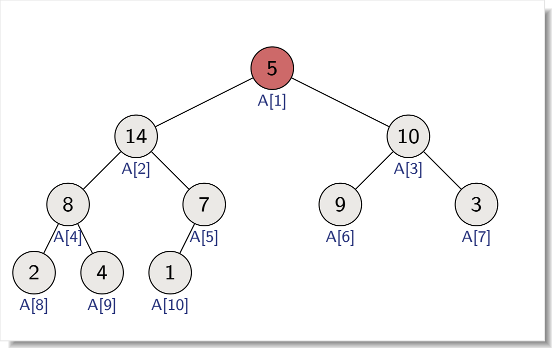

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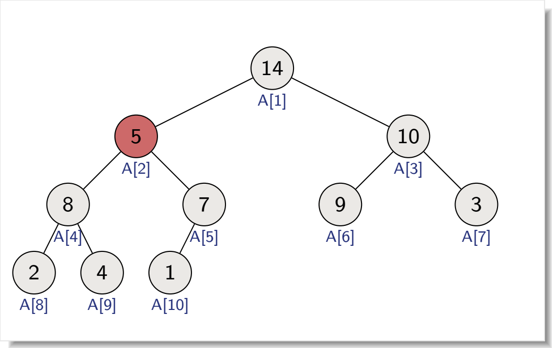

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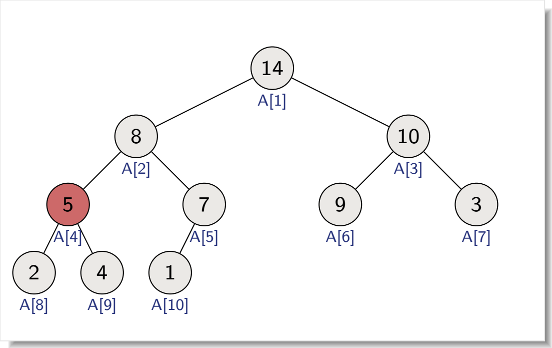

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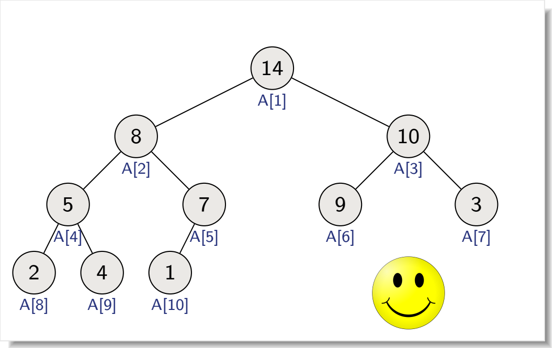
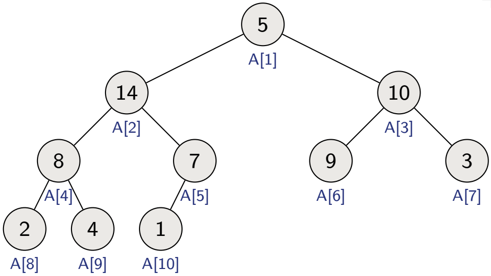


Pseudo-code and analysis

MAX-HEAPIFY(A, i, n)

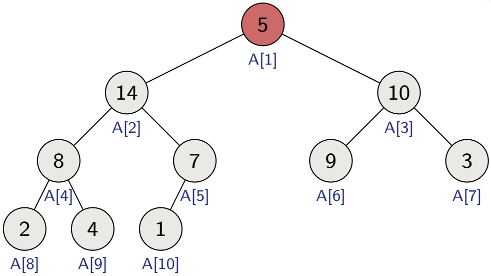
```
l = LEFT(i)
r = RIGHT(i)
if l ≤ n and A[l] > A[i]
    largest = l
else largest = i
if r ≤ n and A[r] > A[largest]
    largest = r
if largest ≠ i
    exchange A[i] with A[largest]
    MAX-HEAPIFY(A, largest, n)
```



Pseudo-code and analysis

MAX-HEAPIFY(A, i, n)

$l = \text{LEFT}(i)$
$r = \text{RIGHT}(i)$
if $l \leq n$ and $A[l] > A[i]$
  $largest = l$
else $largest = i$
if $r \leq n$ and $A[r] > A[largest]$
  $largest = r$
if $largest \neq i$
  exchange $A[i]$ with $A[largest]$
  MAX-HEAPIFY$(A, largest, n)$



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if $l \leq n$ and $A[l] > A[i]$
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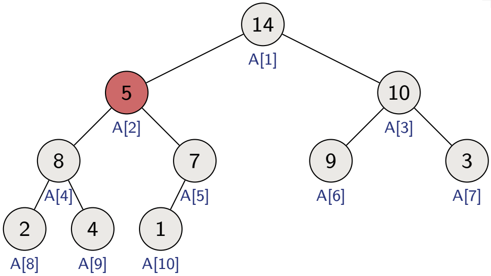
if $r \leq n$ and $A[r] > A[largest]$
  $largest = r$
if $largest \neq i$
  exchange $A[i]$ with $A[largest]$
  MAX-HEAPIFY$(A, largest, n)$



Pseudo-code and analysis

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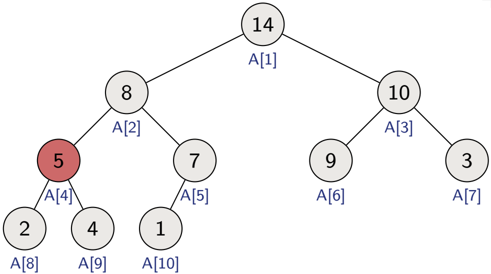

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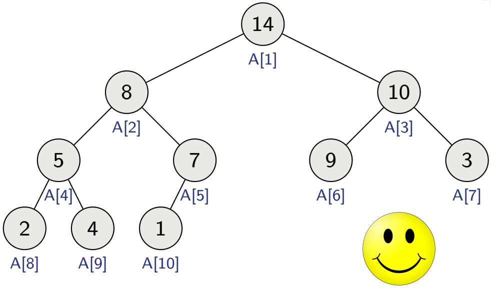

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Running time?

Space?

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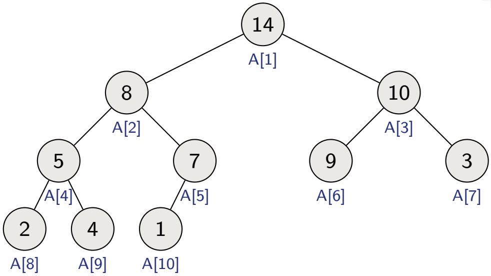

Pseudo-code and analysis

Running time?

$\Theta(\text{height of } i) = O(\log n)$

Space?

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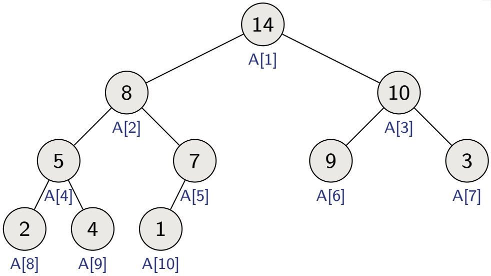

Pseudo-code and analysis

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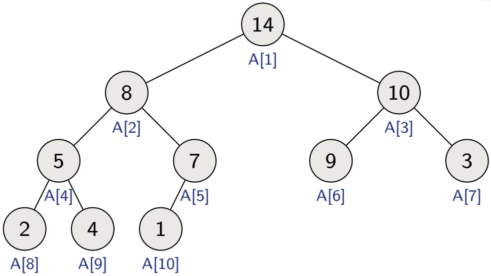
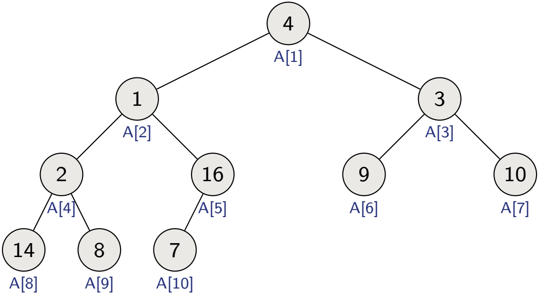


Building a heap

Build-Max-Heap(A, n)
for $i = \lfloor n/2 \rfloor$ **downto** 1
    Max-Heapify(A, i, n)

Given unordered array A of length n, Build-Max-Heap outputs a heap

4	1	3	2	16	9	10	14	8	7



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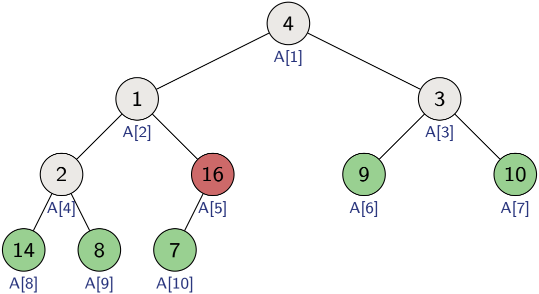

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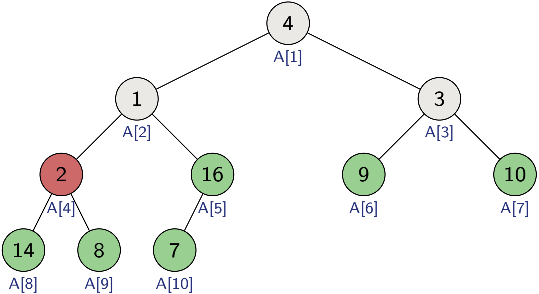

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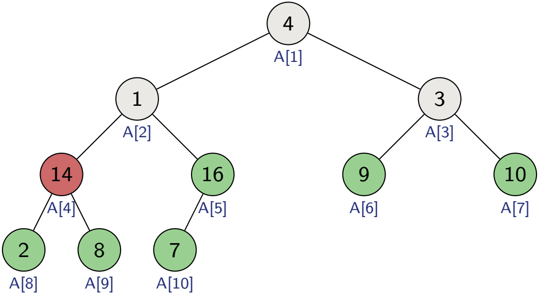

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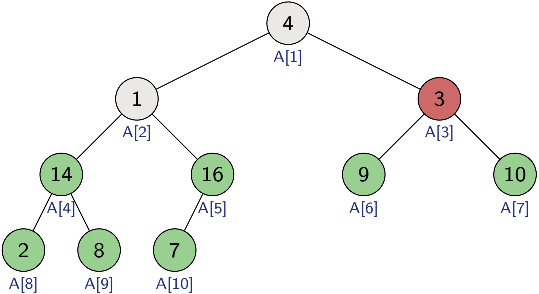

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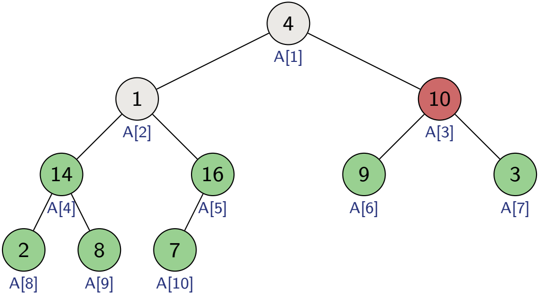

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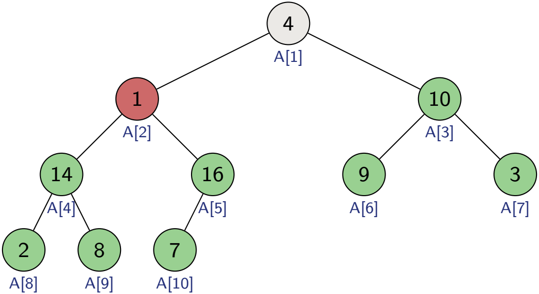

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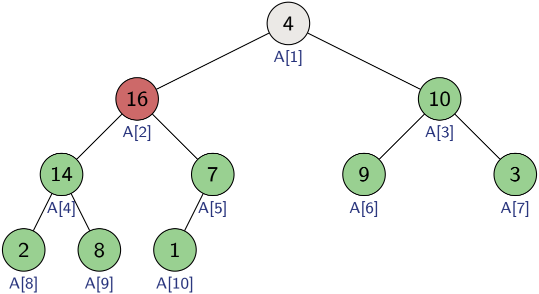

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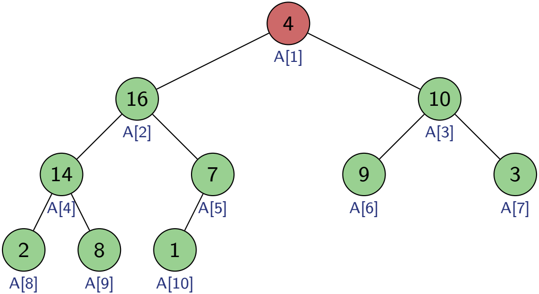

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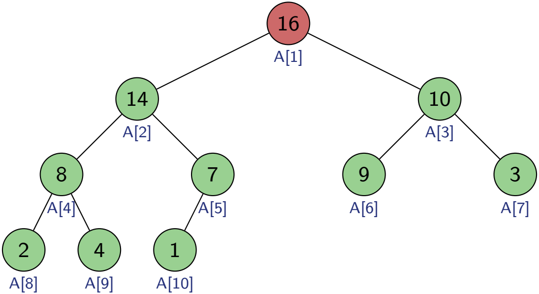
16	14	10	8	7	9	3	2	4	1



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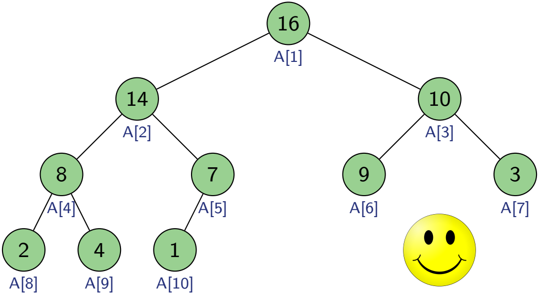

Analysis

$\textsc{Build-Max-Heap}(A, n)$
  for $i = \lfloor n/2 \rfloor$ **downto** 1
    $\textsc{Max-Heapify}(A, i, n)$

What is the worst-case running time of $\textsc{Build-Max-Heap}$?

Analysis

$\text{BUILD-MAX-HEAP}(A, n)$
for $i = \lfloor n/2 \rfloor$ **downto** 1
    $\text{MAX-HEAPIFY}(A, i, n)$

What is the worst-case running time of BUILD-MAX-HEAP?

Simple bound: $O(n)$ calls to MAX-HEAPIFY, each of which takes $O(\lg n)$ time $\Rightarrow O(n \lg n)$ in total

Analysis

$\textsc{Build-Max-Heap}(A, n)$

for $i = \lfloor n/2 \rfloor$ **downto** 1

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What is the worst-case running time of $\textsc{Build-Max-Heap}$?

Simple bound: $O(n)$ calls to $\textsc{Max-Heapify}$, each of which takes $O(\lg n)$ time $\Rightarrow O(n \lg n)$ in total

Tighter analysis: Time to run $\textsc{Max-Heapify}$ is linear in the height of the node it's run on. Hence, the time is bounded by

$$\sum_{h=0}^{\lg n} \{\text{\# nodes of height } h\} O(h) = O\left(n \sum_{h=0}^{\lg n} \frac{h}{2^h}\right),$$

which is $O(n)$ since $\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$.