Algorithms: Divide-and-Conquer (Merge-Sort)

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Recall Last Lecture: Loop Invariant

CalculateSum(n):

1. $ans = 0$
2. **for** $i = 1, 2, \ldots, n$
3. $\quad ans = ans + i$
4. **return** ans

Often used for proof of correctness in presence of loops

Loop invariant = "a statement that is satisfied during the loop"

Ex: At the start of each iteration $ans = (i-1) * i/2$

Need to verify (similar to induction)

Initialization: True at the beginning of the 1st iteration of the loop

Maintenance: If it is true before an iteration of the loop, it remains true before the next iteration.

Termination: When the loop terminates, the invariant — usually along with the reason that the loop terminated — gives us a useful property that helps show that the algorithm is correct.

The difficulty is often to come up with the right loop invariant

```
INSERTION-SORT(A, n)
  for j = 2 to n
      key = A[j]
      // Insert A[j] into the sorted sequence A[1 .. j − 1].
      i = j − 1
      while i > 0 and A[i] > key
          A[i + 1] = A[i]
          i = i − 1
      A[i + 1] = key
```

```
Linear-Search(A,v)
    for i ← 1 to length(A)
        if A[i] = v then
            return i
    return NIL
```

Loop invariant:

At the start of each iteration of the “outer” **for** loop – the loop indexed by j – the subarray $A[1\ldots,j-1]$ consists of the elements originally in $A[1,\ldots,j-1]$ but in sorted order.

Loop invariant:

At the start of each iteration of the **for** loop we have $A[j] \neq v$ for all $j < i$.

Recall Last Lecture: Time Analysis

Random-access machine (RAM) model

- Instructions are executed one after another
- Simplification basic instructions take constant ($O(1)$) time
  - Arithmetic: add, subtract, multiply, divide, remainder, floor, ceiling
  - Data movement: load, store, copy.
  - Control: conditional/unconditional branch, subroutine call and return

Running time: on a particular input, it is the number of primitive operations (steps) executed

We usually concentrate on finding the **worst-case running time:** the longest running time for *any* input of size n

Order of growth: Focus on the important features

- Drop lower-order terms
- Ignore the constant coefficient in the leading term

Recall Last Lecture: Analysis of insertion sort

INSERTION-SORT(A, n)	*cost*	*times*
for $j = 2$ **to** n	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,.\,.\,j-1]$.	0	$n-1$
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

number of times line executed based on the value of j

Worst case: The array is in reverse sorted

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \frac{n(n+1)-2}{2}$$

$$+ (c_6 + c_7)\frac{n \cdot (n-1)}{2} + c_8(n-1) = \Theta(n^2)$$

DIVIDE-AND-CONQUER

Merge Sort

Divide-and-Conquer

Powerful algorithmic approach:

recursively divide problem into smaller subproblems

Divide the problem into a number of subproblems that are smaller instances of the same problem

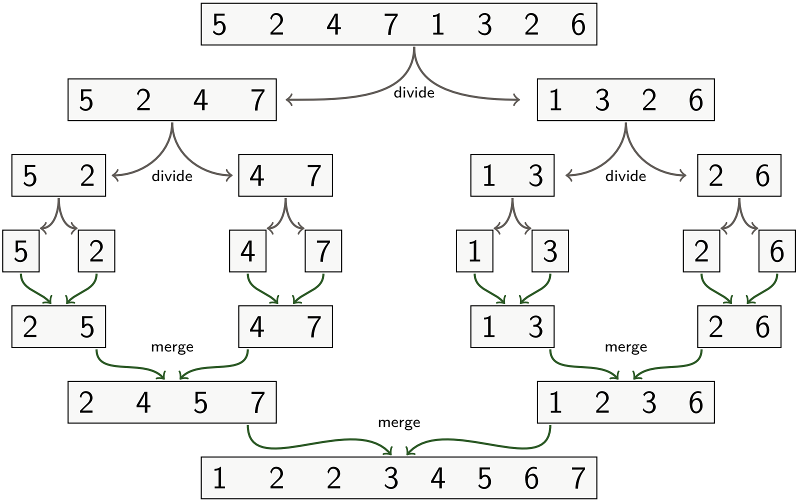
Conquer the subproblems by solving them recursively.

Base case: If the subproblems are small enough, just solve them by brute force

Combine the subproblem solutions to give a solution to the original problem

Merge Sort = D & C applied to sorting

Example $\langle 5, 2, 4, 7, 1, 3, 2, 6 \rangle$



Merge sort

To sort $A[p \ldots r]$:

Divide by splitting into two subarrays $A[p \ldots q]$ and $A[q+1, \ldots, r]$, where q is the halfway point of $A[p \ldots r]$

Conquer by recursively sorting the two subarrays $A[p \ldots q]$ and $A[q+1, \ldots r]$

Combine by merging the two sorted subarrays $A[p \ldots q]$ and $A[q+1, \ldots, r]$ to produce a singe sorted subarray $A[p \ldots r]$

```
MERGE-SORT(A, p, r)
  if p < r                        // check for base case
      q = ⌊(p + r)/2⌋             // divide
      MERGE-SORT(A, p, q)         // conquer
      MERGE-SORT(A, q + 1, r)     // conquer
      MERGE(A, p, q, r)           // combine
```

Merging

What remains is the MERGE procedure to solve the “merge” problem:

Definition

INPUT: Array A and indices $p \leq q < r$ such that subarrays $A[p \dots q], A[q+1 \dots r]$ are sorted.

OUTPUT: The two subarrays are merged into a single sorted subarray in $A[p \dots r]$.

Example:

2	4	5	7	1	2	3	6

$\downarrow$

1	2	2	3	4	5	6	7

Correctness of Merge-Sort

Assuming MERGE is correct

```
MERGE-SORT(A, p, r)
  if p < r
      q = ⌊(p + r)/2⌋
      MERGE-SORT(A, p, q)
      MERGE-SORT(A, q + 1, r)
      MERGE(A, p, q, r)
```

Theorem

Assuming that the implementation of the MERGE *procedure is correct,* MERGE-SORT*(A,p,r) correctly sorts the numbers in* $A[p \ldots r]$

Proof by induction on $n = r - p$

Base case $n = 0$: In this case $r = p$ so $A[p \ldots r]$ is trivially sorted.

Inductive case: Assume statement true $\forall n \in \{0, 1, \ldots, k-1\}$ and prove the statement for $n = k$.

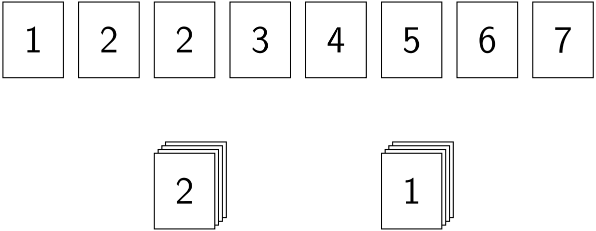
- By induction hypothesis MERGE-SORT(A, p, q) and MERGE-SORT(A, q+1, r) successfully sort the two subarrays.
- Therefore a correct merge procedure will successfully sort $A[p \ldots q]$ as required.

	p			q				r	
$\cdots$	1	2	2	3	4	5	6	7	$\cdots$

Idea behind linear-time merging

Think of two pile of cards that are placed face up

- Basic step: pick the smaller of the two cards and place it in the output pile



Idea behind linear-time merging

Think of two pile of cards that are placed face up

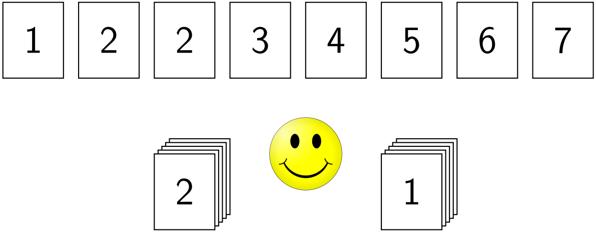
- Basic step: pick the smaller of the two cards and place it in the output pile
- There are $\leq n$ basic steps, since each basic step removes one card from the input piles, and we started with n cards in the input pile
- Therefore the procedure should take $\theta(n)$ time

1	2	2	3	4	5	6	7

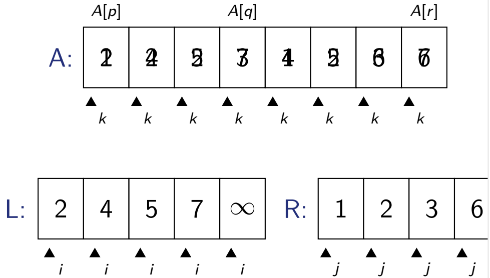
Implementation Simplification

Instead of checking whether a pile is empty:

- Put in the bottom of each input pile a special **sentinel** card of value ∞
- Stop once we have performed $n = r - p + 1$ basic steps (picked n cards)



Merging Algorithm



MERGE(A, p, q, r)

$n_1 = q - p + 1$
$n_2 = r - q$
let $L[1 \,.\,.\, n_1 + 1]$ and $R[1 \,.\,.\, n_2 + 1]$ be new arrays
for $i = 1$ **to** n_1
  $L[i] = A[p + i - 1]$
for $j = 1$ **to** n_2
  $R[j] = A[q + j]$
$L[n_1 + 1] = \infty$
$R[n_2 + 1] = \infty$
$i = 1$
$j = 1$
for $k = p$ **to** r
  if $L[i] \leq R[j]$
    $A[k] = L[i]$
    $i = i + 1$
  else $A[k] = R[j]$
    $j = j + 1$

MERGE(A, p, q, r)

$n_1 = q - p + 1$
$n_2 = r - q$
let $L[1 \,.\,.\, n_1 + 1]$ and $R[1 \,.\,.\, n_2 + 1]$ be new arrays
for $i = 1$ **to** n_1
  $L[i] = A[p + i - 1]$
for $j = 1$ **to** n_2
  $R[j] = A[q + j]$
$L[n_1 + 1] = \infty$
$R[n_2 + 1] = \infty$
$i = 1$
$j = 1$
for $k = p$ **to** r
  if $L[i] \leq R[j]$
    $A[k] = L[i]$

Merging Algorithm

- Runtime analysis?

MERGE(A, p, q, r)

```
n_1 = q - p + 1
n_2 = r - q
let L[1..n_1 + 1] and R[1..n_2 + 1] be new arrays
for i = 1 to n_1
    L[i] = A[p + i - 1]
for j = 1 to n_2
    R[j] = A[q + j]
L[n_1 + 1] = ∞
R[n_2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

Analyzing divide-and-conquer algorithms

Use a recurrence equation to describe the running time:

- Let $T(n) =$ "running time on a problem of size n"
- If n is small enough say $n \leq c$ for some constant c then $T(n) = \Theta(1)$ (by brute force)
- Otherwise, suppose we divide into a sub problems each of size n/b.
- Let $D(n)$ be the time to divide and let $C(n)$ the time to combine solutions.
- We get the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c, \\ aT(n/b) + D(n) + C(n) & \text{otherwise.} \end{cases}$$

Analysis of Merge Sort

```
MERGE-SORT(A, p, r)
  if p < r                          // check for base case
      q = ⌊(p + r)/2⌋               // divide
      MERGE-SORT(A, p, q)           // conquer
      MERGE-SORT(A, q + 1, r)       // conquer
      MERGE(A, p, q, r)             // combine
```

Divide: takes constant time, i.e., $D(n) = \Theta(1)$

Conquer: recursively solve two subproblems, each of size $n/2 \Rightarrow 2T(n/2)$.

Combine: Merge on an n-element subarray takes $\Theta(n)$ time $\Rightarrow C(n) = \Theta(n)$.

Recurrence for merge sort running time is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise.} \end{cases}$$

Comparing the Two Sorting Algorithms

	worst-case running time	in-place
Insertion Sort	$\Theta(n^2)$	YES
Merge Sort	$\Theta(n \log n)$	NO

- A sorting algorithm is in-place if the numbers are rearranged within the array (while using at most a constant amount of additional space)
- Insertion sort is incremental: having sorted the subarray $A[1 \ldots j-1]$, we inserted the single element $A[j]$ into its proper place, yielding the sorted subarray $A[1 \ldots j]$.
- Merge sort is divide-and-conquer: break the problem into smaller subproblems and then combine the solutions to the subproblems

SOLVING RECURRENCES

INDUCTION

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Analysing Recurrences

As an example, we shall consider the following recurrence

$$T(n) = \begin{cases} c & \text{if } n = 1, \\ 2T(n/2) + c \cdot n & \text{otherwise.} \end{cases}$$

Note that this recurrence upper bounds and lower bounds the recurrence for MERGE-SORT by selecting c sufficiently large and small, respectively.

We shall solve recurrences by using three techniques:

- The substitution method
- Recursion trees
- Master method

The substitution method

- Guess the form of the solution
- Use mathematical induction to find the constants and show that the solution works.

$$T(n) = 2T(n/2) + c \cdot n$$

A qualified guess is that $T(n) = \Theta(n \log n)$

The substitution method: proof of guess

Upper bound

There exists a constant $a > 0$ such that $T(n) \leq a \cdot n \log n$ for all $n \geq 2$

Proof by induction on n

Base cases: For any constant $n \in \{2, 3, 4\}$, $T(n)$ has a constant value, selecting a larger than this value will satisfy the base cases when $n \in \{2, 3, 4\}$.

Inductive step: Assume statement true $\forall n \in \{2, 3, \ldots, k-1\}$ and prove the statement for $n = k$.

$$T(n) = 2T(n/2) + cn$$

We can thus select a to be a positive constant so that both the base cases and the inductive step holds. Hence, $T(n) = O(n \log n)$

The substitution method: proof of guess

Lower bound

There exists a constant $b > 0$ such that $T(n) \geq b \cdot n \log n$ for all $n \geq 0$

Proof by induction on n

Base case: For $n = 1$, $T(n) = c$ and $b \cdot n \log n = 0$ so the base case is satisfied for any b.

Inductive step: Assume statement true $\forall n \in \{0, 1, \ldots, k-1\}$ and prove the statement for $n = k$.

$$T(n) = 2T(n/2) + cn$$

We can thus select b to be a positive constant so that both the base cases and the inductive step holds. Hence, $T(n) = \Omega(n \log n)$

Common mistake using the substitution method

Be careful when using asymptotic notation!

The false proof for the recurrence $T(n) = 4T(n/4) + n$, that $T(n) = O(n)$:

$$\begin{aligned} T(n) &\leq 4(c(n/4)) + n \\ &\leq cn + n = O(n) \qquad \text{wrong!} \end{aligned}$$

Because we haven't proven the *exact form* of our inductive hypothesis (which is that $T(n) \leq cn$), this proof is false

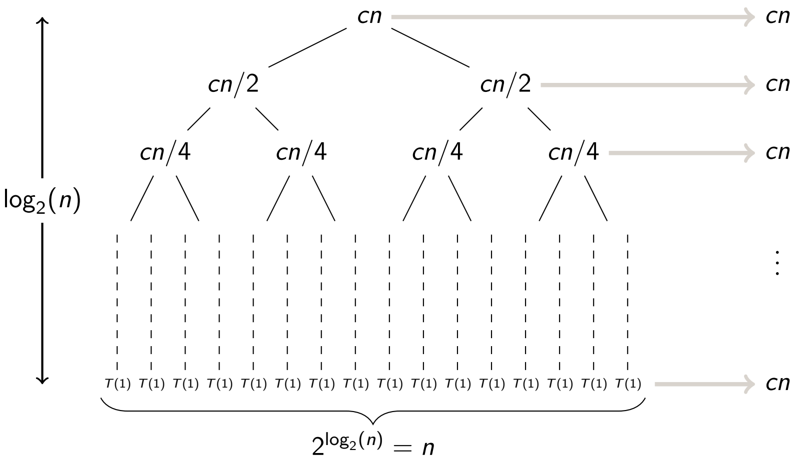
Recursion trees

Another way to generate a guess. Then verify by substitution method.

- Each node corresponds to the cost of a subproblem
- We sum the costs within each level of the tree to obtain a set of per-level costs,
- then we sum all the per-level costs to determine the total cost of all levels of the recursion.

Recursion trees

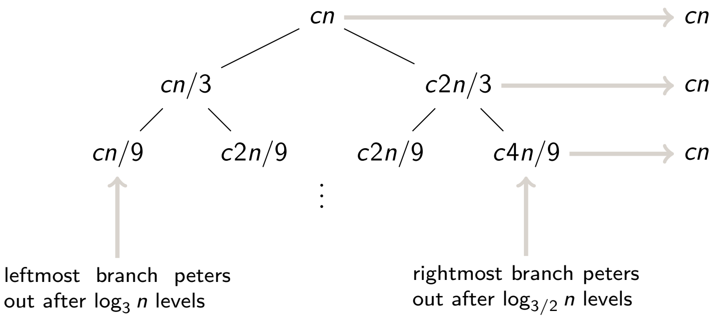
Our favorite example: $T(1) = c$ and $T(n) = 2T(n/2) + cn$



Qualified guess: $T(n) = cn \log_2 n = \Theta(n \log n)$

Recursion trees

Another interesting example: $T(n) = T(n/3) + T(2n/3) + cn$



- There are $\log_3 n$ full levels and after $\log_{3/2} n$ levels the problem size is down to 1.
- Each level contributes $\approx cn$

Qualified guess: exist positive constants a, b so that

$$a \cdot n \log_3(n) \leq T(n) \leq b \cdot n \log_{3/2} n \Rightarrow T(n) = \Theta(n \log n)$$

Master method

Used to black-box solve recurrences of the form $T(n) = aT(n/b) + f(n)$

Theorem (Master Theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n).$$

Then, $T(n)$ has the following asymptotic bounds

- *If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$*
- *If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$*
- *If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$*

Our favorite example: $T(1) = c$ and $T(n) = 2T(n/2) + cn$

- $f(n) = O(n)$ and $a = b = 2$ so $\log_b(a) = 1$ and $f(n) = \Theta(n^{\log_b(a)})$.
- By Master theorem, we have $T(n) = \Theta(n \log n)$:)

Summary

- Divide-and-conquer simple but powerful algorithmic paradigm
- Solving the recurrence for merge sort shows that it runs in time $\Theta(n \log n)$, i.e., much faster than Insertion sort for large instances
- For small instances insertion sort can still be faster
- Solving recurrences fun but delicate