Algorithms: Sorting + (Time) Analysis

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School of Computer and Communication Sciences

Lecture 2, 19.02.2025

Recall Last Lecture

- CS-250: A lot of interesting and useful material!

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- A computational problem is defined by an input/output relationship
  - Example: INPUT: n OUTPUT: $\sum_{i=1}^{n} i$
- An algorithm describes a specific computational procedure for achieving that input/output relationship
  - Example: return $n(n+1)/2$
- "Time + Space" is crucial for the usefulness of an algorithm

The Growth of Functions

- Three algorithms: A, B, C with different running times in ms.

	A ($1000 \log_2 n$ ms)	B ($2n^2$ ms)	C (2^n ms)

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$n = 64$	6 s	8 s 192 ms	$>$ age of the universe

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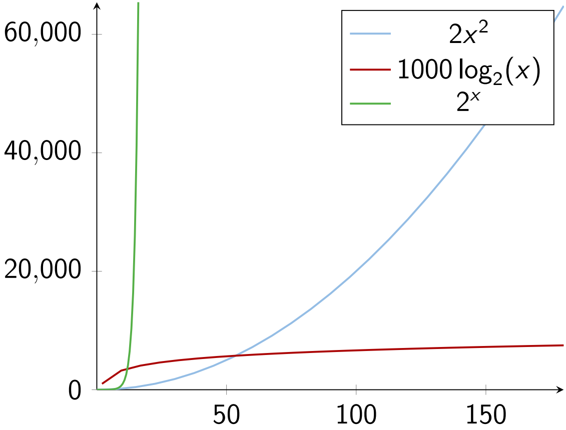

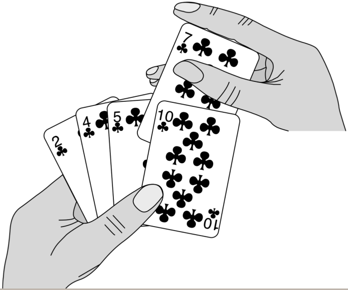

SORTING

Insertion Sort

The sorting problem

Definition

INPUT: A sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$.

OUTPUT: A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

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For example

- Given the input $\langle 5, 2, 4, 6, 1, 3 \rangle$

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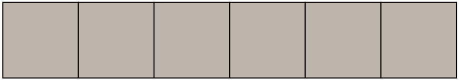
For example

- Given the input $\langle 5, 2, 4, 6, 1, 3 \rangle$
- a correct output is $\langle 1, 2, 3, 4, 5, 6 \rangle$

Insertion Sort - The Idea

Like sorting a hand of playing cards

- Start with an empty left hand of playing cards and the cards face down on the table



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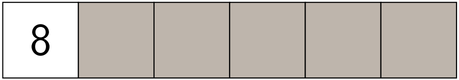

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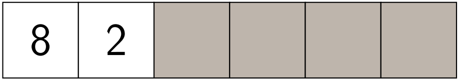

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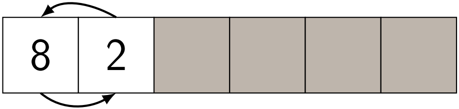

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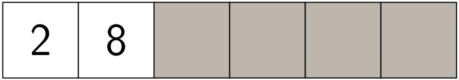

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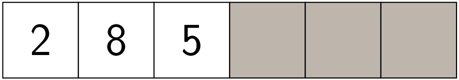

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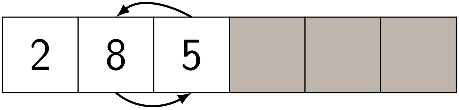

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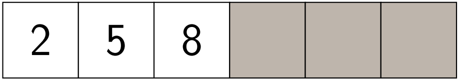

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2	5	8	10		

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2	5	8	10	1	

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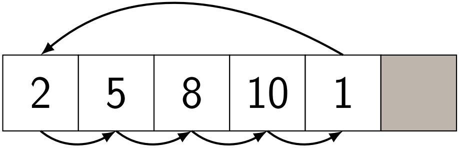

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1	2	5	8	10	

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1	2	5	8	10	3

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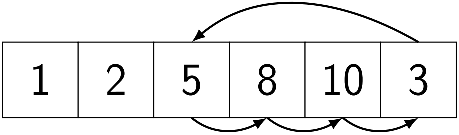

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- To find the correct position for a card, compare it with each of the cards already in the hand, from right to left.
- At all times, the cards, held in the left hand are sorted, and these cards were originally the top cards of the pile on the table

1	2	3	5	8	10

Insertion Sort

The Algorithm

- Takes as parameters an array $A[1 \dots n]$ and the length n of the array

```
INSERTION-SORT(A, n)
 for j = 2 to n
     key = A[j]
     // Insert A[j] into the sorted sequence A[1 .. j − 1].
     i = j − 1
     while i > 0 and A[i] > key
         A[i + 1] = A[i]
         i = i − 1
     A[i + 1] = key
```

Insertion Sort

Example on $\langle 8, 2, 5, 10, 1, 3 \rangle$

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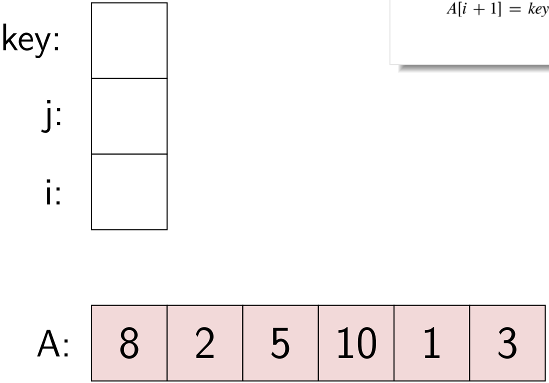

Insertion Sort

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INSERTION-SORT(A, n)

for $j = 2$ **to** n
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  $i = j - 1$
  while $i > 0$ and $A[i] > key$
    $A[i+1] = A[i]$
    $i = i - 1$
  $A[i+1] = key$

key:	0
j:	2
i:	0

A:	8	2	5	10	1	3

Insertion Sort

Example on $\langle 8, 2, 5, 10, 1, 3 \rangle$

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key:	2
j:	2
i:	0

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key: 2

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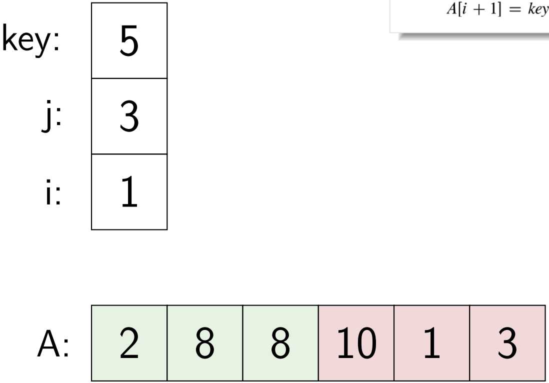

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```

key: 5

j: 3

i: 1

A:						
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And so on. . .

PROVING ALGORITHMS CORRECT

Loop invariants

Insertion Sort

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Loop invariant:

At the start of each iteration of the "outer" **for** loop – the loop indexed by j– the subarrary $A[1\ldots, j-1]$ consists of the elements originally in $A[1,\ldots, j-1]$ but in sorted order.

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Need to verify:

- **Initialization:** It is true prior to the first iteration of the loop.

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Loop invariant:

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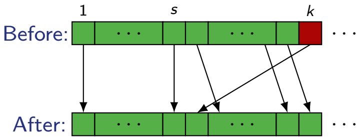

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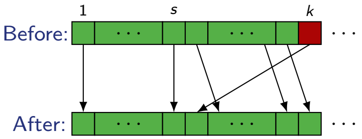

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ANALYZING ALGORITHMS

Computational Model

We want to predict the resources that the algorithm requires. Usually, running time.

For that we need a computational model

Random-access machine (RAM) model

- Instructions are executed one after another
- Simplification basic instructions take constant ($O(1)$) time
  - Arithmetic: add, subtract, multiply, divide, remainder, floor, ceiling
  - Data movement: load, store, copy.
  - Control: conditional/unconditional branch, subroutine call and return
- We don't worry about precision, although it is crucial in certain numerical applications

Analyzing an algorithm's running time (1/2)

Time it takes depend on the input

- Sorting 1000 numbers take longer than sorting 3 numbers

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- Usually, the number of items in the input. Like the size n of the array being sorted
- If multiplying two integers, could be the total number of bits in the two integers
- Could be described by more than one number: e.g. graph algorithm running times are usually expressed in terms of the number of vertices and the number of edges in the input graph.

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- Figure that each line of pseudocode requires a constant amount of time
- One line may take a different amount of time than another, but each execution of line i takes the same amount of time c_i
- This is assuming that the line consists only of primitive operations
  - If the line is a subroutine call, then the actual call takes constant time, but the execution of the subroutine might not
  - If the line specifies operations other than primitive ones, then it might take more than constant time. Example: "sort the points by x-coordinate"

Analysis of insertion sort

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for $j = 2$ **to** n
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INSERTION-SORT(A, n)	*cost*	*times*
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while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
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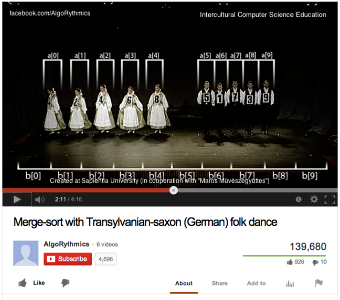
- Gives a guaranteed upper bound on the running time for any input
- For some algorithms, the worst case occurs often. For example, when searching, the worst case often occurs when the item being searched for is not present
- Average case often as bad as worst-case: Suppose that we randomly choose n numbers as the input to insertion sort

Order of growth: Focus on the important features

- Drop lower-order terms
- Ignore the constant coefficient in the leading term



SORTING BY DIVIDE-AND-CONQUER

Merge Sort

Divide-and-Conquer

Powerful algorithmic approach:

- recursively divide problem into smaller subproblems

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Combine the subproblem solutions to give a solution to the original problem

Merge Sort = D & C applied to sorting

Example $\langle 5, 2, 4, 7, 1, 3, 2, 6 \rangle$

5	2	4	7	1	3	2	6

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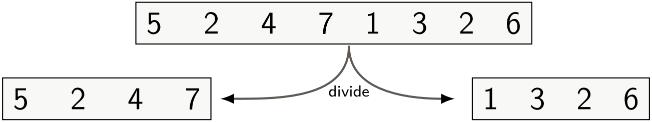

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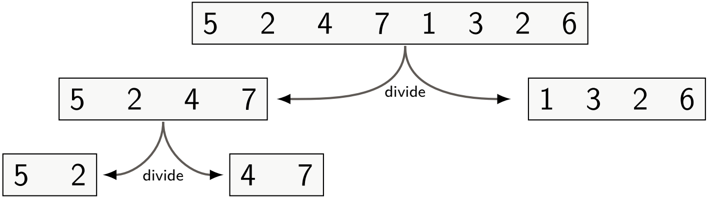

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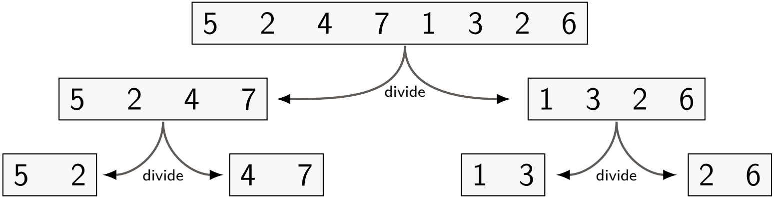

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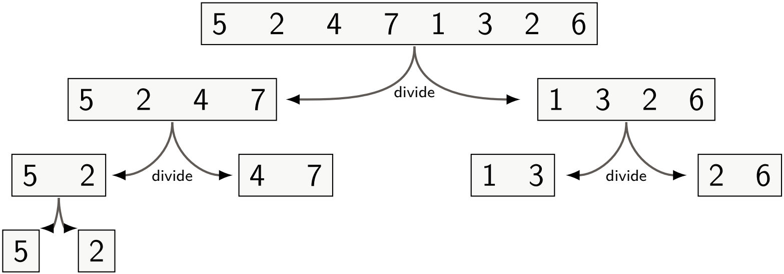

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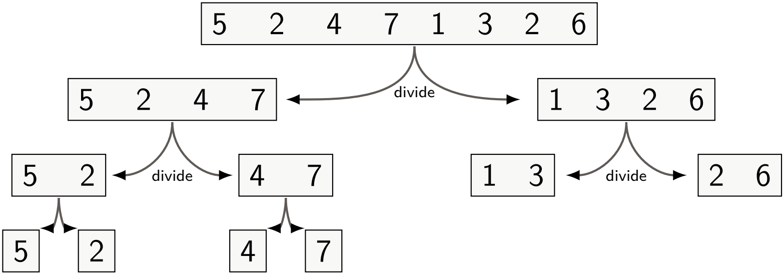

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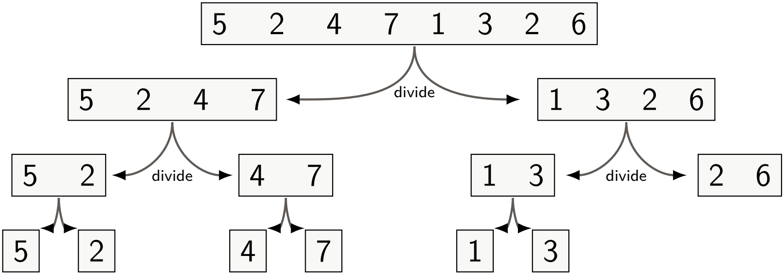

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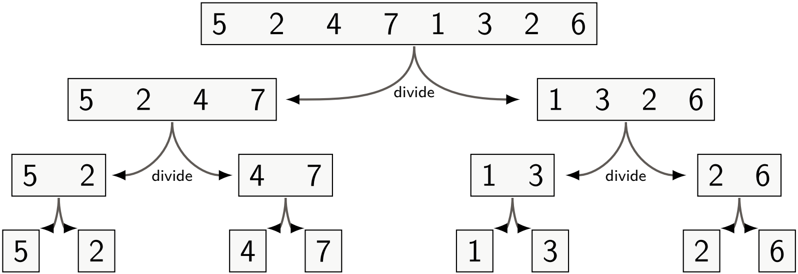

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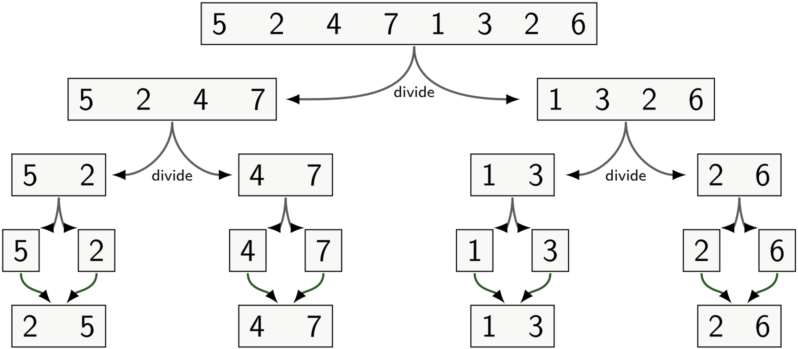

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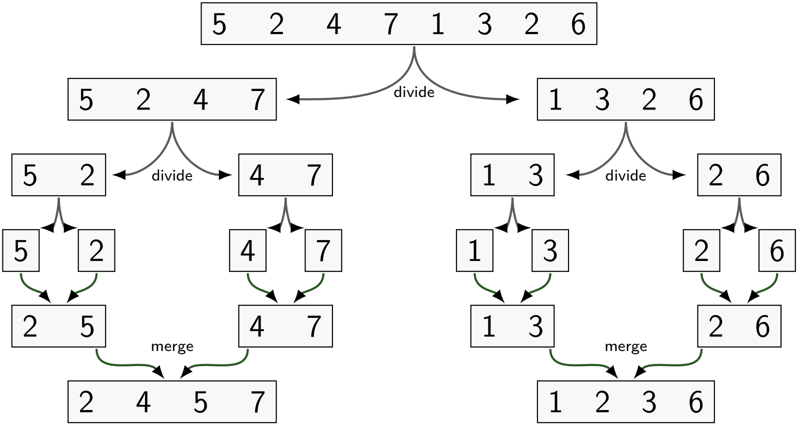

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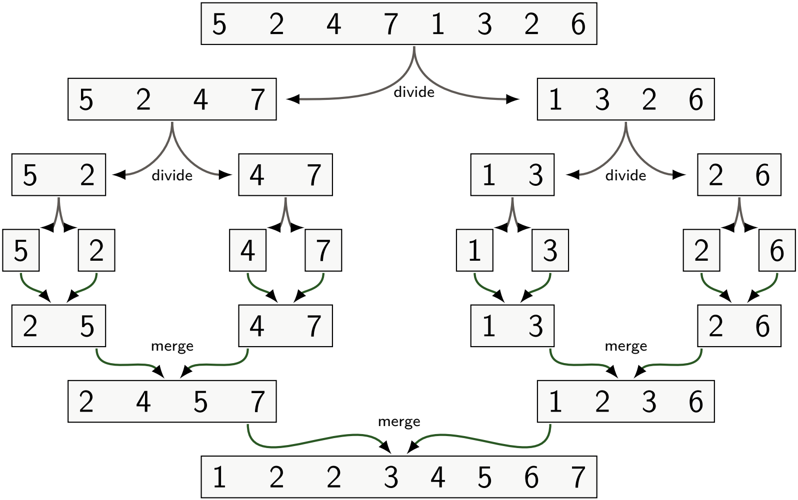

Merge sort

To sort $A[p \dots r]$:

Divide by splitting into two subarrays $A[p \dots q]$ and $A[q+1, \dots, r]$, where q is the halfway point of $A[p \dots r]$

Conquer by recursively sorting the two subarrays $A[p \dots q]$ and $A[q+1, \dots r]$

Combine by merging the two sorted subarrays $A[p \dots q]$ and $A[q+1, \dots, r]$ to produce a singe sorted subarray $A[p \dots r]$

Merge sort

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```
MERGE-SORT(A, p, r)
  if p < r                          // check for base case
      q = ⌊(p + r)/2⌋               // divide
      MERGE-SORT(A, p, q)           // conquer
      MERGE-SORT(A, q + 1, r)       // conquer
      MERGE(A, p, q, r)             // combine
```

Merging

What remains is the Merge procedure to solve the "merge" problem:

Definition

INPUT: Array A and indices $p \leq q < r$ such that subarrays $A[p \dots q], A[q+1 \dots r]$ are sorted.

OUTPUT: The two subarrays are merged into a single sorted subarray in $A[p \dots r]$.

We will give a procedure that solves this problem in time $\Theta(n)$ where n is the size of the subproblem, i.e.,

$$n = r - p + 1$$

Idea behind linear-time merging

Think of two pile of cards that are placed face up

- Basic step: pick the smaller of the two cards and place it in the output pile

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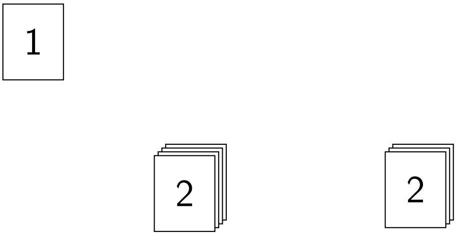

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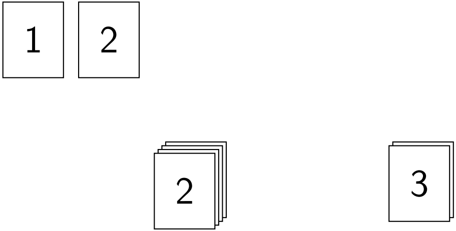

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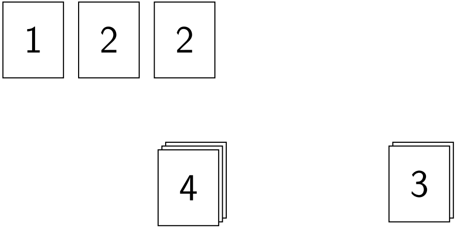

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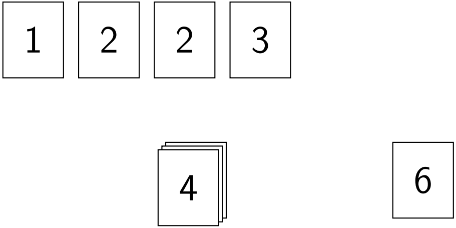

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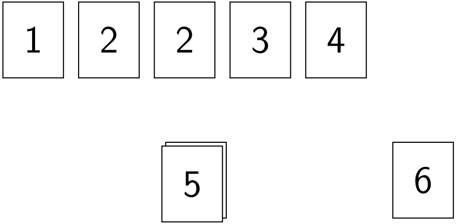

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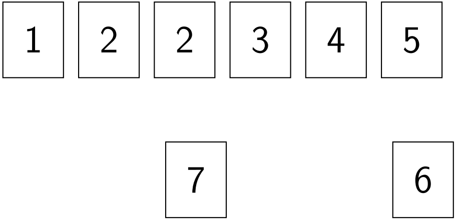

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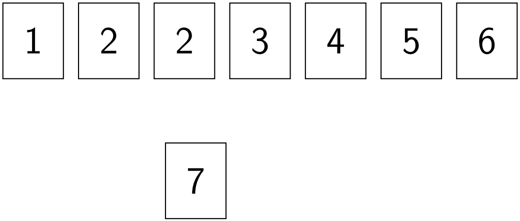

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Think of two pile of cards that are placed face up

- Basic step: pick the smaller of the two cards and place it in the output pile

1	2	2	3	4	5	6	7

Idea behind linear-time merging

Think of two pile of cards that are placed face up

- Basic step: pick the smaller of the two cards and place it in the output pile
- There are $\leq n$ basic steps, since each basic step removes one card from the input piles, and we started with n cards in the input pile
- Therefore the procedure should take $\theta(n)$ time

1	2	2	3	4	5	6	7

Implementation Simplification

Instead of checking whether a pile is empty:

- Put in the bottom of each input pile a special **sentinel** card of value ∞
- Stop once we have performed $n = r - p + 1$ basic steps
  (picked n cards)

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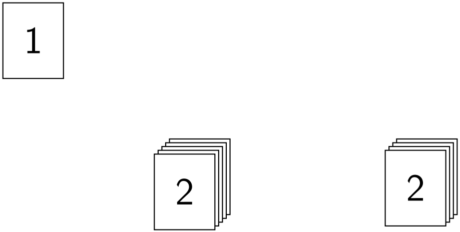

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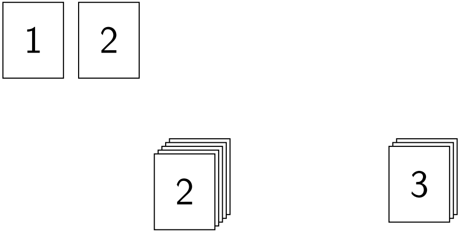

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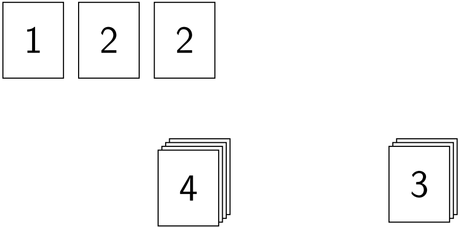

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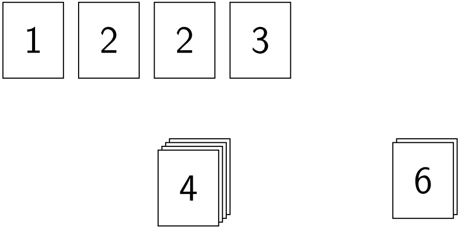

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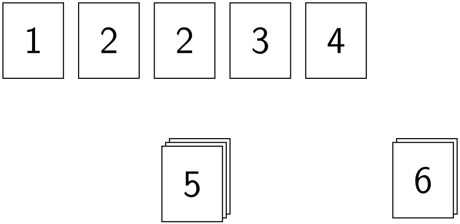

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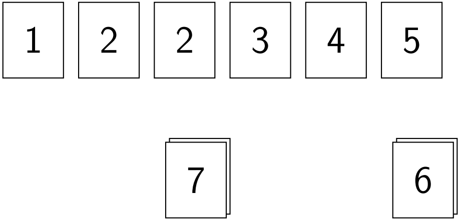

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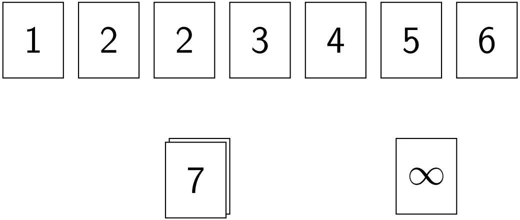

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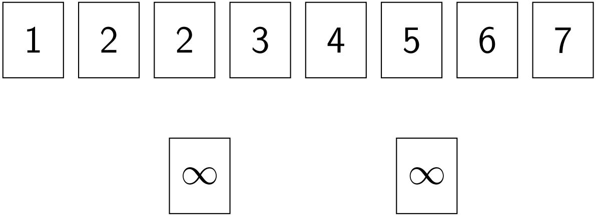

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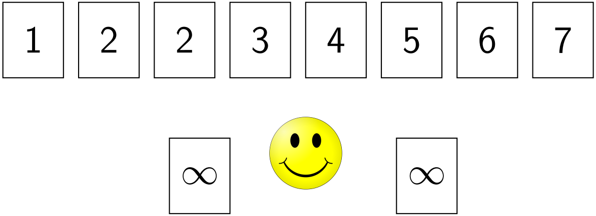

Merging Algorithm

	$A[p]$			$A[q]$				$A[r]$
A:	2	4	5	7	1	2	3	6

```
MERGE(A, p, q, r)
  n1 = q − p + 1
  n2 = r − q
  let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
  for i = 1 to n1
      L[i] = A[p + i − 1]
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  for k = p to r
      if L[i] ≤ R[j]
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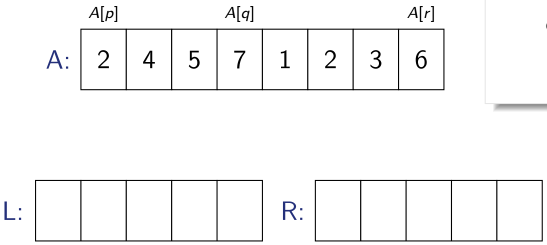

Merging Algorithm

A[p] … A[q] … A[r]

A:								
	2	4	5	7	1	2	3	6

L:					
	2	4	5	7	

R:					

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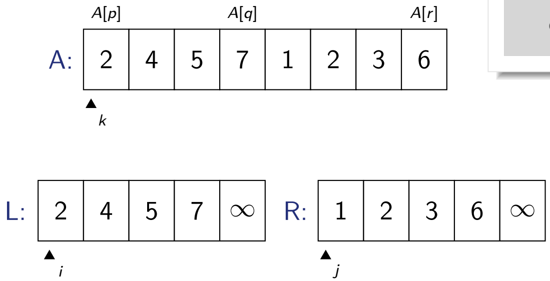

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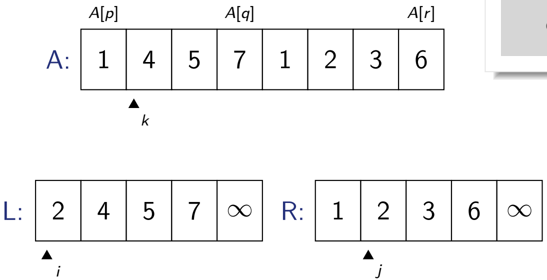

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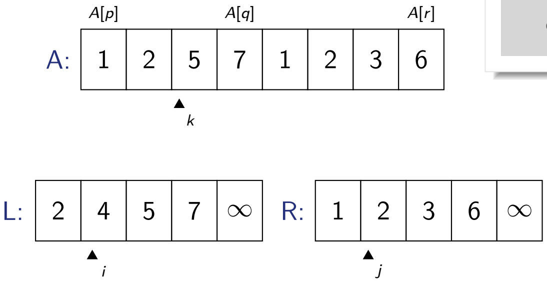

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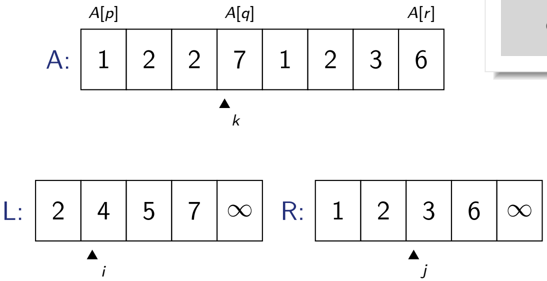

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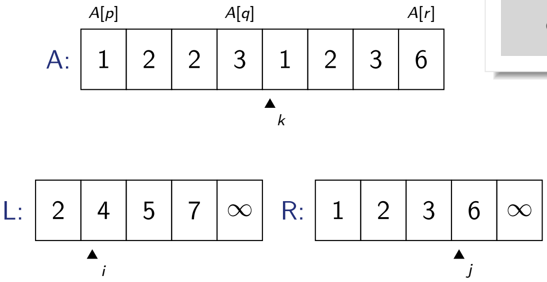

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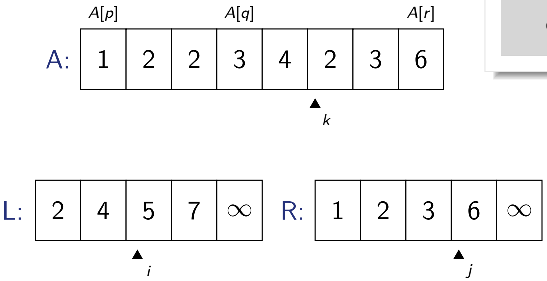

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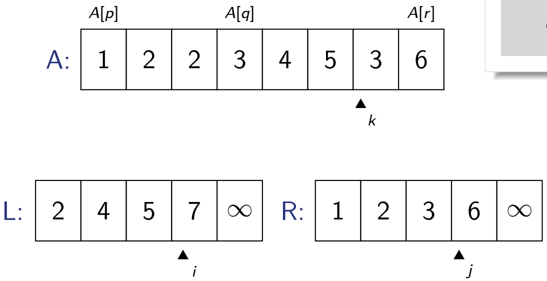

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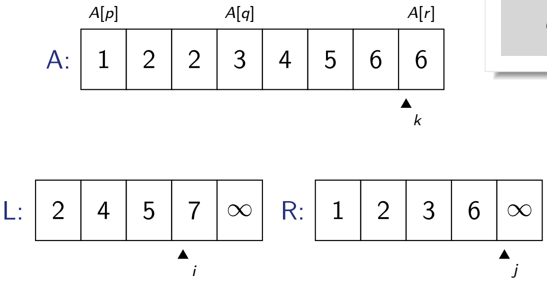

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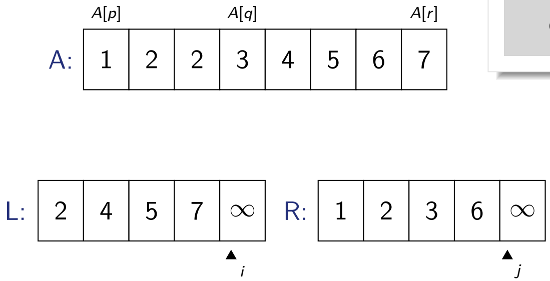

Merging Algorithm

- Runtime analysis?

MERGE(A, p, q, r)

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- Runtime analysis?

Merge runs in time $\Theta(n)$ where n is the number of elements in the subarray, i.e.,

$$n = r - p + 1$$

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- We get the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c, \\ aT(n/b) + D(n) + C(n) & \text{otherwise.} \end{cases}$$

Analysis of Merge Sort

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MERGE-SORT(A, p, r)
  if p < r                              // check for base case
      q = ⌊(p + r)/2⌋                   // divide
      MERGE-SORT(A, p, q)               // conquer
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Recurrence for merge sort running time is

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 2T(n/2) + \Theta(n) & \text{otherwise.} \end{cases}$$

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- For small instances insertion sort can still be faster
- Insertion sort is also **in place**: the numbers are rearranged within the array (with at most a constant number outside the array at any time)
- Merge sort is not in place!