Algorithms: Hashing and Quick Sort

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Hash tables: summary

HASH-tables efficiently implement:

- INSERT: $O(1)$
- DELETE: $O(1)$
- SEARCH: Expected $O(n/m)$ (if good hash function)

Cannot avoid collisions without having $m \gg n^2$

Instead deal with collisions using for example chaining

Quick Sort

- The sorting algorithm of choice in many computer systems
- Easy to implement
- Fast in practice (and as we will see in theory)
- As merge-sort, based on divide-and-conquer paradigm

DIVIDE-AND-CONQUER

Quick Sort

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Quick Sort

Quick Sort Idea

Example $\langle 5, 8, 4, 7, 1, 2, 3, 6 \rangle$

5	8	4	7	1	2	3	6

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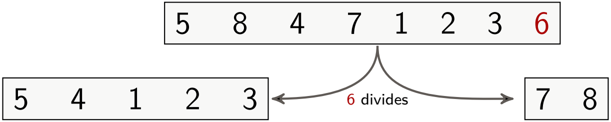

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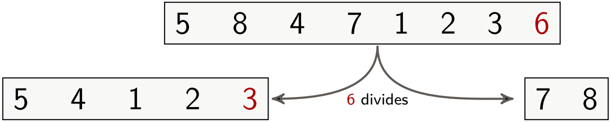

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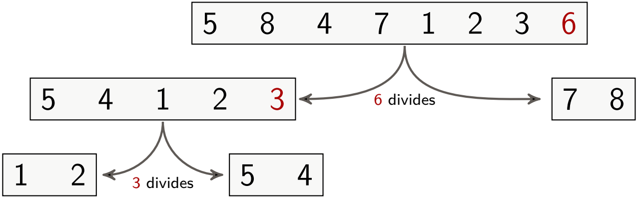

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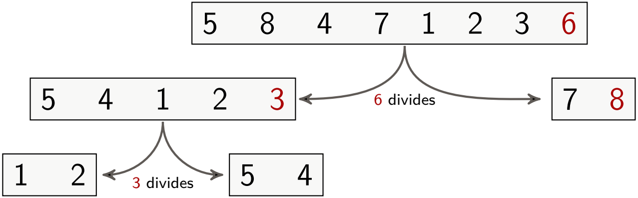

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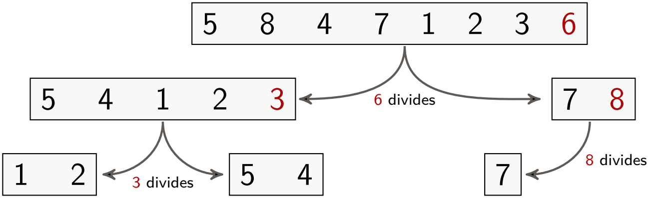

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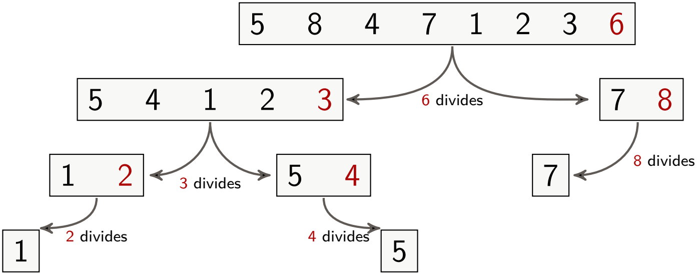

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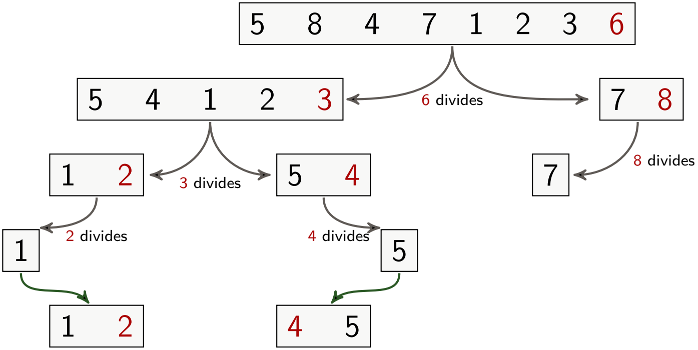

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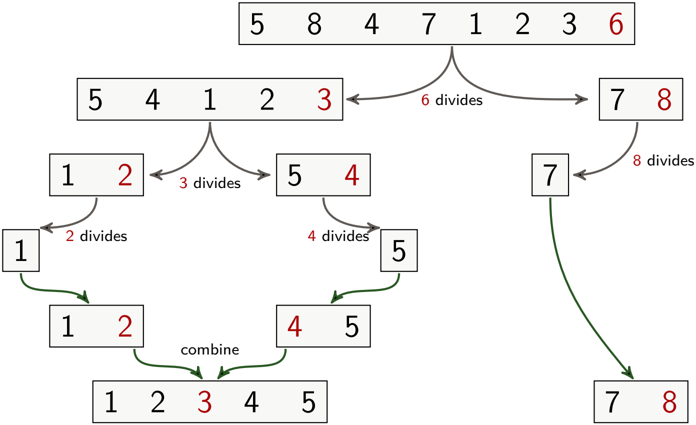

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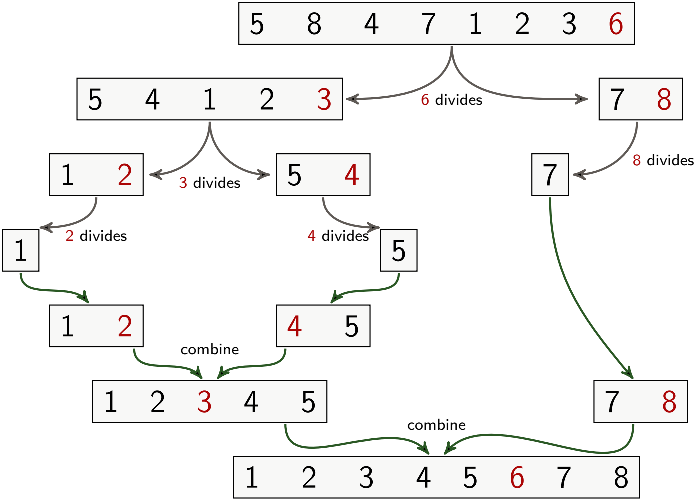

Quick Sort — Divide-and-Conquer

To sort the subarray $A[p \dots r]$:

Divide: Partition $A[p \dots r]$, into two (possibly empty) subarrays $A[p \dots q-1]$ and $A[q+1 \dots r]$, such that each element in the first subarray is $\leq A[q]$ and each element in the second subarray is $\geq A[q]$

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Combine: No work is needed to combine the subarrays, because they are sorted in place

Partitioning (divide step)

PARTITION always selects the last element $A[r]$ in the subarray $A[p \dots r]$ as the **pivot** — the element around which to partition

```
PARTITION(A, p, r)
  x = A[r]
  i = p - 1
  for j = p to r - 1
      if A[j] ≤ x
          i = i + 1
          exchange A[i] with A[j]
  exchange A[i + 1] with A[r]
  return i + 1
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Loop Invariant:

1. All entries in $A[p \dots i]$ are $\leq$ pivot
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▲ i

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i (before $A[p]$), j (at $A[p]$)

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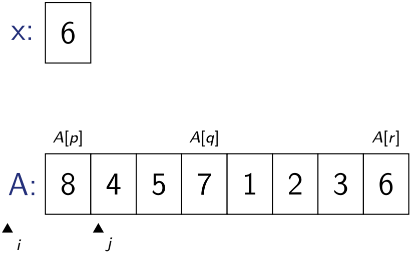

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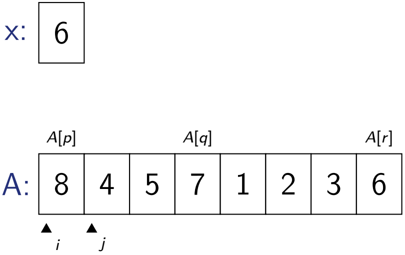

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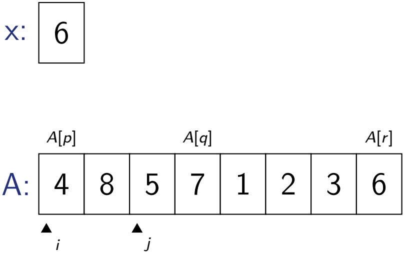

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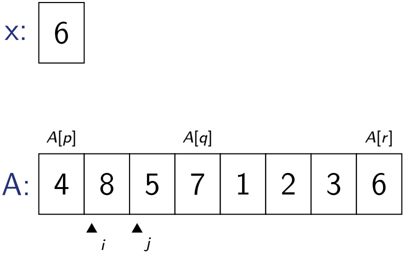

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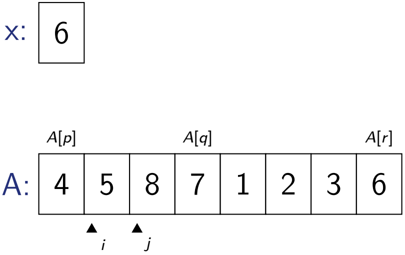

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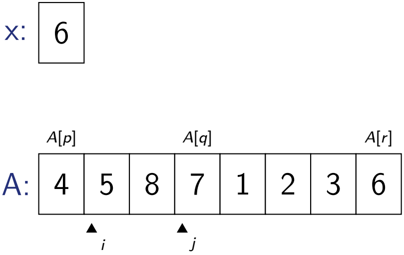

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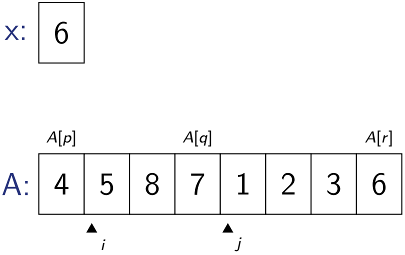

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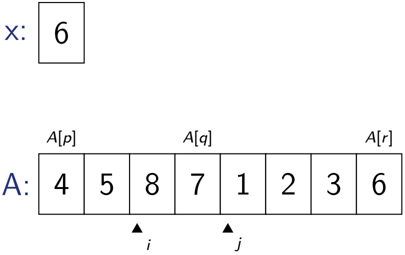

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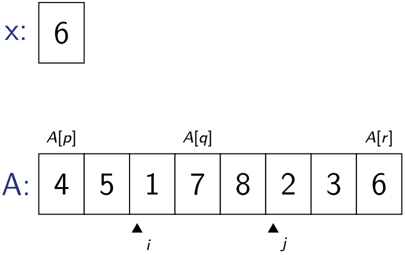

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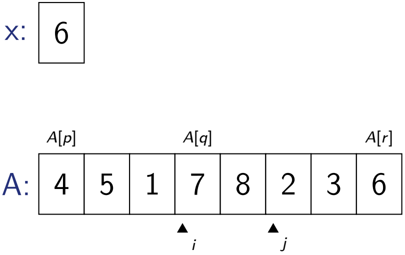

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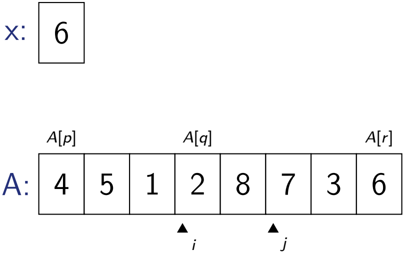

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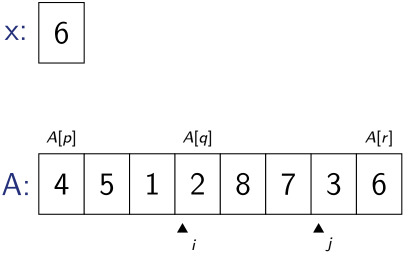

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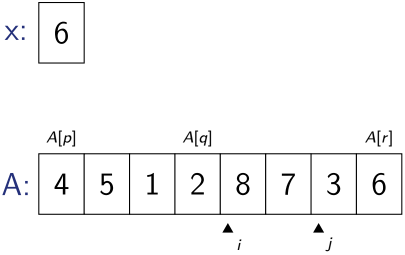

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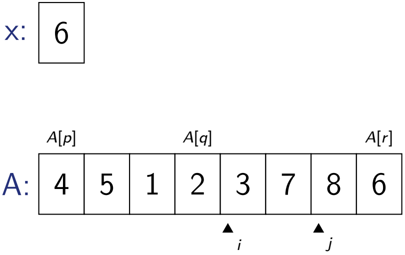

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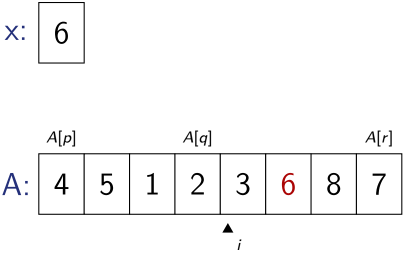

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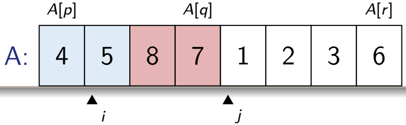

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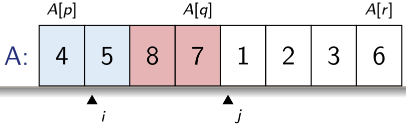

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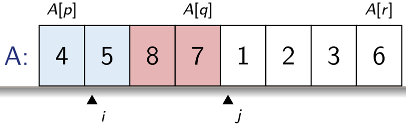
Correctness of Partitioning



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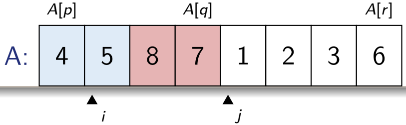

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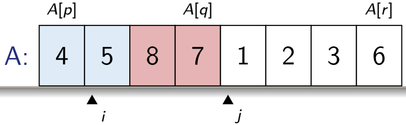

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The last two lines of PARTITION moves the pivot element to the "right" place by swapping $A[i+1]$ and $A[r]$

Time for partitioning

PARTITION(A, p, r)

```
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i + 1
```

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- Total running time is $\Theta(n)$ for an array of length n.
- Note that the number of comparisons made is $\approx n$

Quick Sort Algorithm

```
QUICKSORT(A, p, r)
  if p < r
      q = PARTITION(A, p, r)
      QUICKSORT(A, p, q − 1)
      QUICKSORT(A, q + 1, r)
```

Worst case running time of quick sort

1	2	3	4	...	n-2	n-1	n

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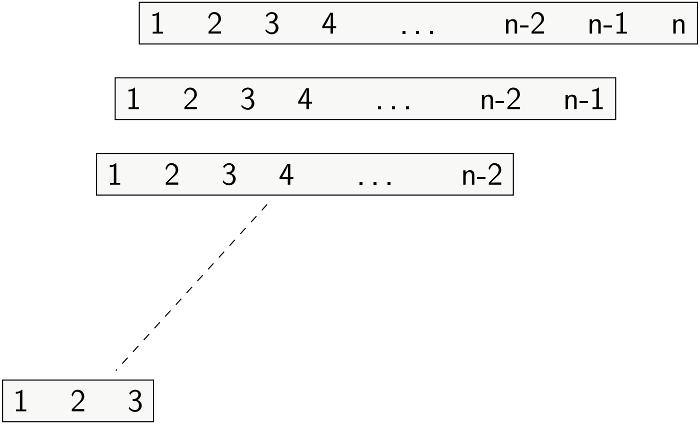

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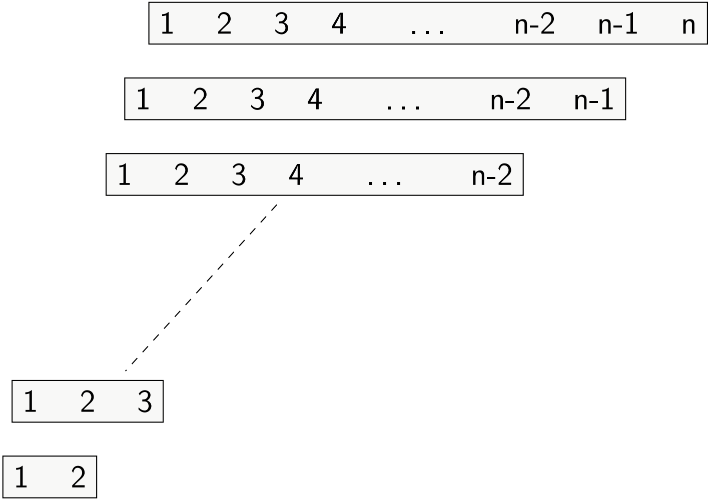

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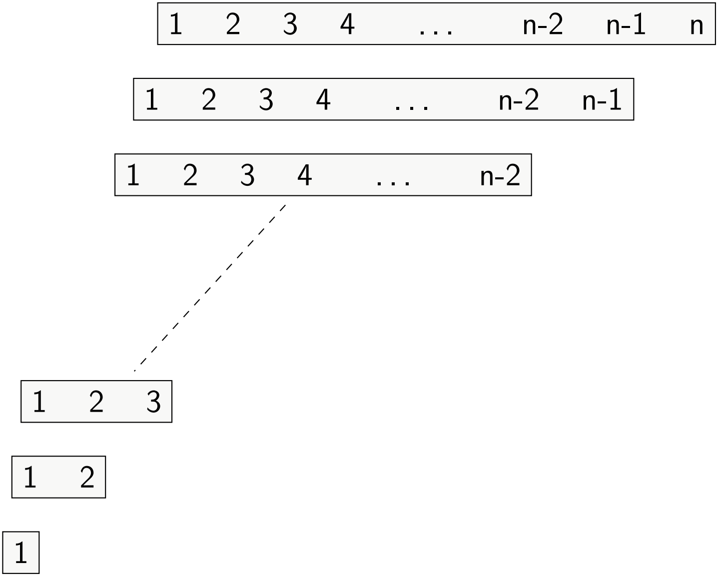

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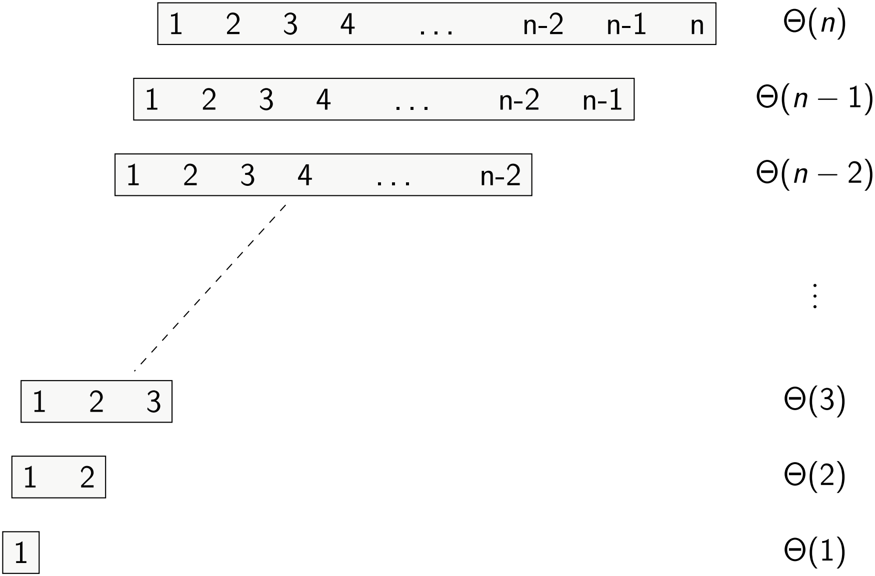

Worst case running time of quick sort

1 2 3 4 … n-2 n-1 n	$\Theta(n)$
1 2 3 4 … n-2 n-1	$\Theta(n-1)$
1 2 3 4 … n-2	$\Theta(n-2)$
	$\vdots$
1 2 3	$\Theta(3)$
1 2	$\Theta(2)$
1	$\Theta(1)$

Total running time: $\Theta(n^2)$

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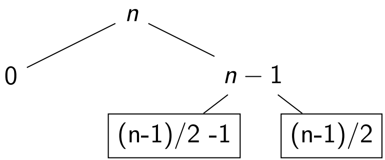

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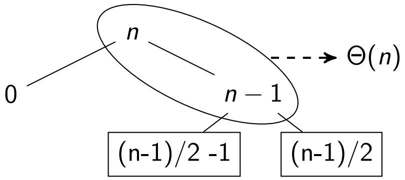

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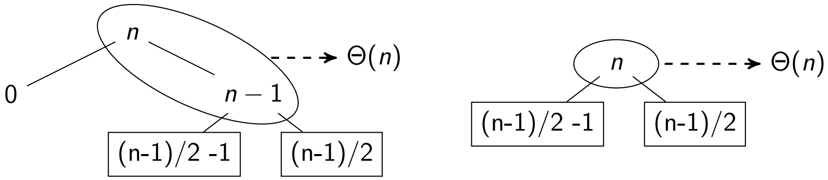

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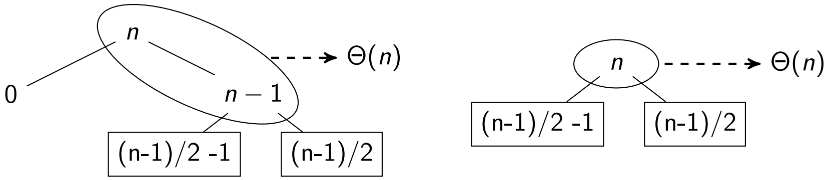


Both trees have the same asymptotic running time: $\Theta(n \lg n)$

RANDOMIZED VERSION

OF QUICK SORT

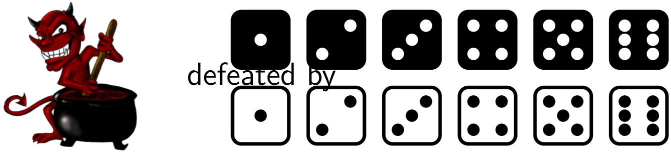
RANDOMIZED VERSION

OF QUICK SORT

defeated by

RANDOMIZED VERSION

OF QUICK SORT



RANDOMIZED VERSION

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Randomized version of quick sort

Advantages

- We saw intuition for good running time when all permutations of input are equally likely
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- This is not always true
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- **HUGE difference between**

Expected running time over all inputs

and

Expected running time for any input

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- We could randomly permute input array
- Instead we use **random sampling** or picking one element in random
- Don't always use $A[r]$ as the pivot. Instead, randomly pick an element from the subarray that is being sorted

Randomized quick sort

```
RANDOMIZED-PARTITION(A, p, r)
  i = RANDOM(p, r)
  exchange A[r] with A[i]
  return PARTITION(A, p, r)
```

```
RANDOMIZED-QUICKSORT(A, p, r)
  if p < r
      q = RANDOMIZED-PARTITION(A, p, r)
      RANDOMIZED-QUICKSORT(A, p, q - 1)
      RANDOMIZED-QUICKSORT(A, q + 1, r)
```

Analysis

Time to wake up!

Example

$A[1]$	$A[2]$	$A[3]$	$A[4]$	$A[5]$	$A[6]$	$A[7]$	$A[8]$	$A[9]$	$A[10]$	$A[11]$
7	6	2	3	1	5	10	12	9	15	4

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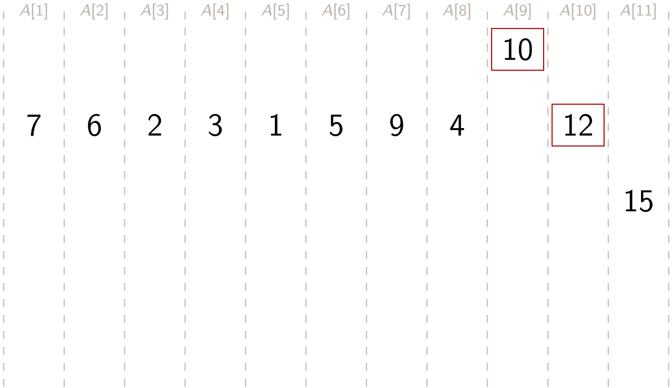

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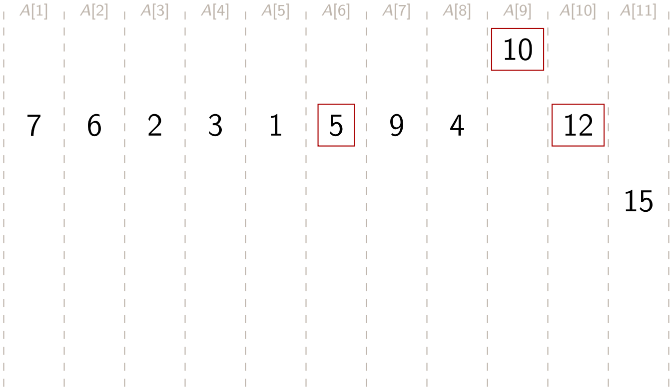

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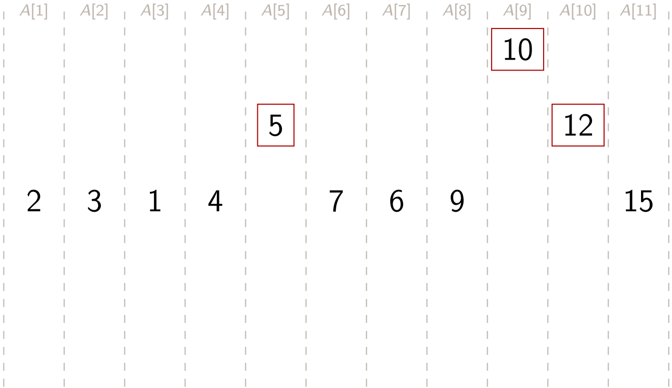

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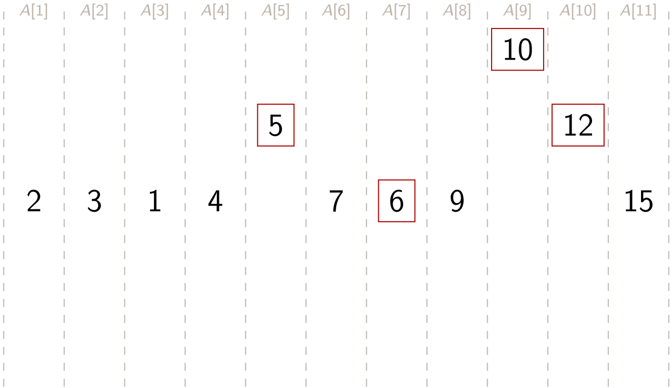

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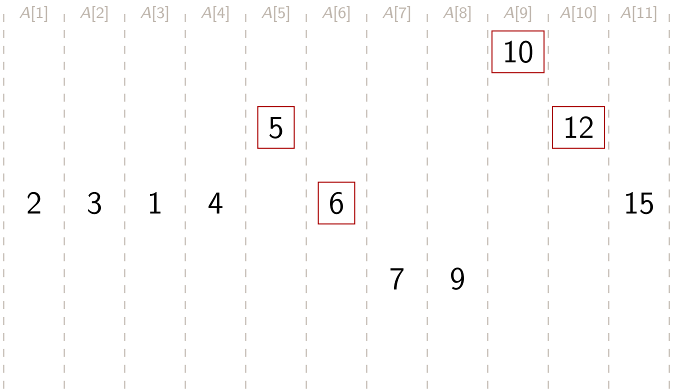

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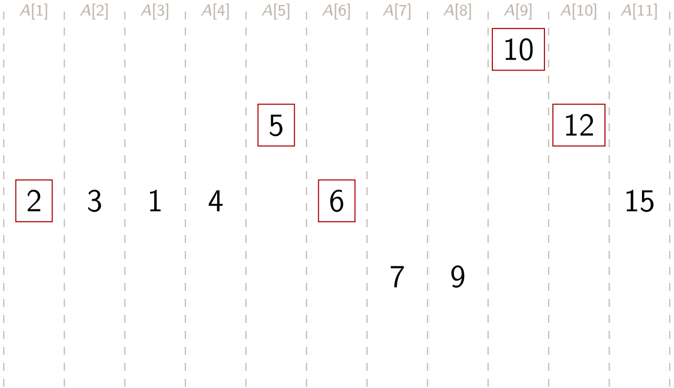

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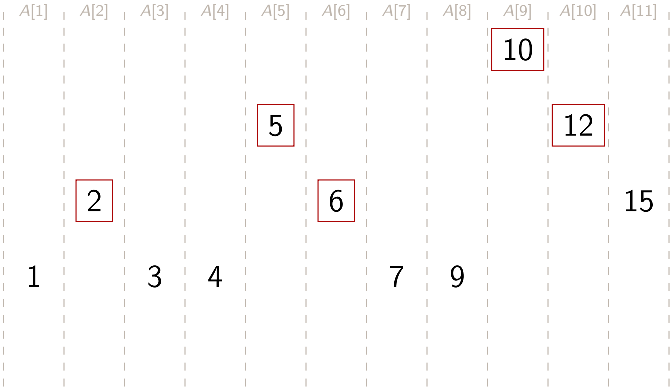

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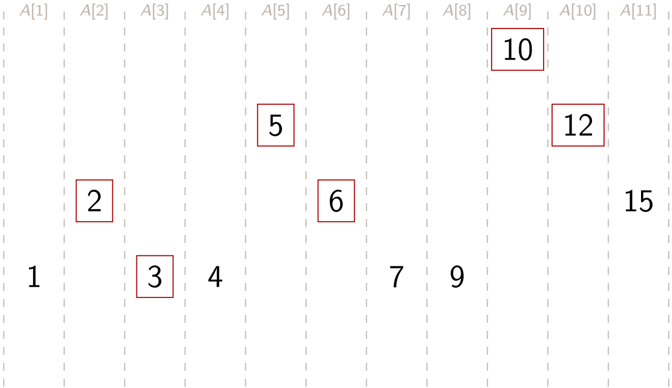

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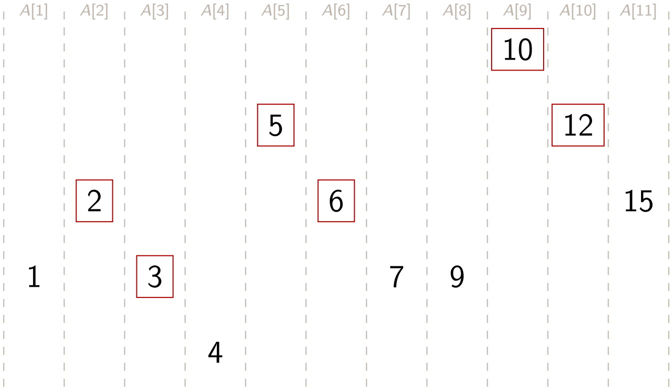

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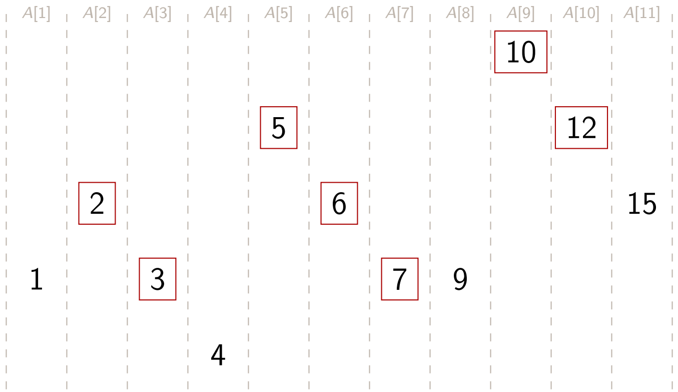

Time it takes is number of calls to PARTITION + total number of comparisons

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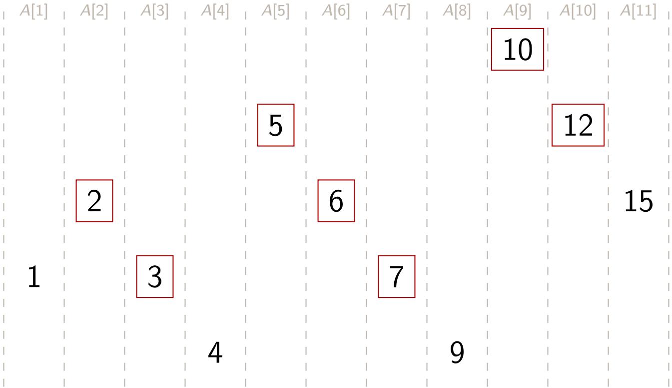

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- We proceed by bounding (the expected value of) X

Bound on the overall number of comparisons

For ease of notation:

- Rename elements of A as $z_1, \ldots, z_n$, with z_i being the ith smallest element
- Define the set $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$

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Random indicator variables:

- Let $X_{ij} = I\{z_i \text{ is compared to } z_j\}$
- As each pair is compared at most once (when one of them is the pivot), the total number of comparisons formed by the algorithm is

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

Applying linearity of expectation

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$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \Pr[z_i \text{ is compared to } z_j]$$

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- Therefore

$$\Pr[z_i \text{ is compared to } z_j] = \frac{2}{j - i + 1}$$

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$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

Wrapping up

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[z_i \text{ is compared to } z_j] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} \\
&< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n-1} O(\lg n) = O(n \lg n)
\end{aligned}
$$

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