# Algorithms: Hashing and Quick Sort

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# Hash tables: summary

HASH-tables efficiently implement:

INSERT: $O(1)$
DELETE: $O(1)$
SEARCH: Expected $O(n/m)$ (if good hash function)

Cannot avoid collisions without having $m \gg n^2$

Instead deal with collisions using for example chaining

# Quick Sort

- The sorting algorithm of choice in many computer systems
- Easy to implement
- Fast in practice (and as we will see in theory)
- As merge-sort, based on divide-and-conquer paradigm


# DIVIDE-AND-CONQUER

## Quick Sort

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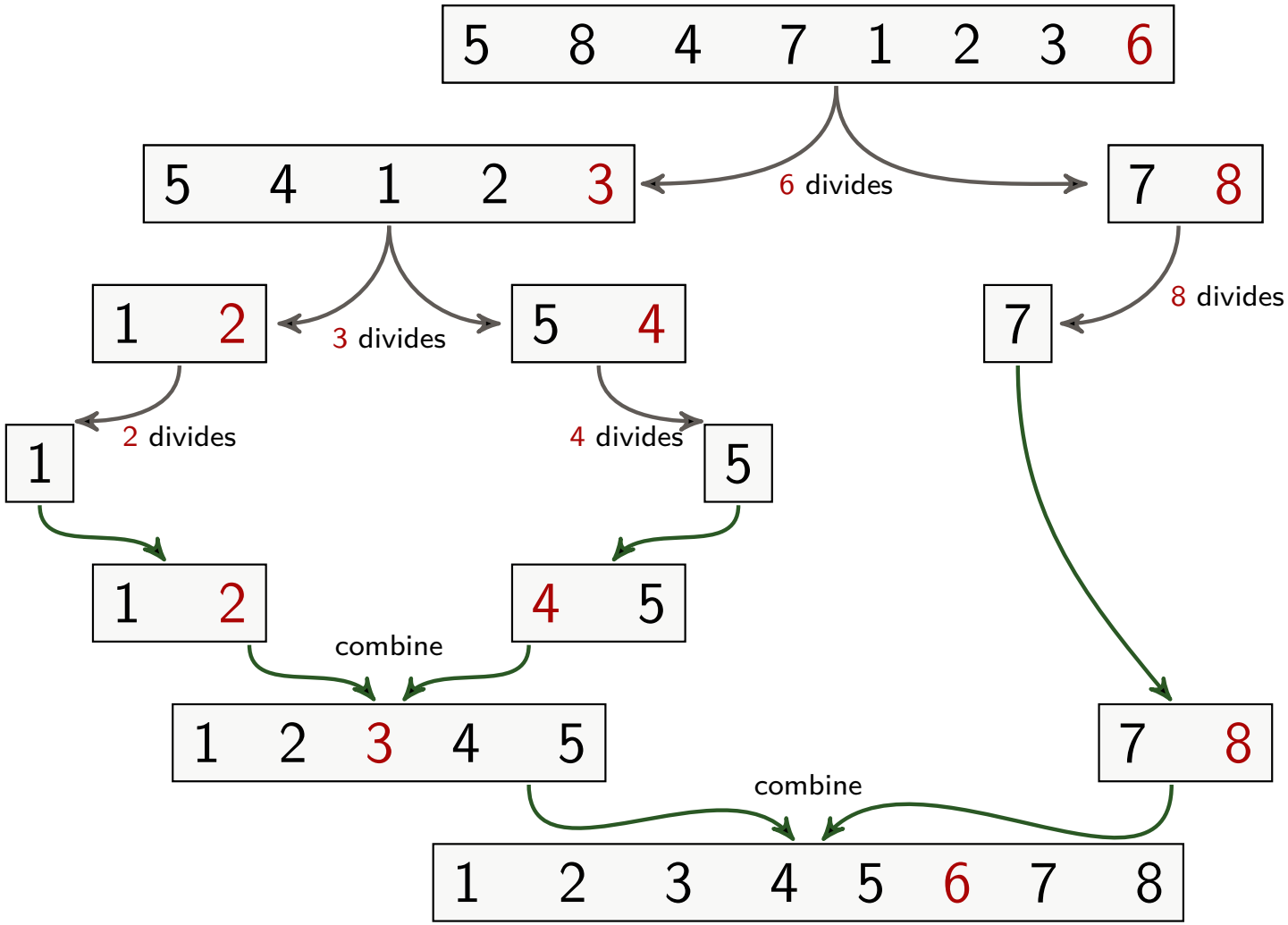
# Quick Sort Idea

## Example $\langle 5, 8, 4, 7, 1, 2, 3, 6 \rangle$



# Quick Sort — Divide-and-Conquer

To sort the subarray $A[p \ldots r]$:

**Divide:** Partition $A[p \ldots r]$, into two (possibly empty) subarrays $A[p \ldots q-1]$ and $A[q+1 \ldots r]$, such that each element in the first subarray is $\leq A[q]$ and each element in the second subarray is $\geq A[q]$

**Conquer:** Sort the two subarrays by recursive calls to QUICKSORT

**Combine:** No work is needed to combine the subarrays, because they are sorted in place

# Partitioning (divide step)

PARTITION always selects the last element $A[r]$ in the subarray $A[p \ldots r]$ as the **pivot** — the element around which to partition

```
PARTITION(A, p, r)
  x = A[r]
  i = p − 1
  for j = p to r − 1
      if A[j] ≤ x
          i = i + 1
          exchange A[i] with A[j]
  exchange A[i + 1] with A[r]
  return i + 1
```

# Partitioning (divide step)

**Loop Invariant:**

1. All entries in $A[p \ldots i]$ are $\leq$ pivot
2. All entries in $A[i+1 \ldots j-1]$ are $>$ pivot
3. $A[r] =$ pivot

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PARTITION(A, p, r)
x = A[r]
i = p − 1
for j = p to r − 1
    if A[j] ≤ x
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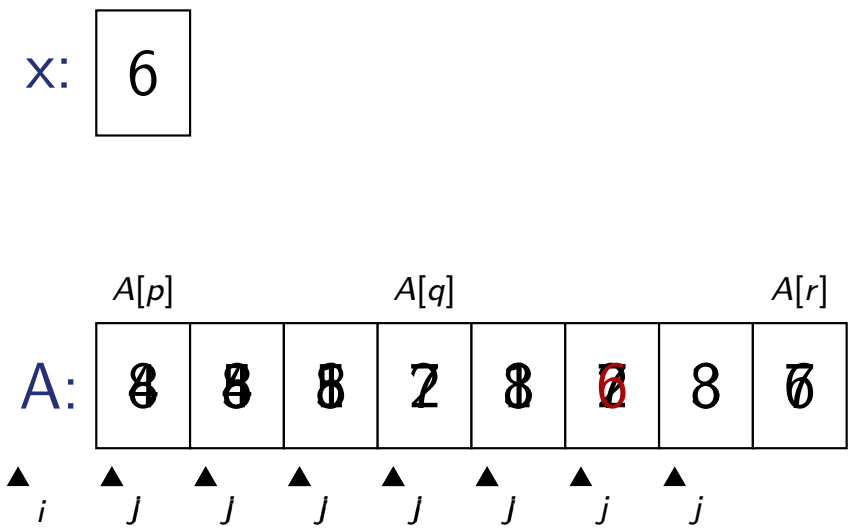
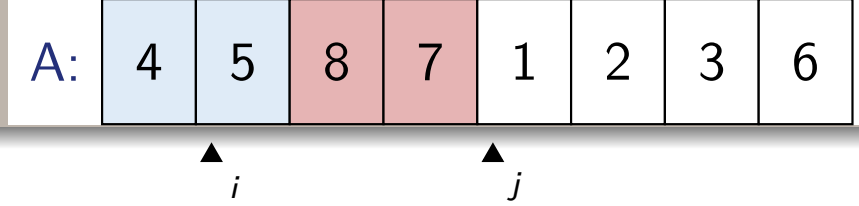


# Correctness of Partitioning




**Loop Invariant:**

1. All entries in $A[p \ldots i]$ are $\leq$ pivot
2. All entries in $A[i+1 \ldots j-1]$ are $>$ pivot
3. $A[r] =$ pivot

Initialization: Before the loop starts, loop invariant satisfied, because r is the pivot and the subarrays $A[p \ldots i]$ and $A[i+1 \ldots j-1]$ are empty

Maintenance: If $A[j] \leq$ pivot, then $A[j]$ and $A[i+1]$ are swapped and then $i$ and $j$ are incremented. If $A[j] >$ pivot then increment only $j$

Termination: When the loop terminates, $j = r$ so all elements in $A$ are partitioned into $A[p \ldots i] \leq$ pivot, $A[i+1 \ldots r-1] >$ pivot and $A[r] =$ pivot

The last two lines of PARTITION moves the pivot element to the "right" place by swapping $A[i+1]$ and $A[r]$

# Time for partitioning

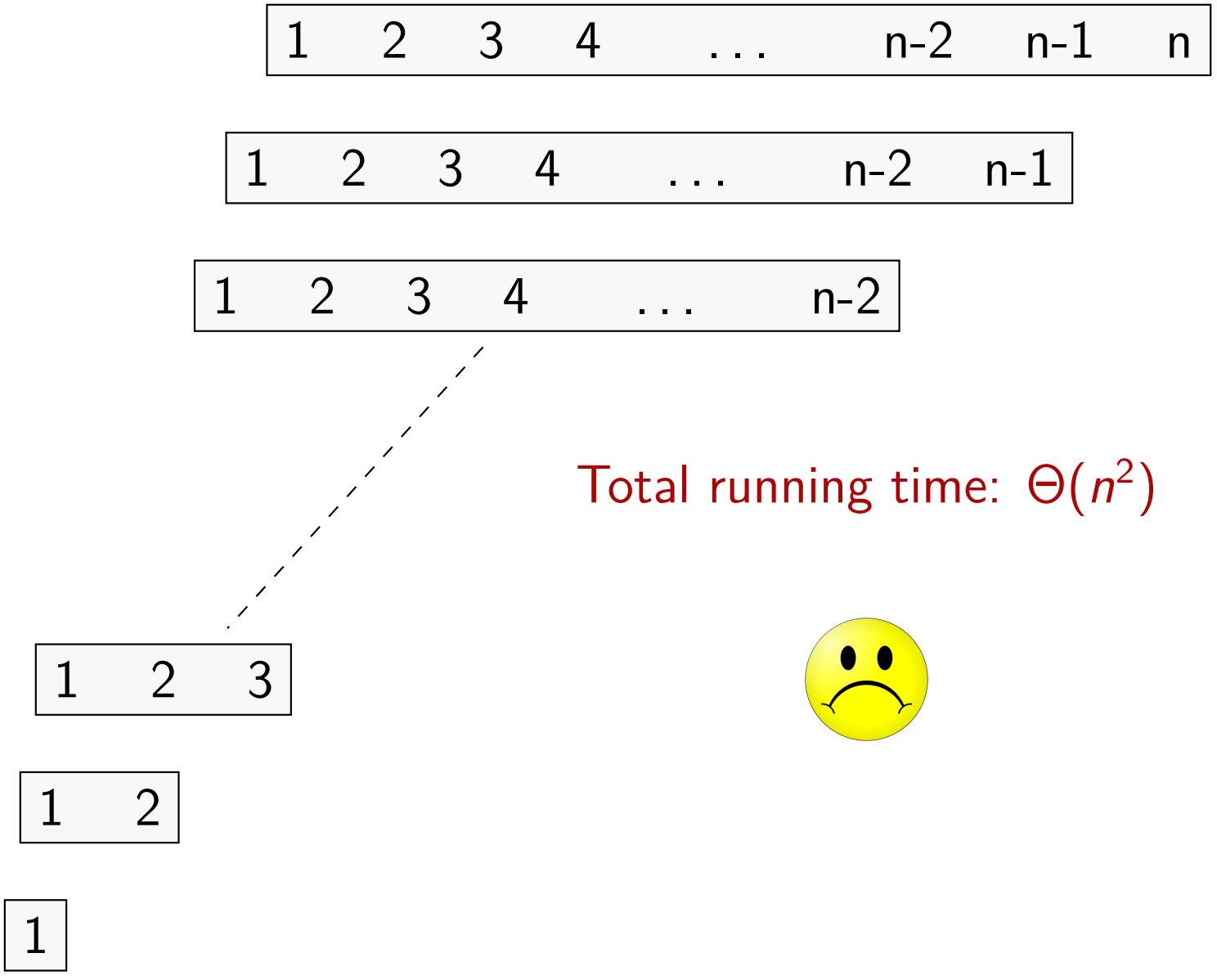
```
PARTITION(A, p, r)
  x = A[r]
  i = p − 1
  for j = p to r − 1
      if A[j] ≤ x
          i = i + 1
          exchange A[i] with A[j]
  exchange A[i + 1] with A[r]
  return i + 1
```

- **for** loop runs $\approx n := r - p + 1$ times.
- Each iteration takes time $\Theta(1)$
- Total running time is $\Theta(n)$ for an array of length $n$.
- Note that the number of comparisons made is $\approx n$

# Quick Sort Algorithm

```
QUICKSORT(A, p, r)
  if p < r
        q = PARTITION(A, p, r)
        QUICKSORT(A, p, q − 1)
        QUICKSORT(A, q + 1, r)
```

# Worst case running time of quick sort



# Best case running time of quick sort

- Occurs when the subarrays are completely balanced every time = the pivots always split the array into two subarrays of equal size
- Get the recurrence

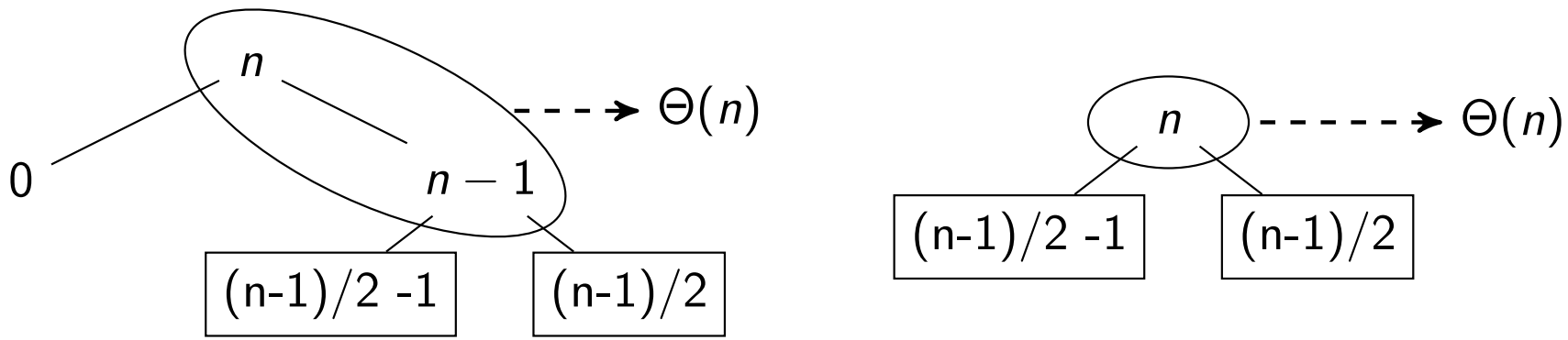
$$T(n) = 2T(n/2) + \Theta(n)$$

$$= \Theta(n \lg n)$$

# Average case over all inputs

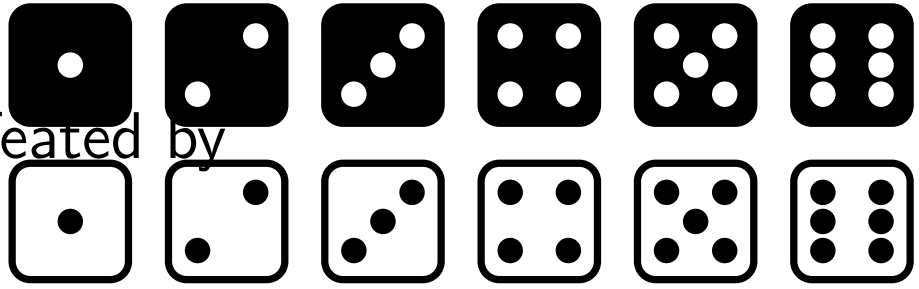
## Intuition

- Imagine that PARTITION always produces a 9-to-1 split.
- Get the recurrence $T(n) = T(9n/10) + T(n/10) + \Theta(n) = \Theta(n \lg n)$
- Splits in the recursion tree will not always be good, there will be usually a mix of good and bad splits
- For intuition why this not affect the running time, suppose we alternate between best-case and worst-case flips




Both trees have the same asymptotic running time: $\Theta(n \lg n)$

defeated by



# RANDOMIZED VERSION

## OF QUICK SORT

# Randomized version of quick sort

## Advantages

- We saw intuition for good running time when all permutations of input are equally likely
- This is not always true
- To correct this and remove the possibilities for enemies we add randomization
- **HUGE difference between**

**Expected running time over all inputs**

and

**Expected running time for any input**

# How to use randomization

- We could randomly permute input array
- Instead we use **random sampling** or picking one element in random
- Don't always use $A[r]$ as the pivot. Instead, randomly pick an element from the subarray that is being sorted

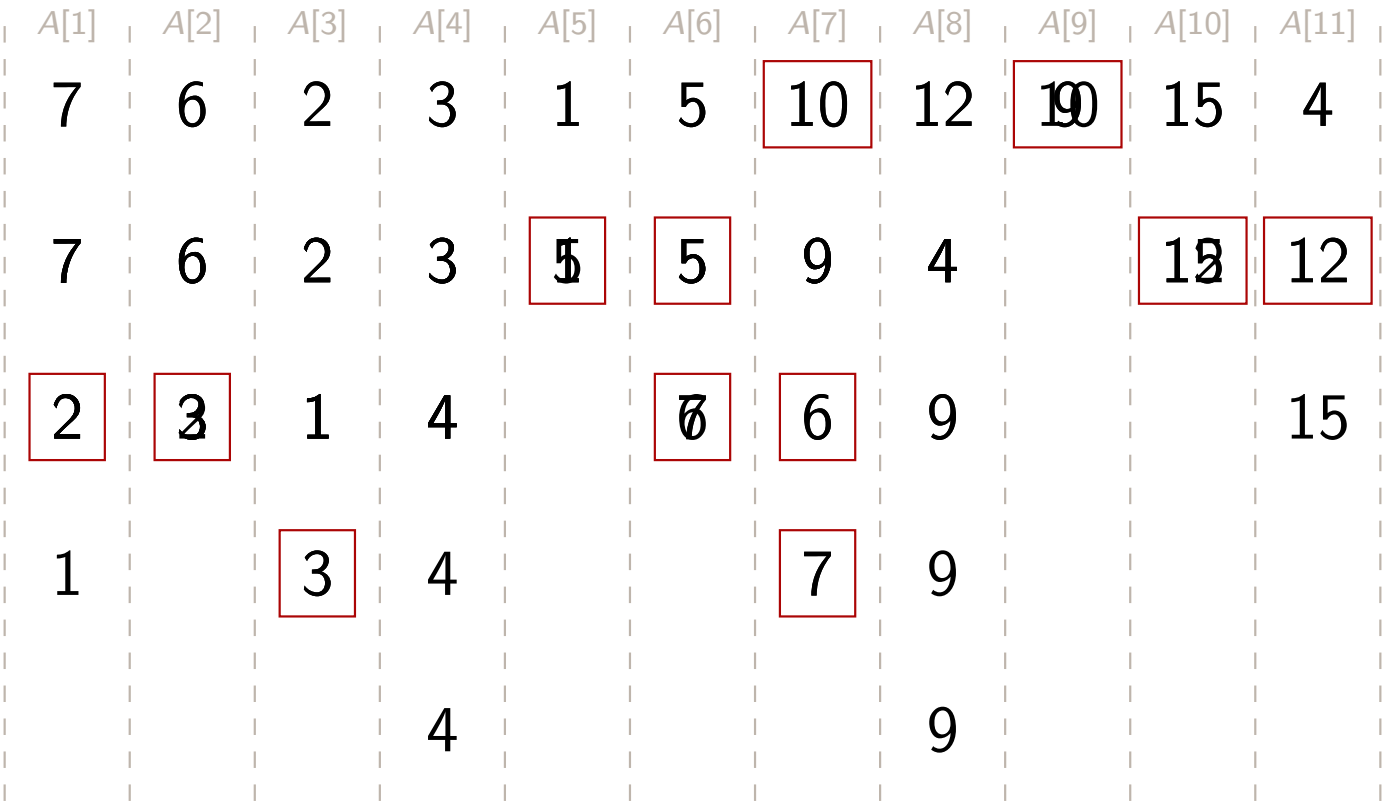
# Randomized quick sort

```
RANDOMIZED-PARTITION(A, p, r)
  i = RANDOM(p, r)
  exchange A[r] with A[i]
  return PARTITION(A, p, r)
```

```
RANDOMIZED-QUICKSORT(A, p, r)
  if p < r
      q = RANDOMIZED-PARTITION(A, p, r)
      RANDOMIZED-QUICKSORT(A, p, q − 1)
      RANDOMIZED-QUICKSORT(A, q + 1, r)
```

Time to wake up!

# Example




Time it takes is number of calls to PARTITION + total number of comparisons

# Average-case analysis

- The dominant cost of the algorithm is partitioning
- Total amount of work of each call to PARTITION is a constant plus the number of comparisons that are performed in the **for** loop
- An element is a pivot at most once $\Rightarrow$ PARTITION is called at most $n$ times
- Let $X =$ the total number of comparisons performed in *all calls* to PARTITION
- Then the total work done over the entire execution is $O(n + X)$
- We proceed by bounding (the expected value of) $X$

# Bound on the overall number of comparisons

For ease of notation:

- Rename elements of $A$ as $z_1, \ldots, z_n$, with $z_i$ being the $i$th smallest element
- Define the set $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$

Random indicator variables:

- Let $X_{ij} = I\{z_i \text{ is compared to } z_j\}$
- As each pair is compared at most once (when one of them is the pivot), the total number of comparisons formed by the algorithm is

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$$

# Applying linearity of expectation

The expected total number of comparisons is

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{E}\left[X_{ij}\right]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \Pr[z_i \text{ is compared to } z_j]$$

# Probability that $z_i$ is compared to $z_j$

- If a pivot $x$ such that $z_i < x < z_j$ is chosen, then $z_i$ and $z_j$ will never be compared at any later time
- If either $z_i$ or $z_j$ is chosen before any other element of $Z_{ij}$, then it will be compared to all the elements of $Z_{ij}$, except itself
- The probability that $z_i$ is compared to $z_j$ is the probability that either $z_i$ or $z_j$ is the element first chosen.
- There are $j - i + 1$ elements and pivots are chosen randomly and independently. Thus the probability that any particular one of them is the first one chose is $1/(j - i + 1)$.
- Therefore

$$\Pr[z_i \text{ is compared to } z_j] = \frac{2}{j - i + 1}$$

# Wrapping up

$$\mathbb{E}[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[z_i \text{ is compared to } z_j]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$$

$$= \sum_{i=1}^{n-1} O(\lg n) = O(n \lg n)$$

# Summary of quick sort

- We have proved that randomized quick sort has expected running time $O(n \lg n)$ for any input.
- The algorithm is in-place
- Very efficient and easy to implement in practice