Algorithms

Dec 13, 2021

PROBABILISTIC ANALYSIS AND RANDOMIZED ALGORITHMS

Motivation

- Worst case does not usually happen
    - Average case analysis
    - Amortized analysis

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  - Choosing the pivot in quick-sort at random
- Randomization necessary in cryptography
- Can we get randomness?
  - How to extract randomness (extractors)
  - Longer "random behaving" strings from small seed (pseudorandom generators)

Probabilistic Analysis: The Hiring Problem

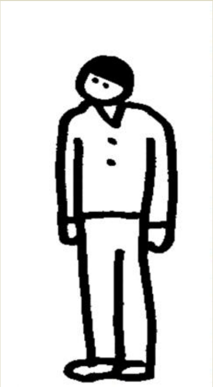
NY Knicks are going to hire one new basketball player

- the taller the better

They have **n** candidates that they call for interview

Strategy: each candidate is hired that is taller than the current best/tallest

Example:



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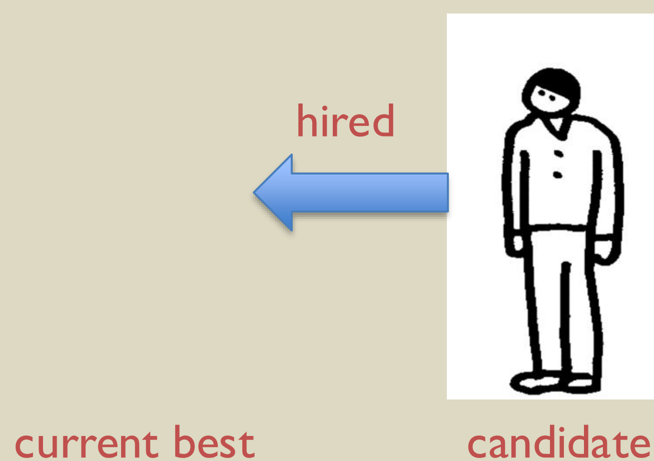

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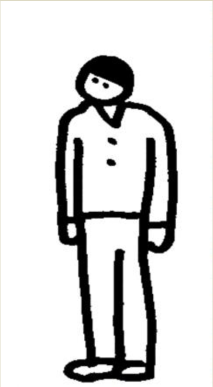

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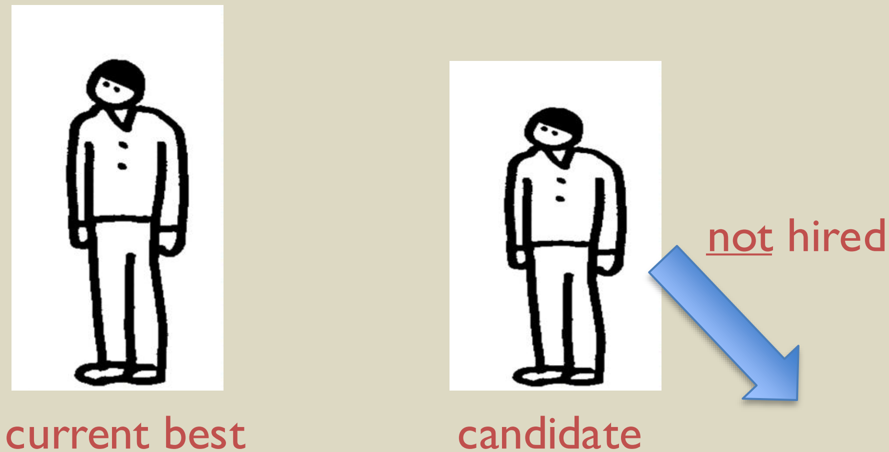

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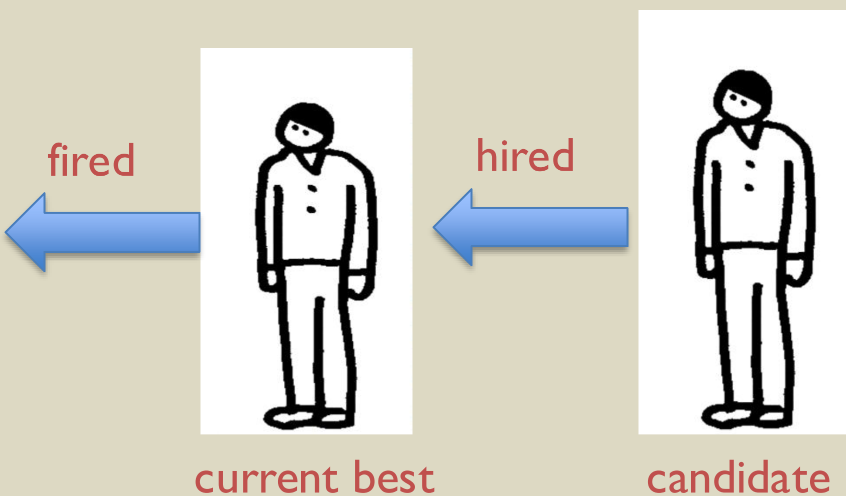

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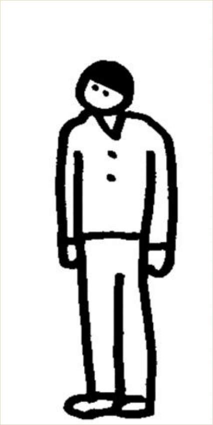

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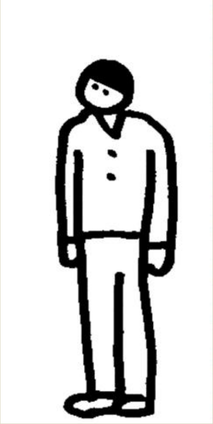

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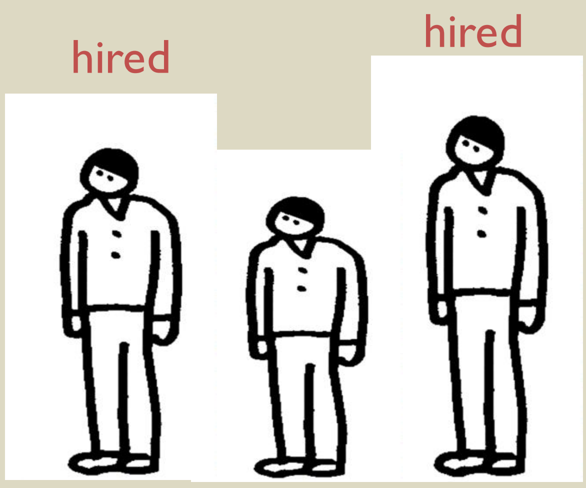

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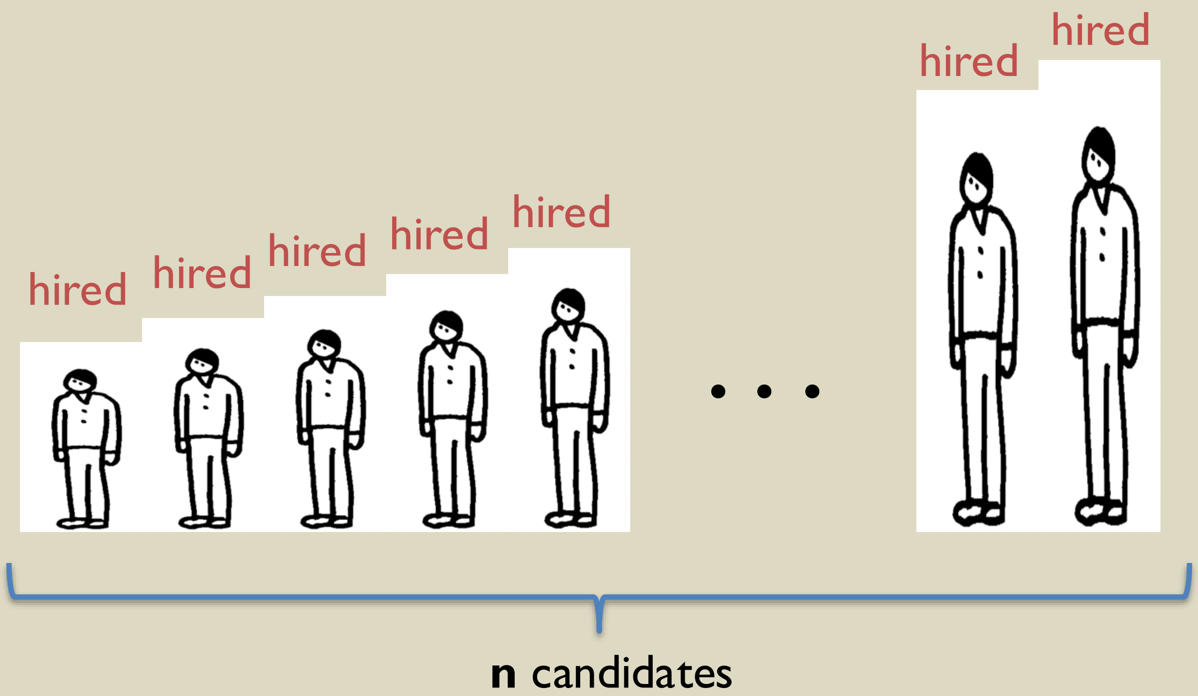


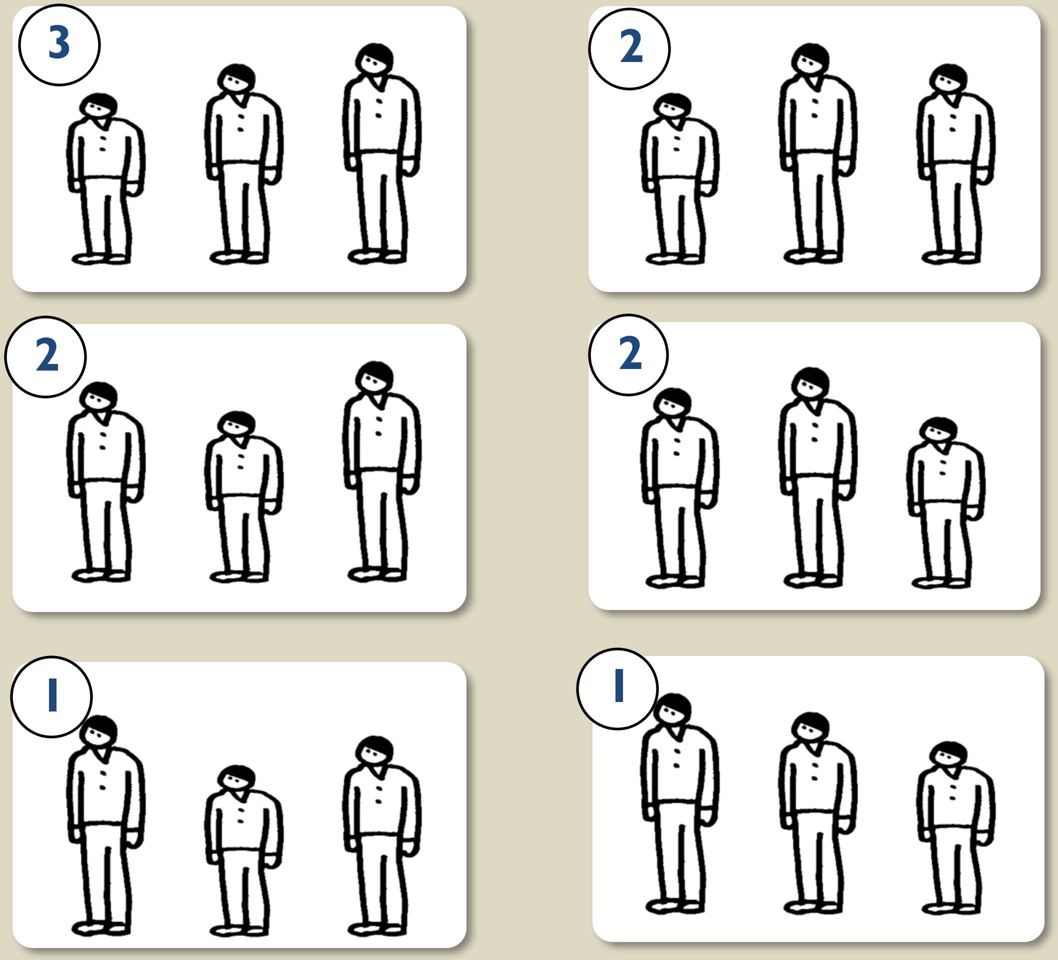
Answer: in the worst case we hire all **n** candidates

Worst-case unlikely to happen

- We only hire all candidates if they arrive in a specific order
- They are likely to arrive in a random order
- More interesting question (probabilistic analysis):

What is the expected number of hires we make over all the permutations of the candidates?

Example



Expected number of hires=

$$\frac{3+2+2+2+1+1}{6}$$

which equals 1+5/6

Calculating the expectation in general 1st trial

- **n!** permutations each equally likely
- Expectation = sum of hires in each permutation divided by **n!**

$$\frac{A_1 + A_2 + \cdots + A_{n!}}{n!}$$

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NEED A MORE CLEVER METHOD

Indicator Random Variables

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DEFINITION: Given a sample space and an event **A**, we define the **indicator random variable**

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$$

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PROOF: $\mathbf{E[X_A] = 1 * Pr\{A\} + 0 * Pr\{\overline{A}\} = Pr\{A\}}$

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- Since **Pr{H} = 1/2**, previous lemma says that $\mathbf{E[X_H] = 1/2}$

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- Could calculate

$$E[X] = \sum_{k=0}^{n} k \cdot \Pr\{X = k\}$$

- … but cumbersome
- Instead use indicator variables

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  *holds even if **X** and **Y** are dependent*

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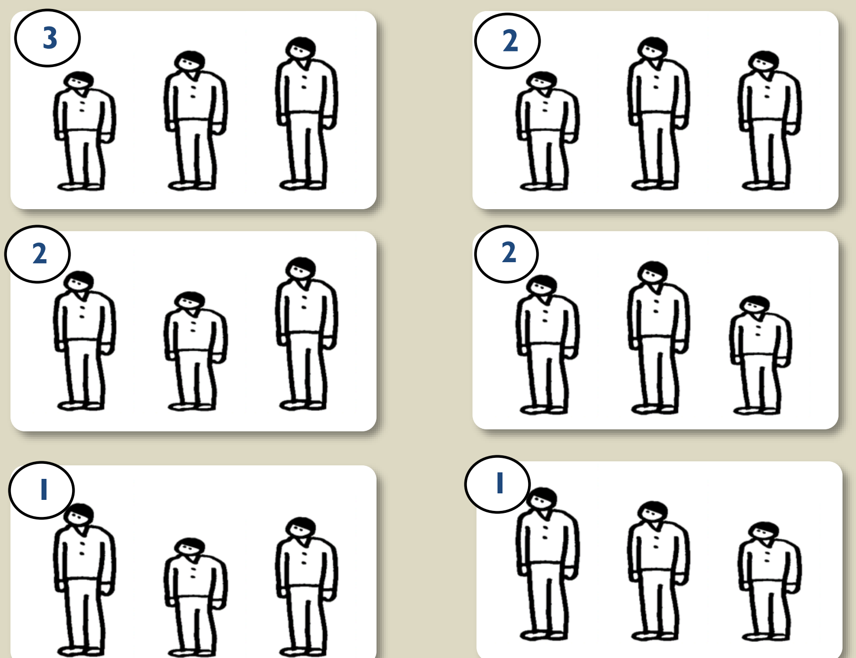

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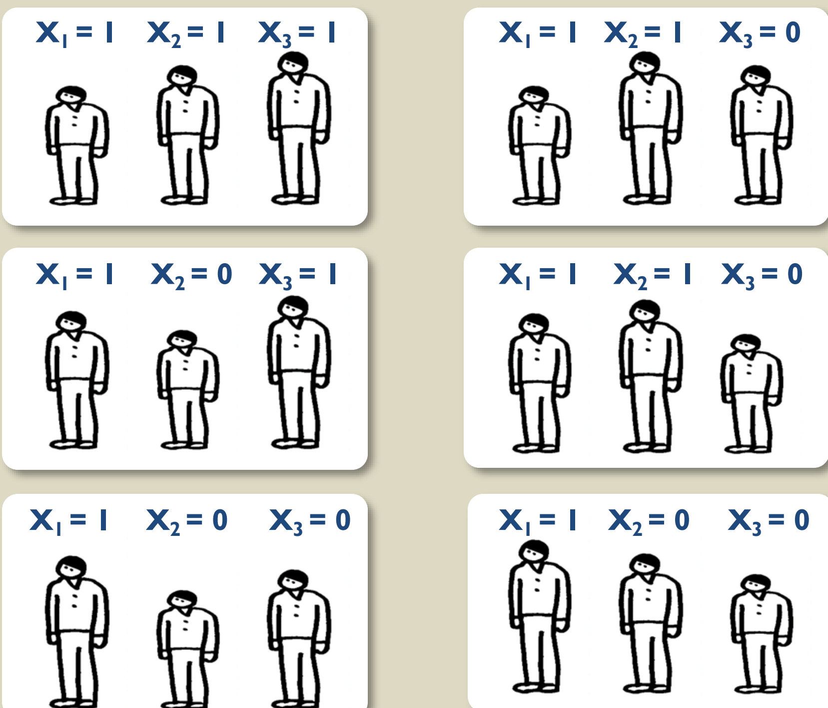

$$\mathbf{E[X] = E[X_1+X_2+X_3]}$$

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$\mathbf{Pr\{}$candidate **1** is hired$\mathbf{\}}$ **+** $\mathbf{Pr\{}$candidate **2** is hired$\mathbf{\}}$ **+ … +** $\mathbf{Pr\{}$candidate **n** is hired$\mathbf{\}}$

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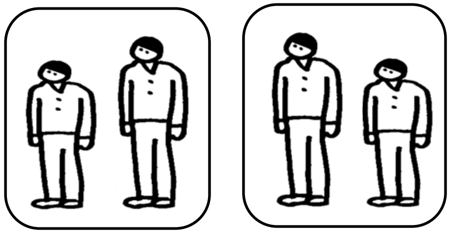

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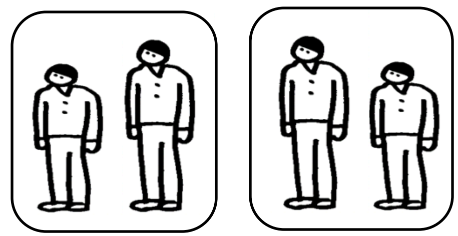
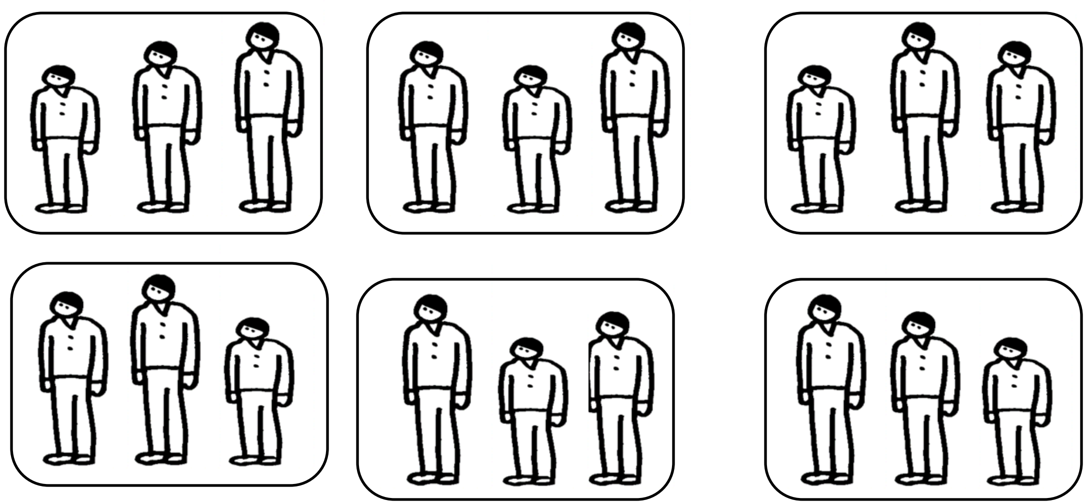


Pr{candidate **3** is hired} **= 1/3**



Probability of Hiring i’th Candidate

- i’th candidate hired iff he is tallest among the first i candidates
- Since they arrive in random order, any one of these first i candidates are equally likely to be the tallest =>

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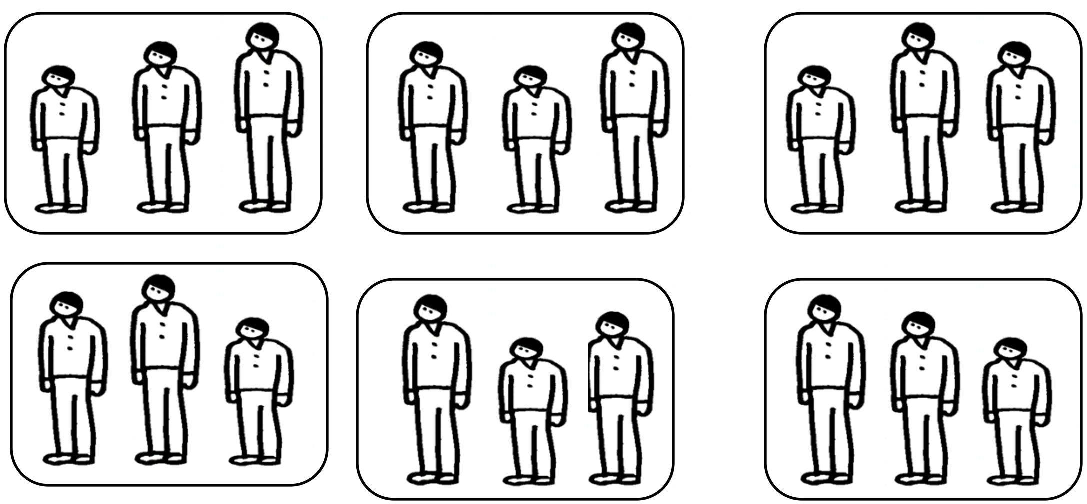

Expected Number of Hires

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n:th harmonic number

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Examples:

- Expected number of hires for **n=6** is **2.45**
- Expected number of hires for **n=100** is **5.1874**
- Expected number of hires for **n=10000** is **9.7876**

Questions

- What is the probability that we hire only one candidate?

- What is the probability that we hire **n** candidates?

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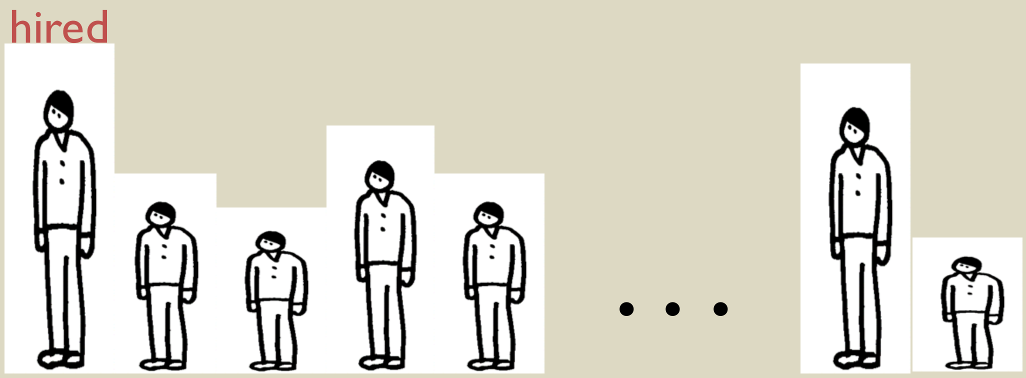

- What is the probability that we hire **n** candidates? **1/n!** (worst case order)

Randomized Algorithm

- Instead of assuming that the candidates arrive in random order

- **We/the algorithm** pick a random order and call the candidates in this order

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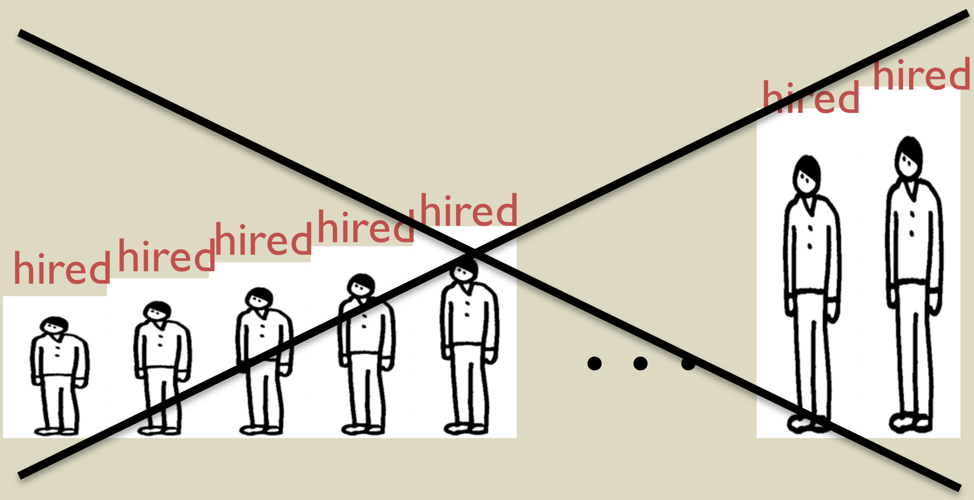
- Instead of assuming that the candidates arrive in random order
- **We/the algorithm** pick a random order and call the candidates in this order
- **In this way we can foul malicious users**



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- Given a function **RANDOM** that returns **1** with probability **p** and **0** with probability **1-p**

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- How to use **RANDOM** for generating an unbiased bit?

- Pick a pair (a,b) of random numbers: a = **RANDOM** and b = **RANDOM**
  - If a≠b return a
  - Otherwise pick a new pair

Two students in class have the same birthday with overwhelming probability

BIRTHDAY PARADOX

Birthday Paradox

- How many students in a room do we need so that the probability that two of them has the same birthday is at least 50%?

  (assuming each of the 365 days is equally likely)

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- Trivially: if we have 366 students then two of them has the same birthday with probability 1

Illustrative Example

2013 YEAR CALENDAR

JANUARY

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

FEBRUARY

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28		

MARCH

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

APRIL

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

MAY

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

JUNE

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30						

JULY

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

AUGUST

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

SEPTEMBER

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

OCTOBER

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

NOVEMBER

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

DECEMBER

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Birthday Paradox

- How many students in a room do we need so that the probability that two of them has the same birthday is at least 50%?

  (assuming each of the 365 days is equally likely)

- Trivially: if we have 366 students then two of them has the same birthday with probability 1

- Surprisingly: 50% probability reached with 23 students and 99% probability reached with just 57 students

Birthday Lemma

If $q > 1.78\sqrt{|M|}$ then the probability that a function chosen uniformly at random $f: \{1, 2, \ldots, q\} \to M$ is injective is at most ½

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Let m = |M|. The probability that the function is injective is

$$\frac{m}{m} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-(q-1)}{m}$$

Since $e^{-x} > 1-x$ we have that this is less than

$$e^{-0} \cdot e^{-1/m} \cdot e^{-2/m} \cdots e^{-(q-1)/m} = e^{-q(q-1)/(2m)}$$

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HASH FUNCTIONS AND TABLES

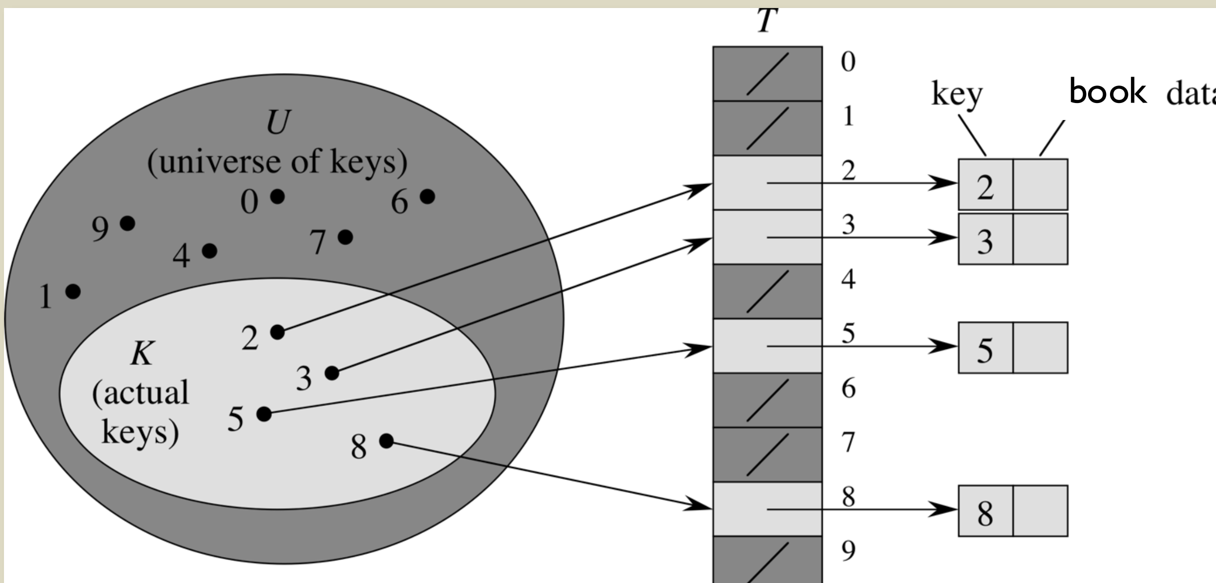
Design a Computer System for a Library

Design a Computer System for a Library

- Insert a new book
- Delete book
- Search book
- All operations in (expected) constant time!

Direct-Address Tables

- Simple technique that allows for simple implementation of constant-time insertion, deletion, and search
- Every book has one unique number (ISBN)
- Construct an array/table T with a position for each book



Direct-Address Tables

Direct-Address-Search(T,k):

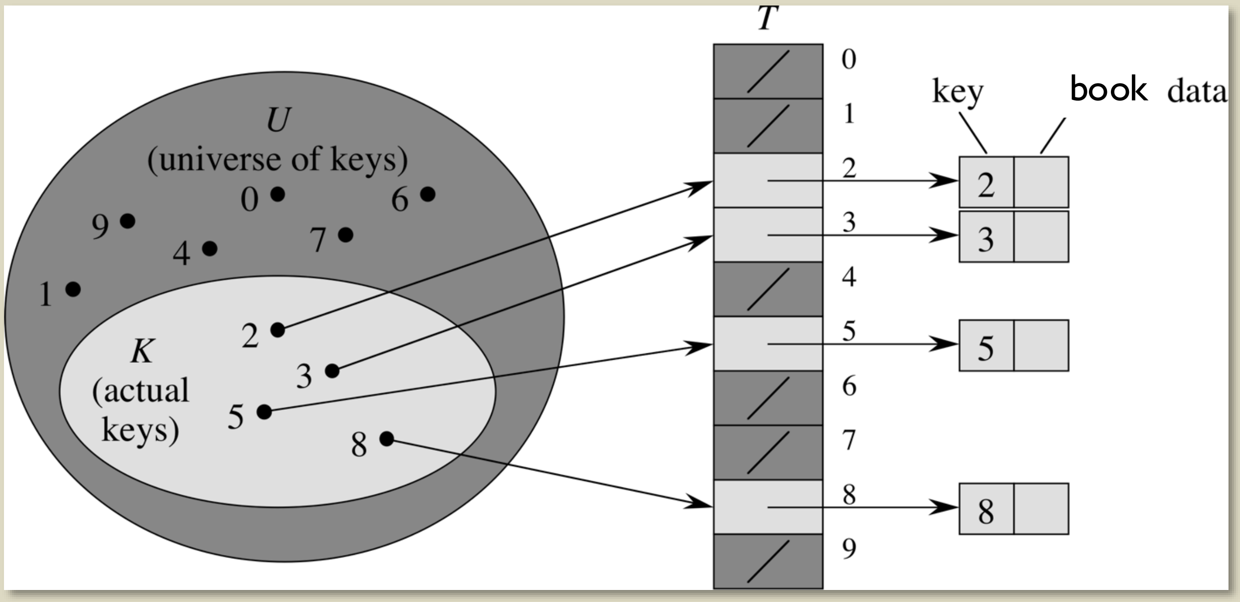
return T[k]

Direct-Address-Insert(T,x):

T[x.key] = x

Direct-Address-Delete(T,x):

T[x.key]= NIL



Direct-Address Tables

- Running time of each operation: **O(1)**

- Space: **O(|U|)**

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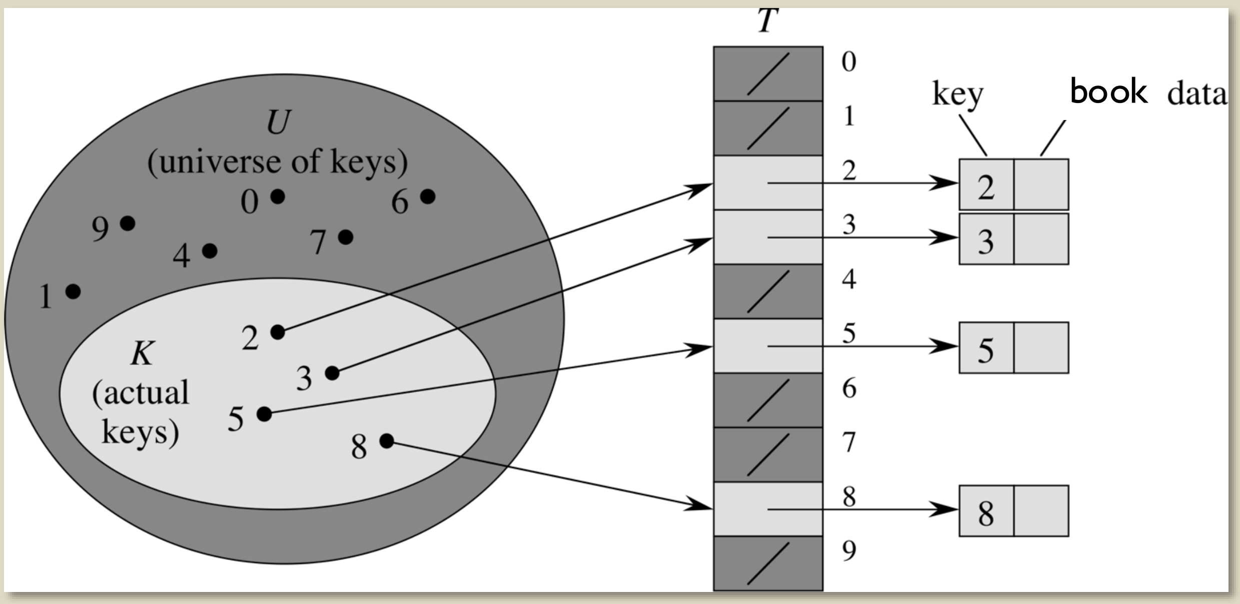

Direct-Address Tables

Positives

Negatives

Direct-Address Tables

Positives

- Running time of each operation: **O(1)**
- Easy implementation

Negatives

- Space: **O(|U|)**
- For most applications (like a Library) we only store a small fraction of all possible items
- Wish to use space proportional to the amount of information stored

Hash Tables

- Uses space proportional to the number **K** of keys stored, i.e., **Θ(K)**
- Implement search, insertion, deletion in time **O(1)** in the average case

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Hash Tables

- Uses space proportional to the number **K** of keys stored, i.e., **$\Theta(K)$**
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- **$h: U \rightarrow \{0,1, \ldots, m-1\}$** is called the ***hash function***

table size

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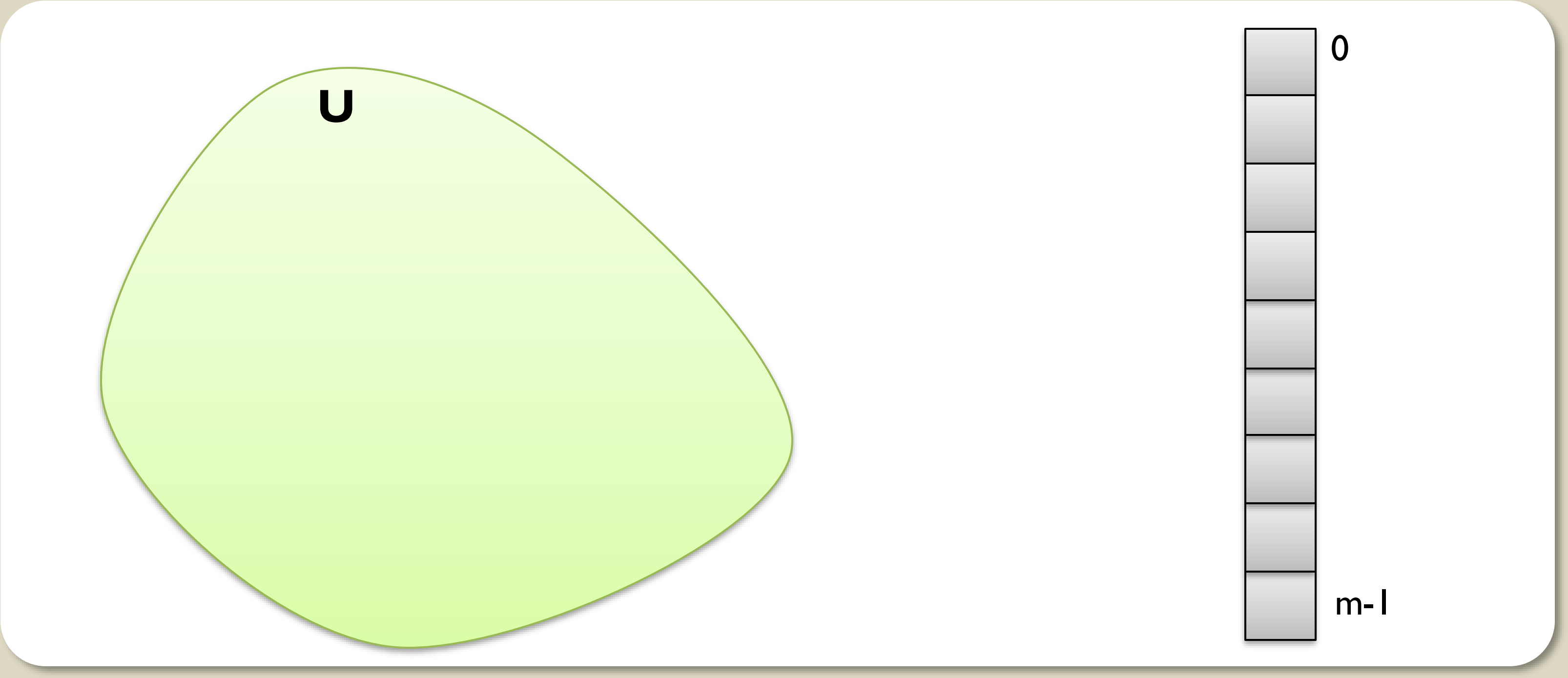

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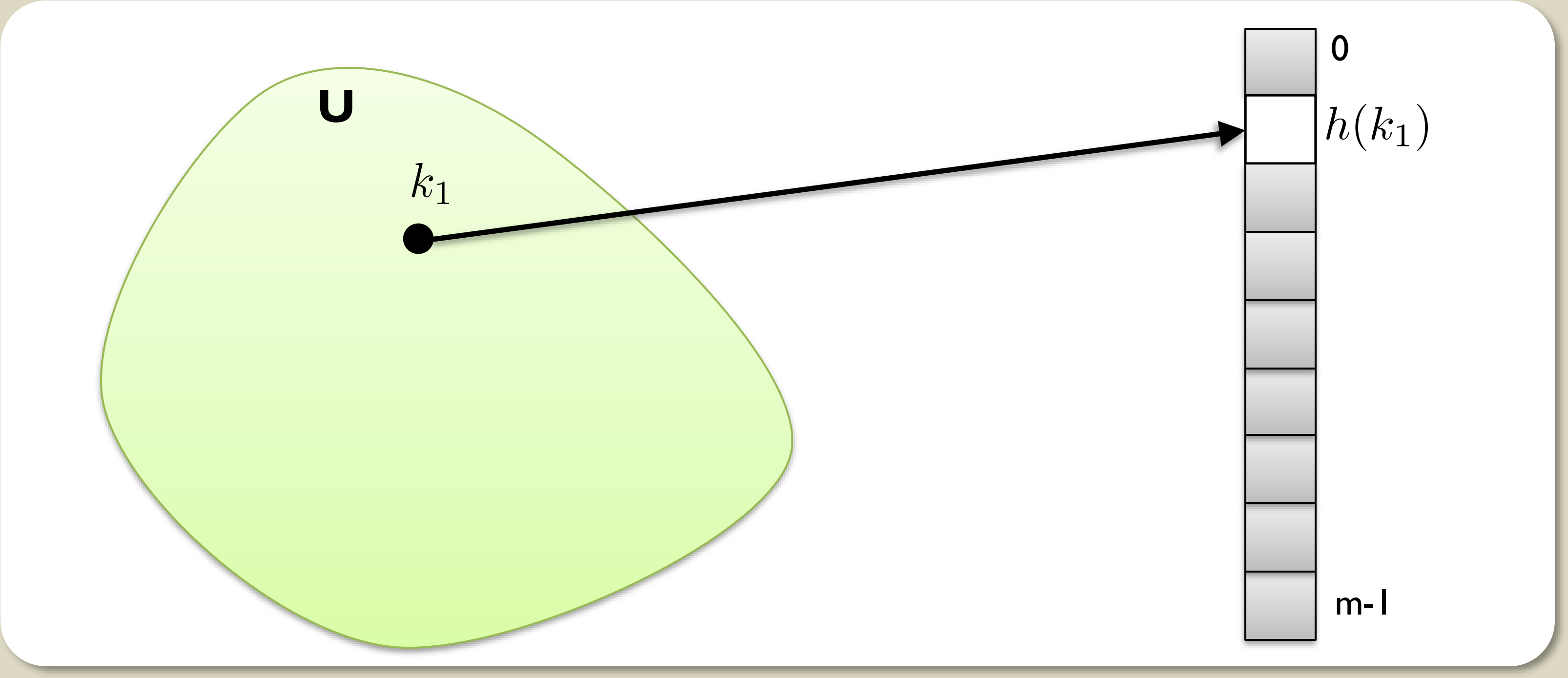

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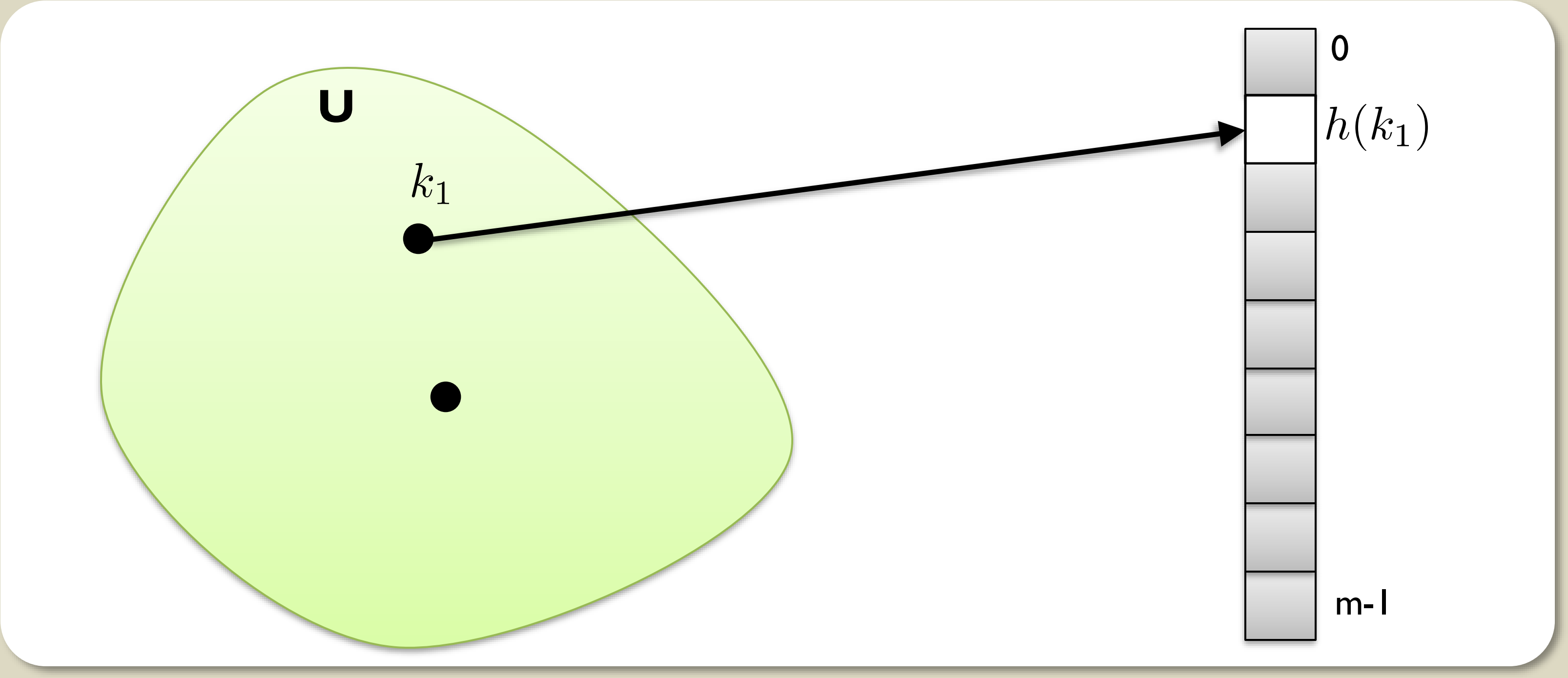

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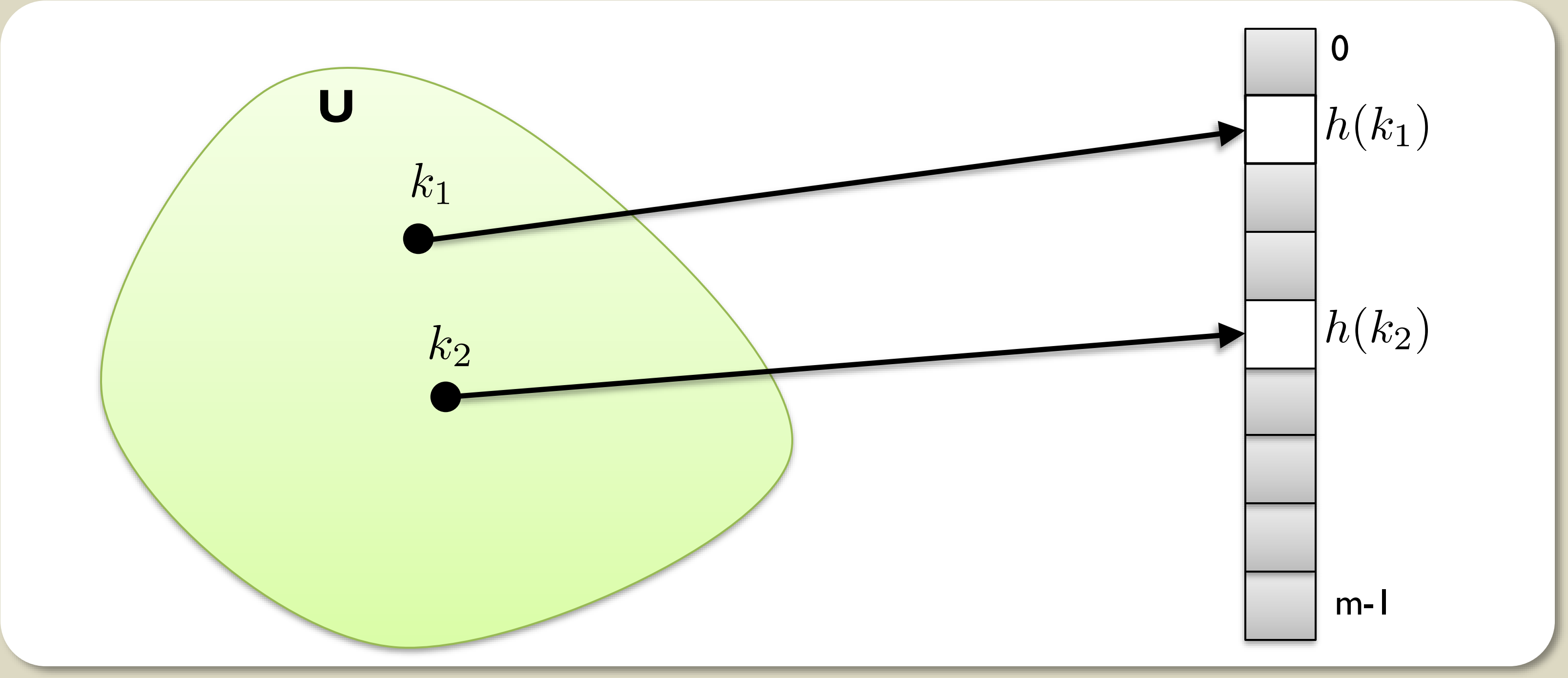

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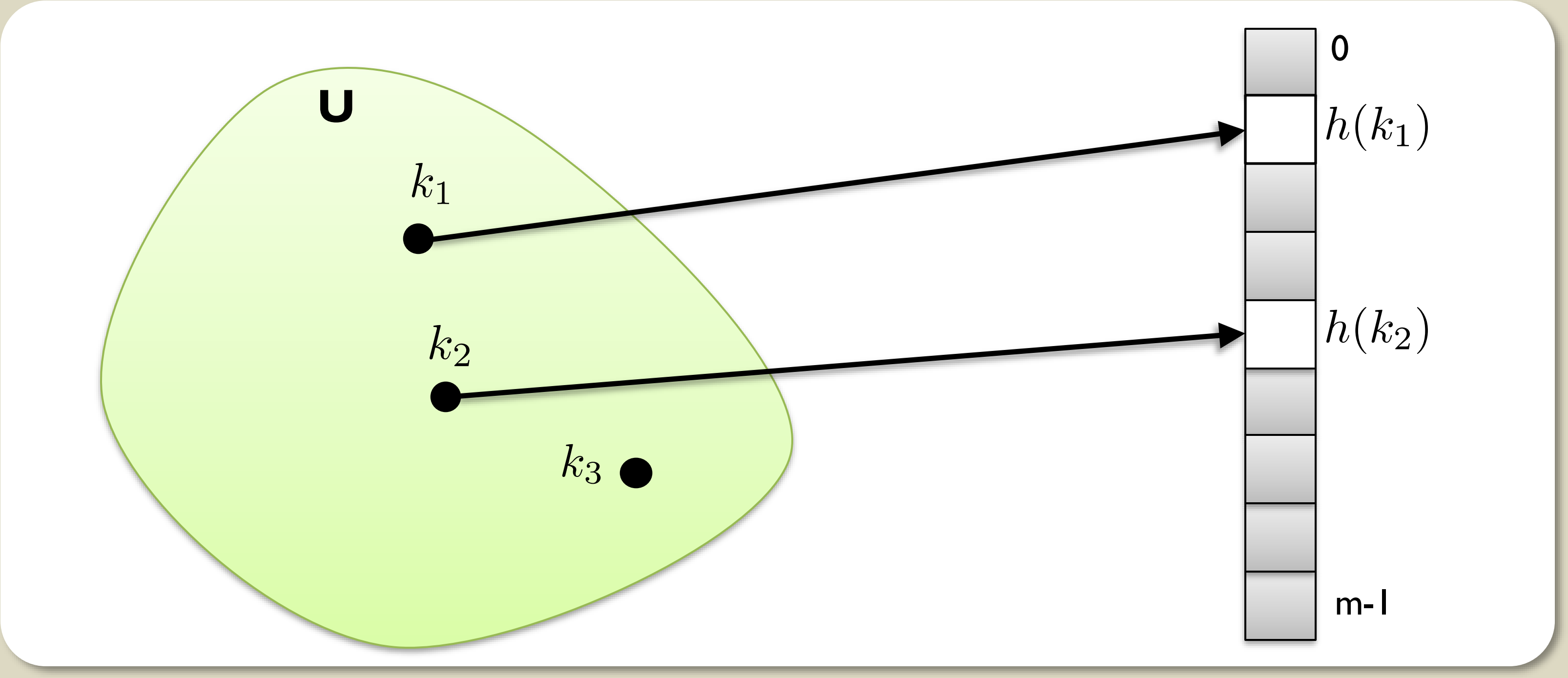

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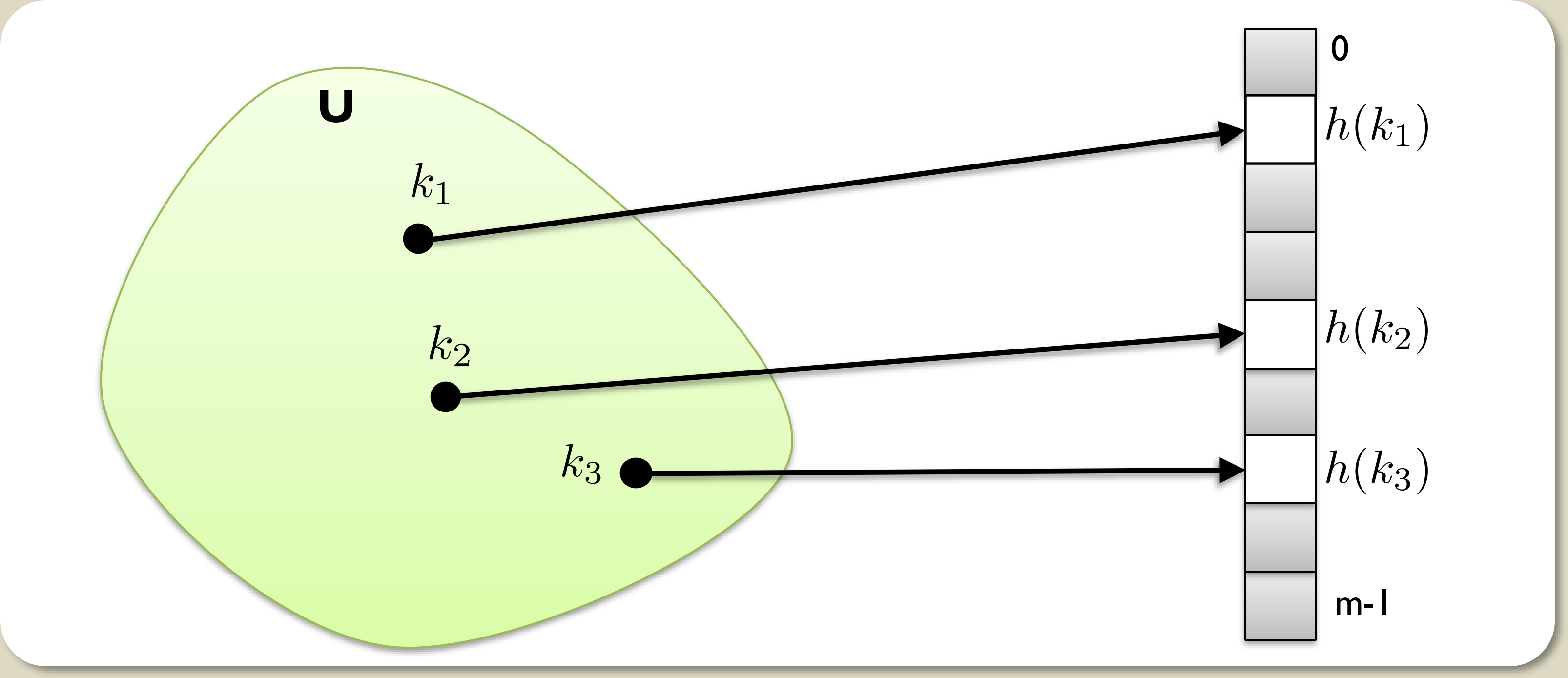

Desired Properties of a Hash Function

"to hash =*Informal* to make a mess of; mangle"

- Efficiently computable
- Distributes keys uniformly (to minimize collisions)
- Deterministic: **h(k)** is always equal to **h(k)**

Simple uniform hashing:

h hashes a new key equally likely to any of the **m** slots independently of where any other has hashed to

Collisions

- Collision: when two items with keys $\mathbf{k}_i$ and $\mathbf{k}_j$ have $h(\mathbf{k}_i) = h(\mathbf{k}_j)$

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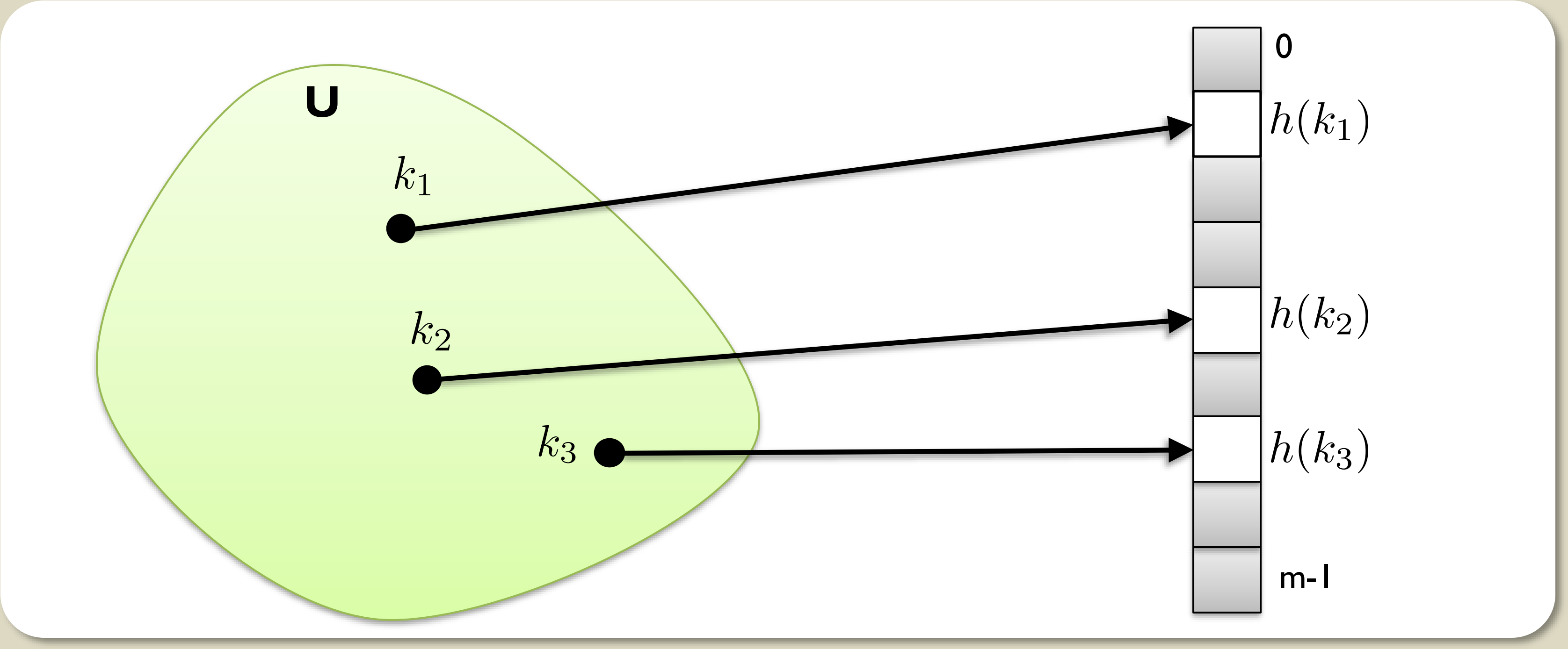

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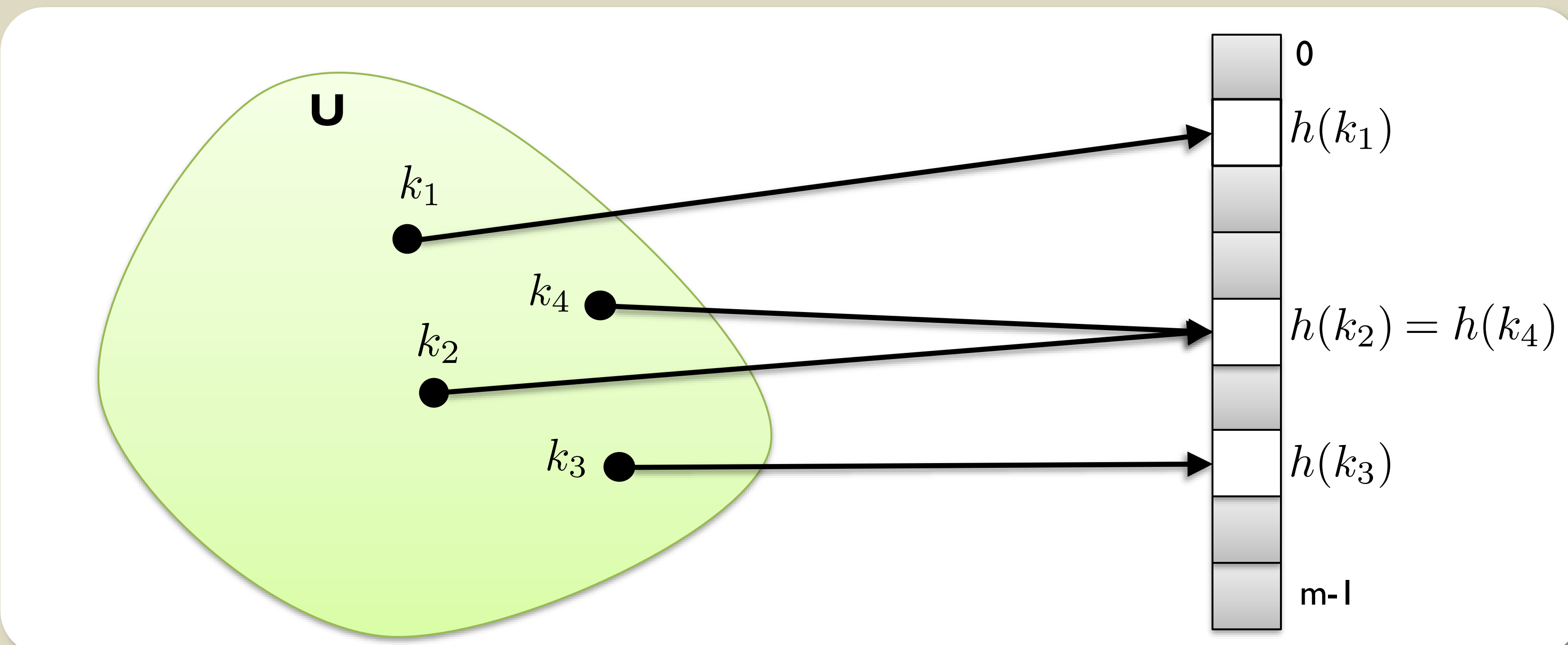

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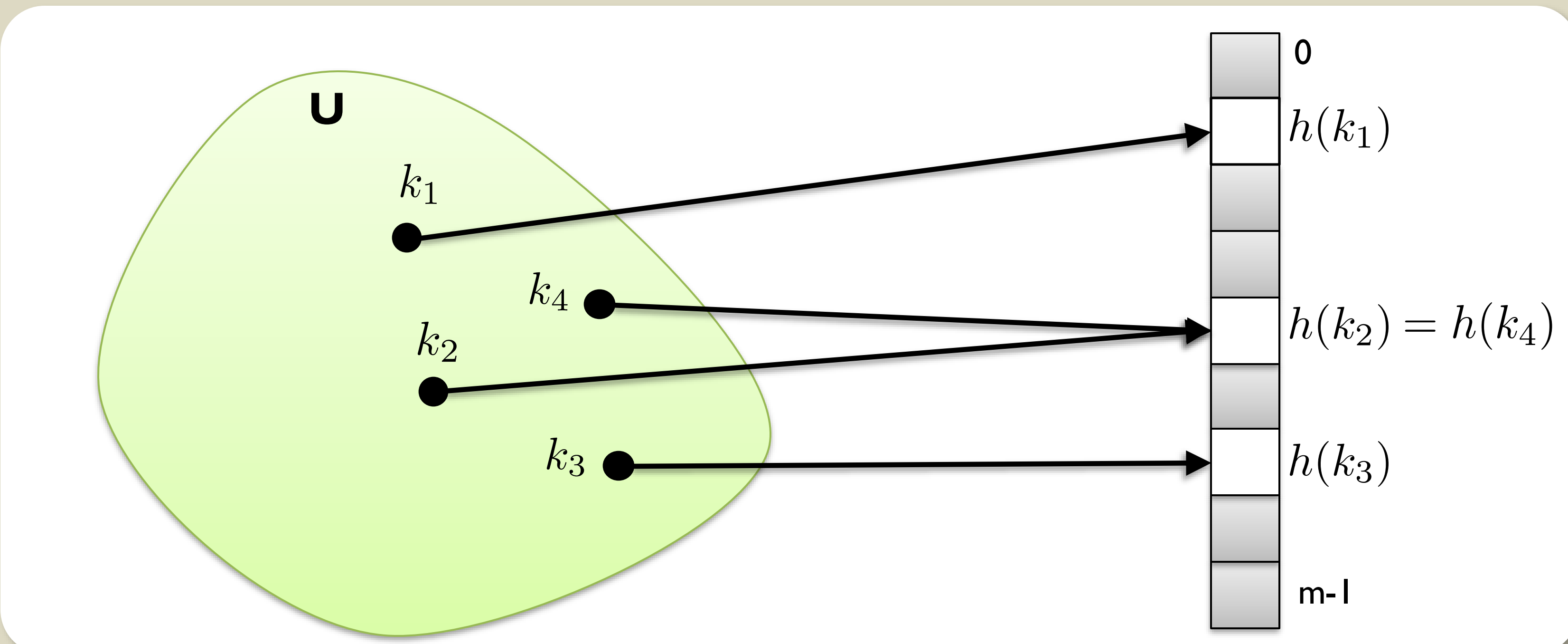

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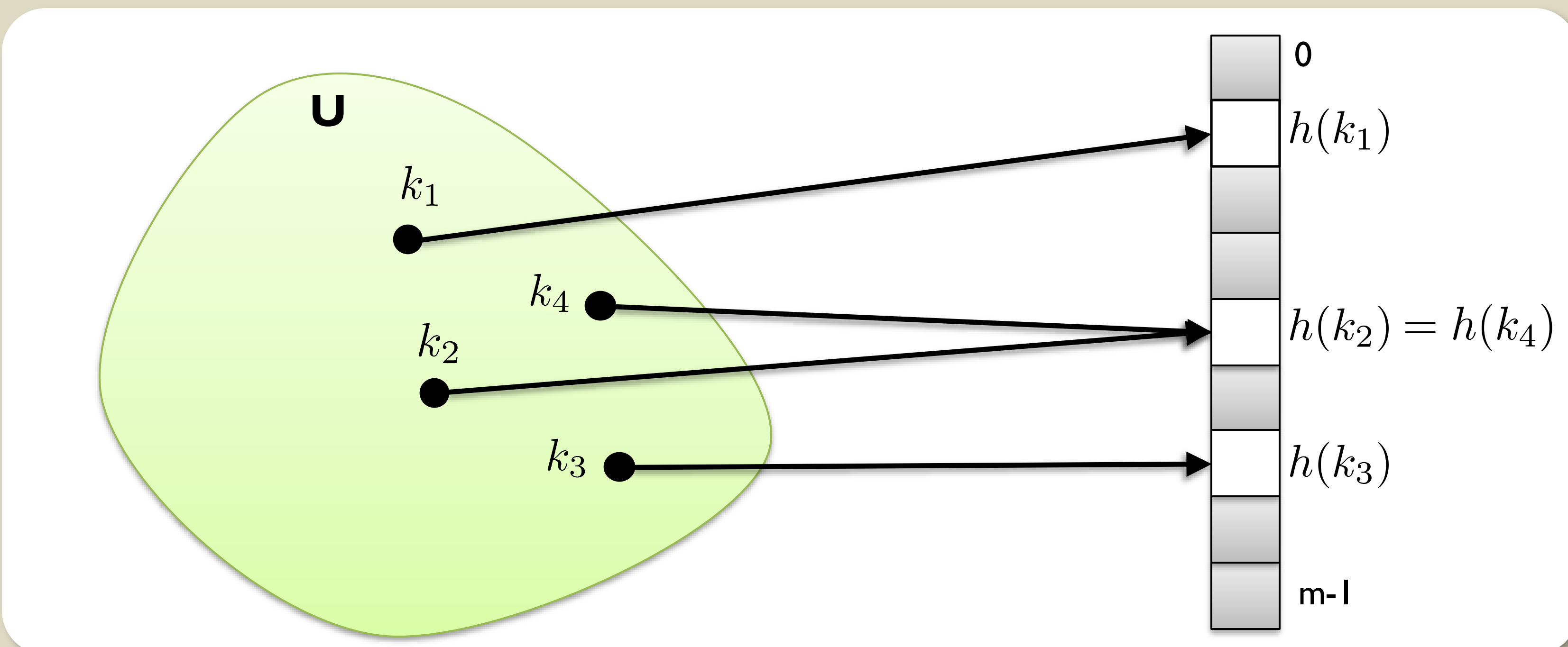

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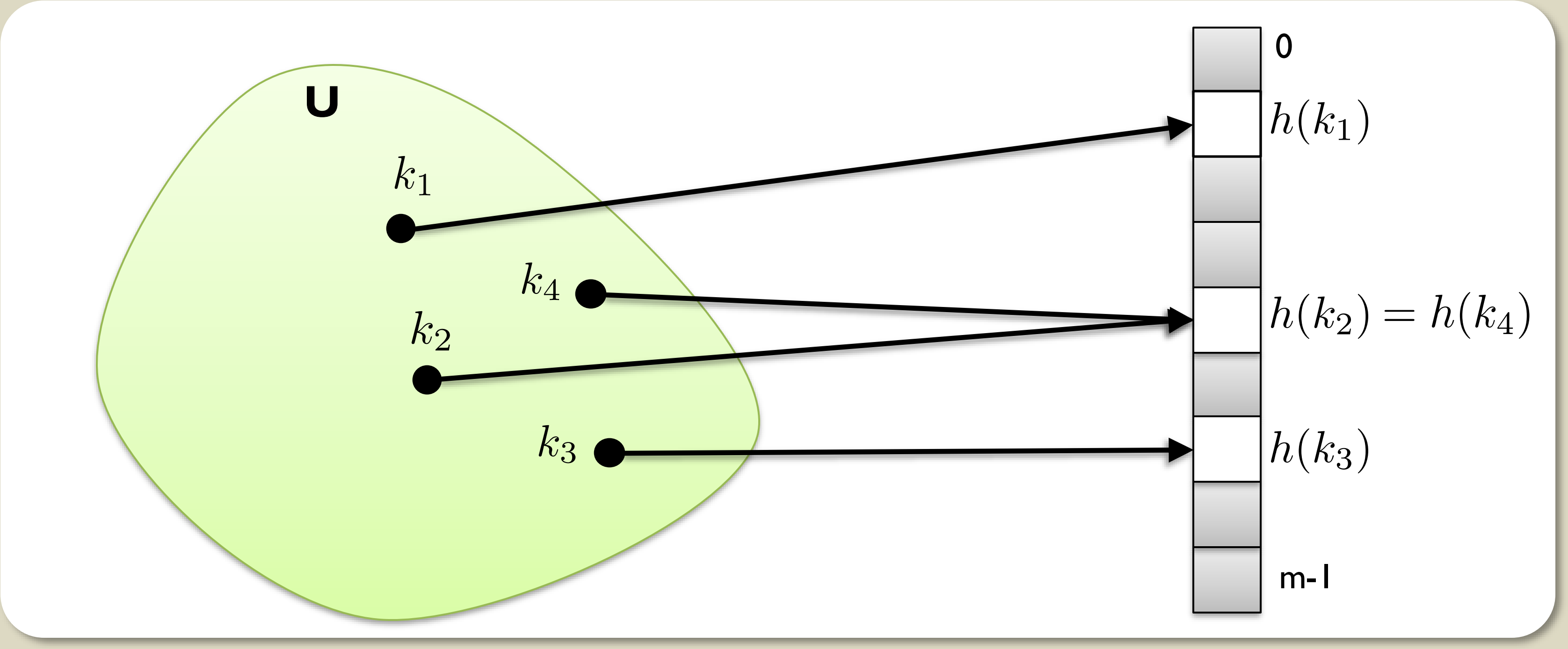

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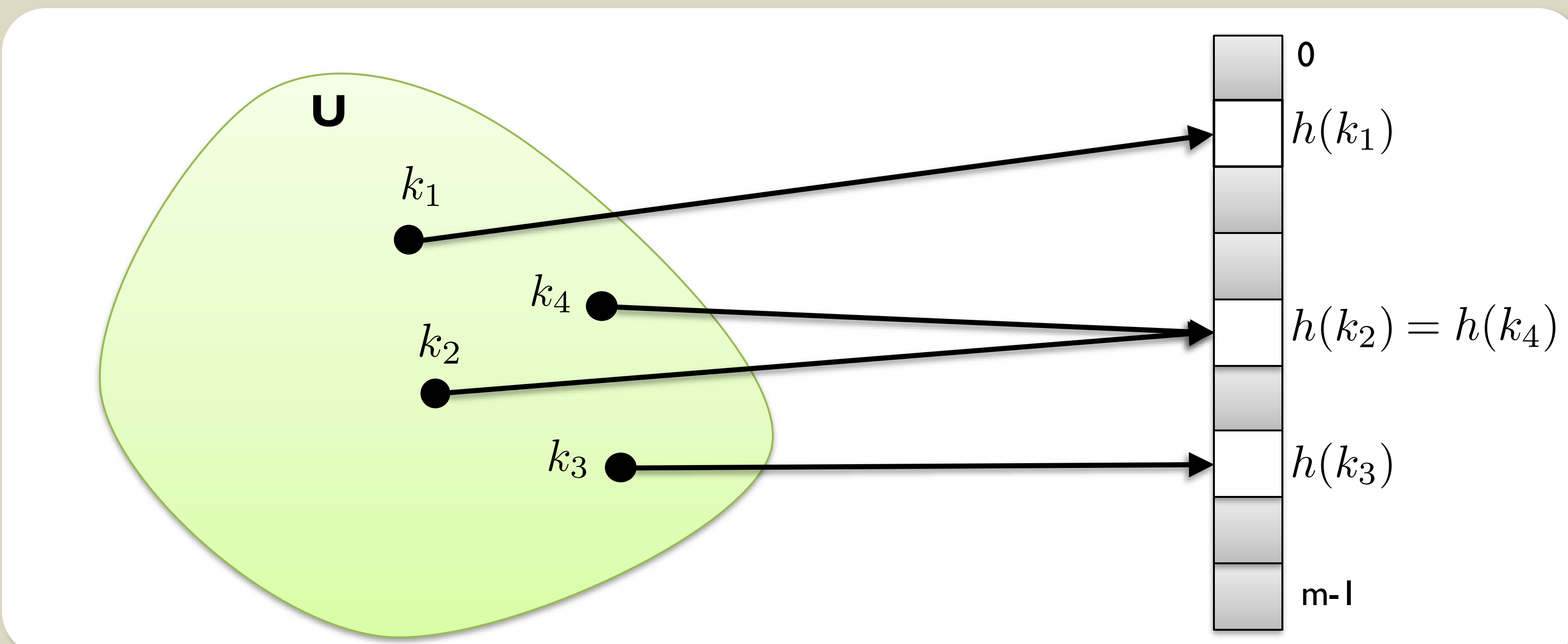

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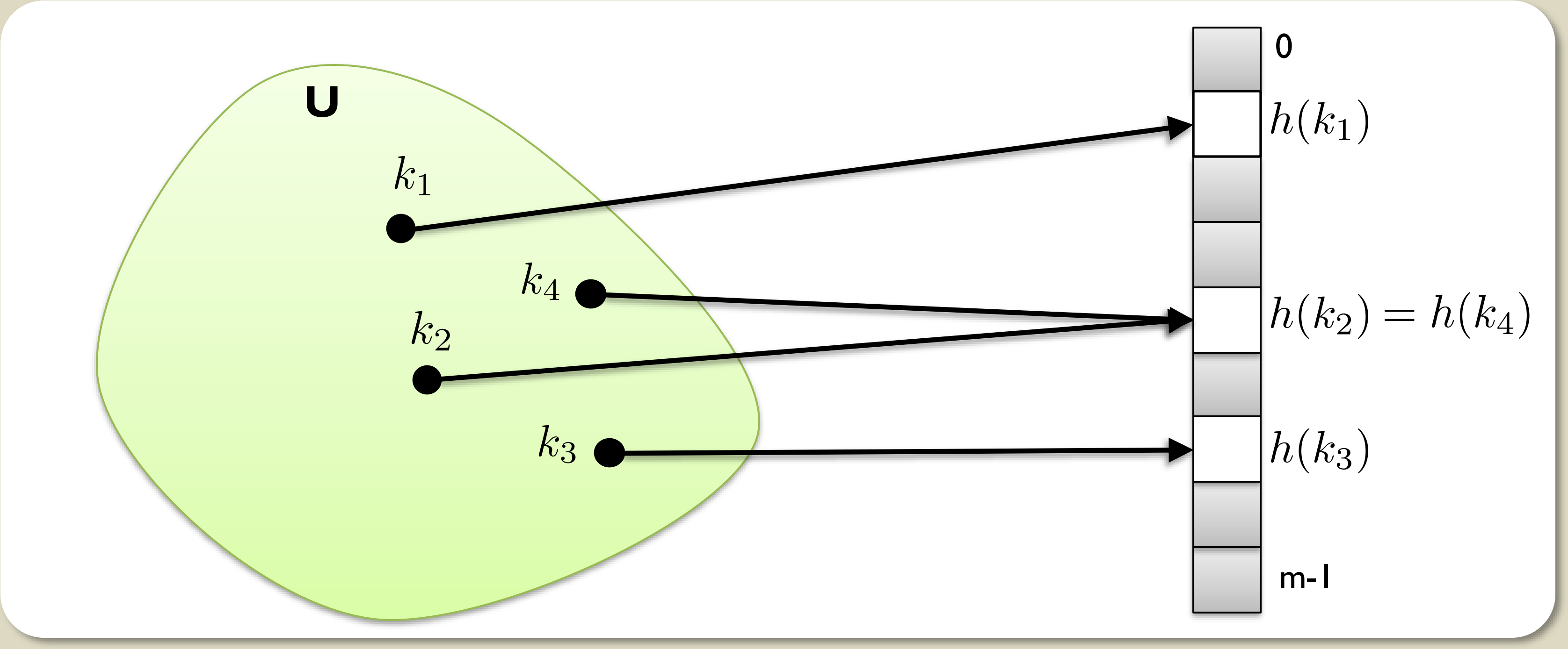

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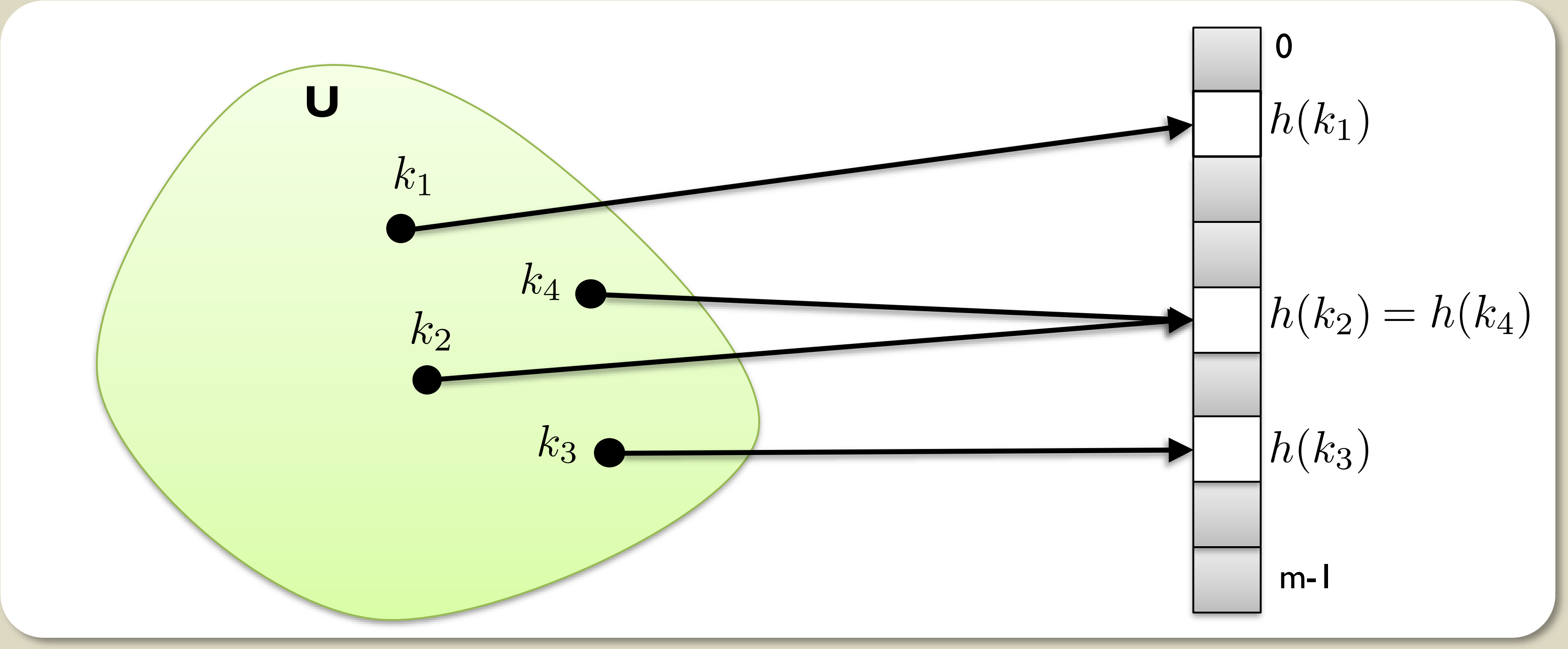
Birthday Lemma says that for **h** to be injective with good probability then we need $\mathbf{m > K^2}$ => if library has **10 000** books need array of size $\mathbf{10^8}$



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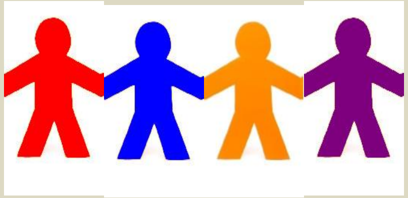

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CHAINING:

place all elements that hash to the same slot into the same linked list

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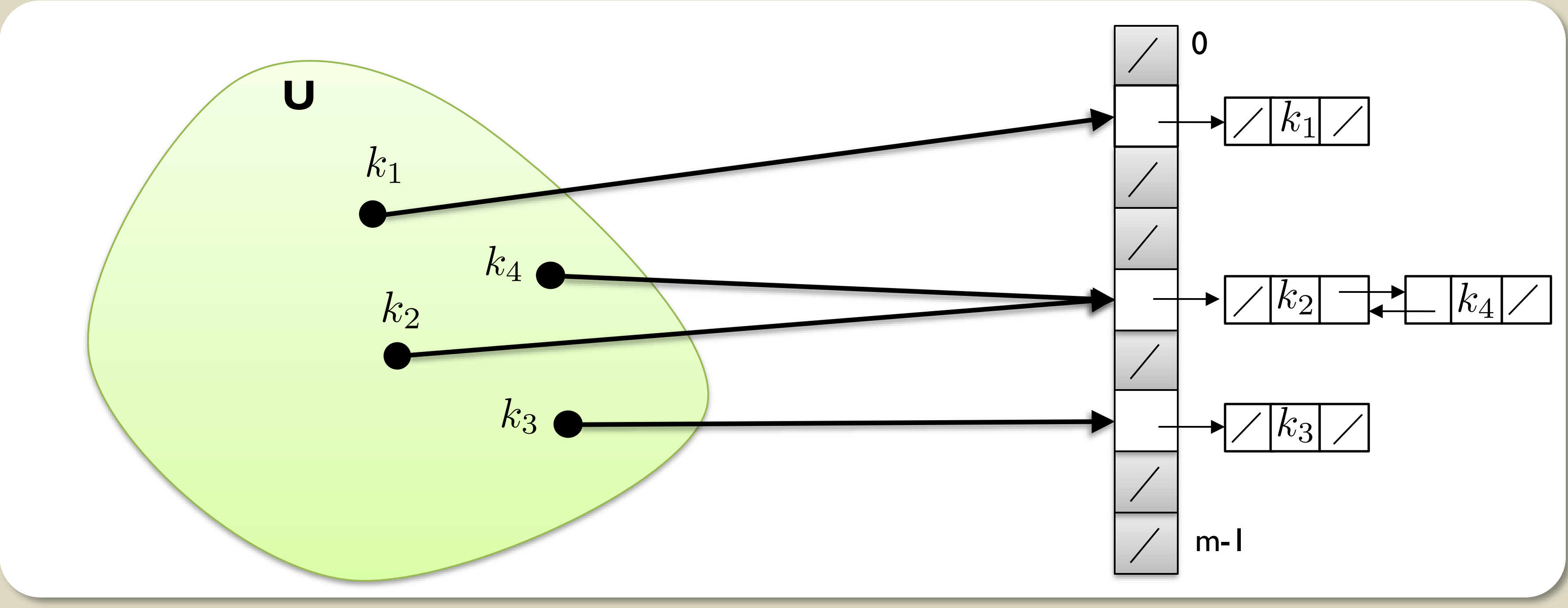

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search for an element with key k in list T[h(k)]

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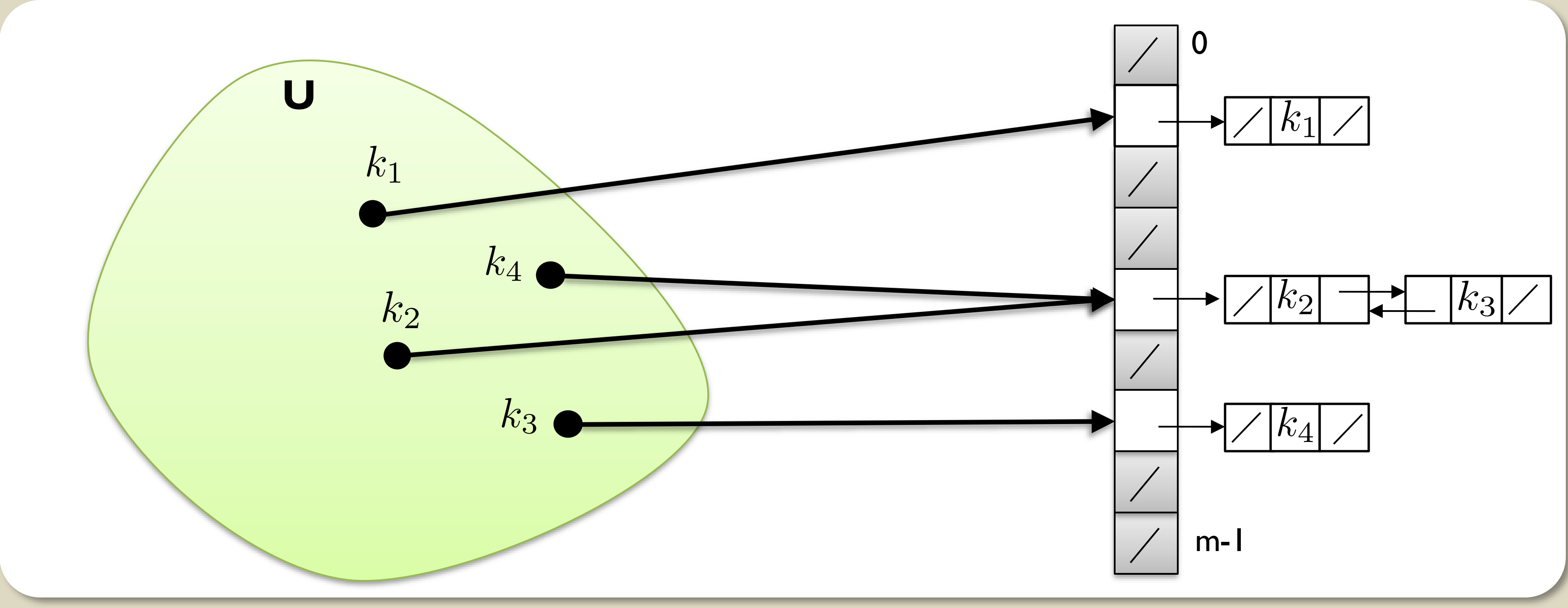

Collisions

Running time?

Space?

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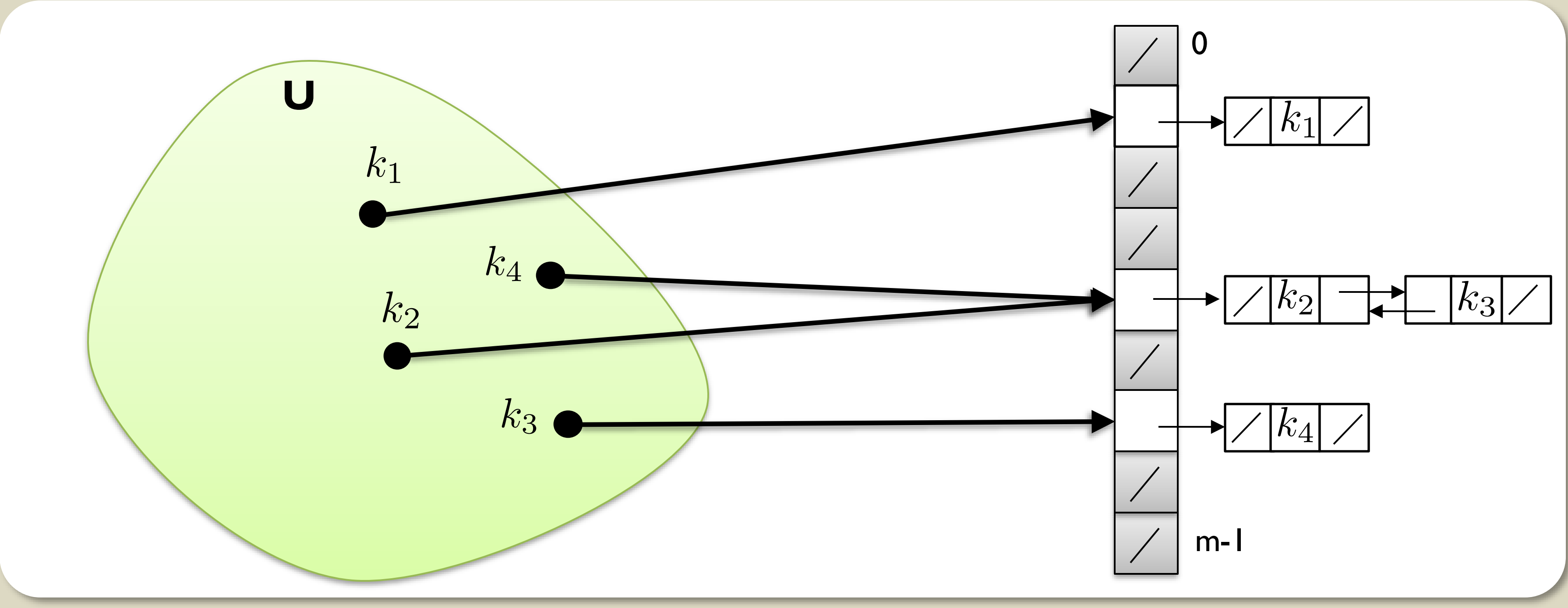

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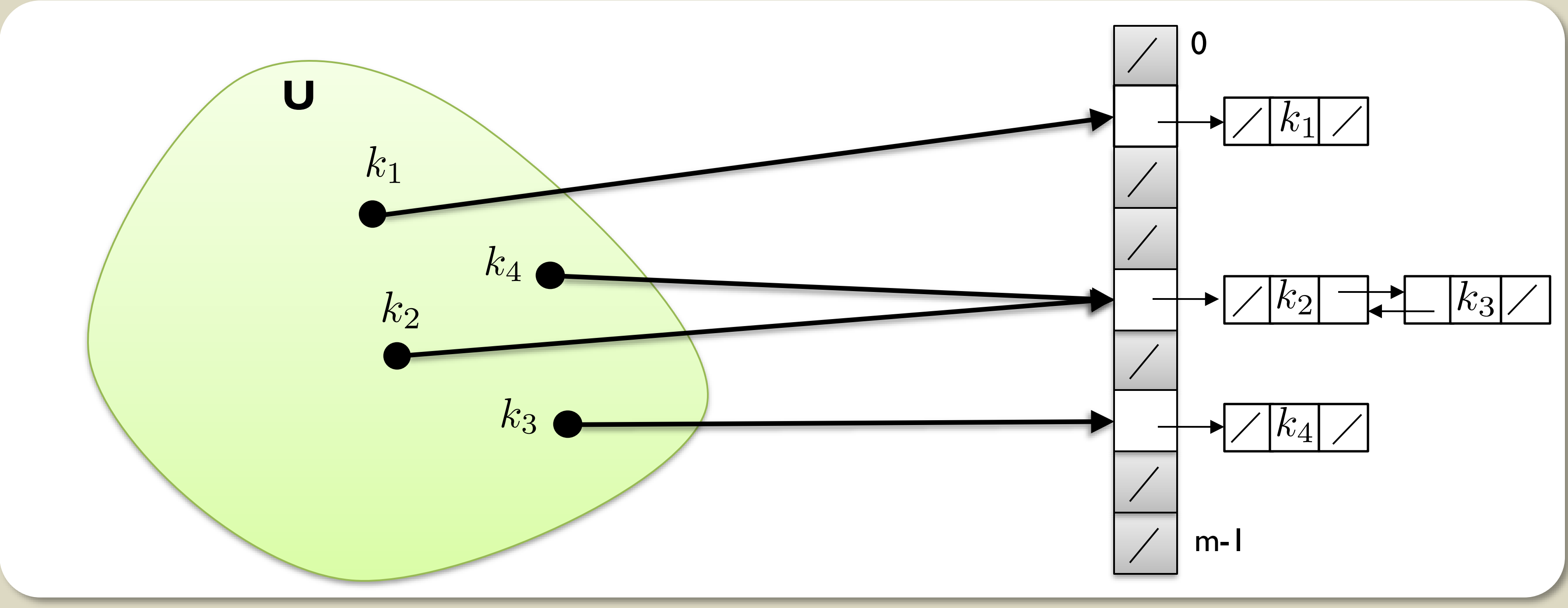

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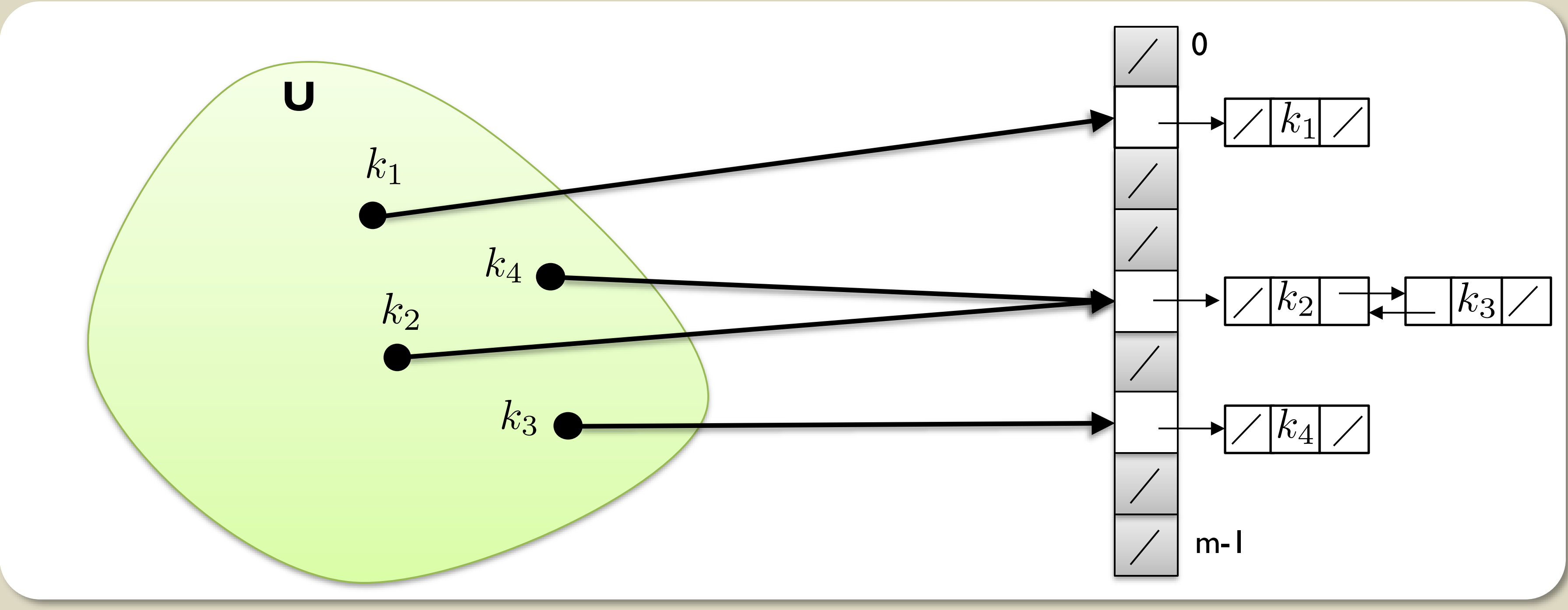
Deletion in time O(1) since

- list is doubly linked
- and we are given a pointer to element and not the key



Running Time of Search

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- Worst case all **n** elements are hashed to same slot
  - Search takes **θ(n)** time in worst case (worse than linked list since we also compute h(k) ☹)
- Analyze average-case behavior
  - We assume we use *simple uniform hashing*

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  - We assume we use *simple uniform hashing*

- Let $\mathbf{n_j}$ denote the length of the list **T[j]**

  - Note that $\mathbf{n = n_0 + n_1 + n_2 + \ldots + n_{m-1}}$

  - And $\mathbf{E[n_j] = Pr[h(k_1) = j] + Pr[h(k_2) = j] + \ldots\ Pr[h(k_n) = j] = \alpha = n/m}$

Running Time of Unsuccessful Search

THEOREM: An unsuccessful search takes expected time **$\theta(1+\alpha)$**

PROOF:

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- To search unsuccessfully for any key **k**, need to search till the end of list T[h(k)].
  - This list has expected length $E[n_{h(k)}] = \alpha$
  - Therefore, the expected number of elements examined in an unsuccessful search is α

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- Adding in the cost for computing the hash function, the total expected time needed is $\Theta(1+\alpha)$

Running Time of Successful Search

- Circumstances are slightly different from unsuccessful search
- The probability that each list is searched is proportional to its length
- (we assume that each element is equally likely to be searched for)

Running Time of Successful Search

THEOREM: A successful search takes expected time **$\theta(1+\alpha)$**

PROOF:

Running Time of Successful Search

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PROOF:

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PROOF:

- Let **x** be the element we search for (picked uniformly at random)
- The number of elements examined during a successful search for **x** is 1 more than the number of elements that appear before **x** in its list
  - These are the elements inserted after **x** was inserted

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  - These are the elements inserted after **x** was inserted
- So we need to find the average, over the **n** elements **x** in the table, of how many elements were inserted into **x**'s list after **x** was inserted

Running Time of Successful Search

THEOREM: A successful search takes expected time **$\theta(1+\alpha)$**

PROOF:

- For i = 1, 2, …, n, let **x_i** be the i'th element inserted into the table and let **k_i = key[x_i]**

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- For all **i** and **j**, define indicator variable
  - $X_{ij} = I\{h(k_i) = h(k_j)\}$
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- Expected number of elements examined in a successful search is

$$E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right]$$

Running Time of Successful Search

THEOREM: A successful search takes expected time **$\theta(1+\alpha)$**

PROOF:

- For i = 1, 2, …, n, let $\mathbf{x_i}$ be the i'th element inserted into the table and let $\mathbf{k_i = key[x_i]}$
- For all **i** and **j**, define indicator variable
  - $\mathbf{X_{ij} = I\{h(k_i) = h(k_j)\}}$
  - Simple uniform hashing => $\mathbf{E[X_{ij}] = 1/m}$
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$$E\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}E[X_{ij}]\right)$$

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$$= \cdots = 1 + \frac{\alpha}{2} - \frac{\alpha}{2n}$$

Consequence of Analysis

- Recall that **$\alpha = n/m$**

- So if we choose the size our hash table to be proportional to the number of elements stored.
  - **$m = \Theta(n)$**

- Then insertion, deletion **$O(1)$** time and search expected **$O(1)$** time

Examples of Hash Functions

- Big area of research
- Depends on data distribution and other properties
- We give two basic examples

Examples of Hash Functions

Division method

$h(k) = k \bmod m$

m often selected to be a prime not too close to a power of 2

Examples of Hash Functions

Division method

h(k) = *k* mod *m*

m often selected to be a prime not too close to a power of 2

Multiplicative method

h(k) = floor[m * FractionalPartOf(Ak)]

Knut suggests to chose $A \approx (\sqrt{5} - 1)/2$

Summary

- Probabilistic analysis (the hiring problem)
  - Random indicator variables
  - Linearity of expectation:

    E[aX + bY] = aE[X] + bE[Y]

    *holds even if **X** and **Y** are dependent*

- Hash tables
  - Very practical method with fast insertion, deletion, and search
  - Performance depends on choice of hash function
  - Resolve conflicts by for example using chaining