Algorithms

May 7, 2025

PROBABILISTIC ANALYSIS AND RANDOMIZED ALGORITHMS

Motivation

- Worst case does not usually happen
  - Average case analysis
  - Amortized analysis

- Randomization helps avoid worst-case and attacks by evil users
  - Choosing the pivot in quick-sort at random
- Randomization necessary in cryptography
- Can we get randomness?
  - How to extract randomness (extractors)
  - Longer "random behaving" strings from small seed (pseudorandom generators)

Probabilistic Analysis: The Hiring Problem

NY Knicks are going to hire one new basketball player

- the taller the better

They have **n** candidates that they call for interview

Strategy: each candidate is hired that is taller than the current best/tallest

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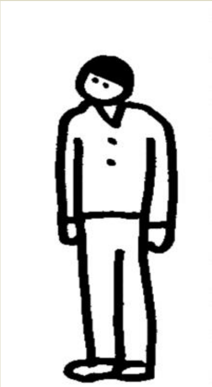

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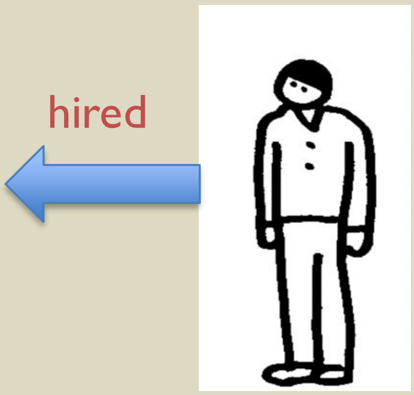

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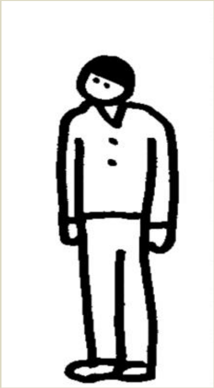

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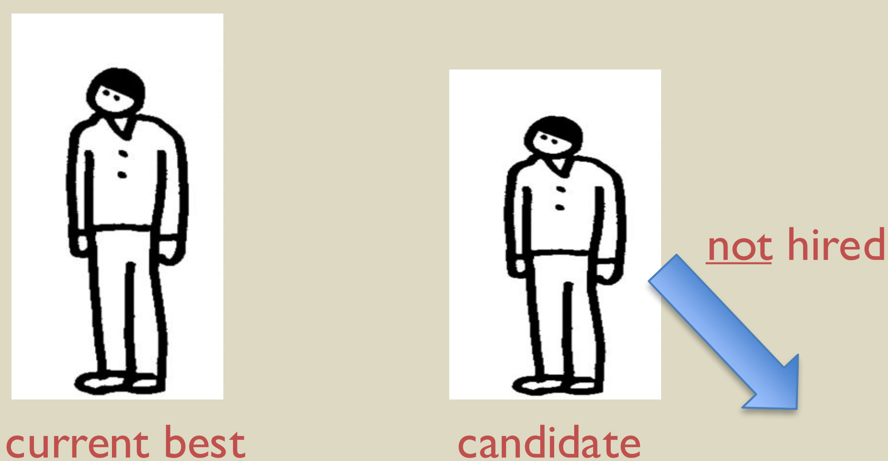

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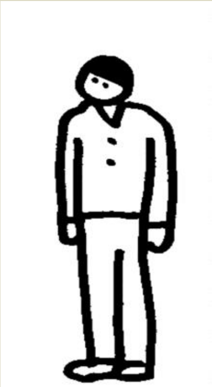

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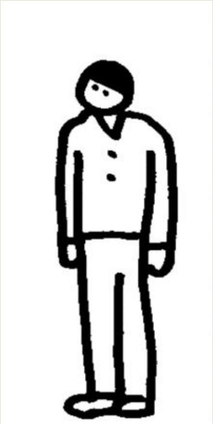

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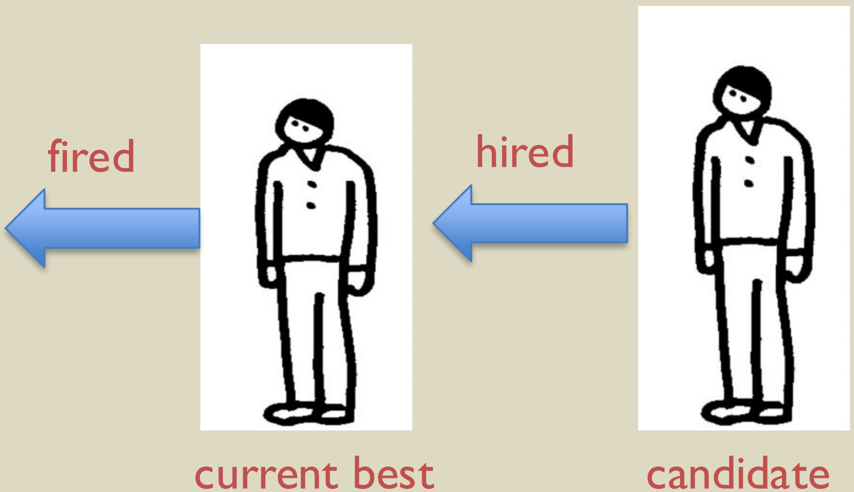

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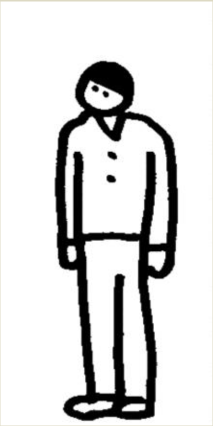

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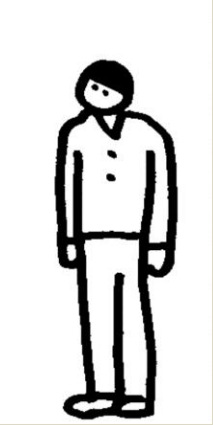

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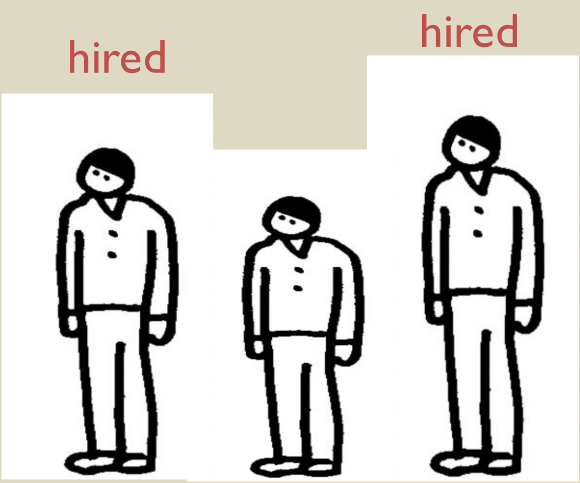

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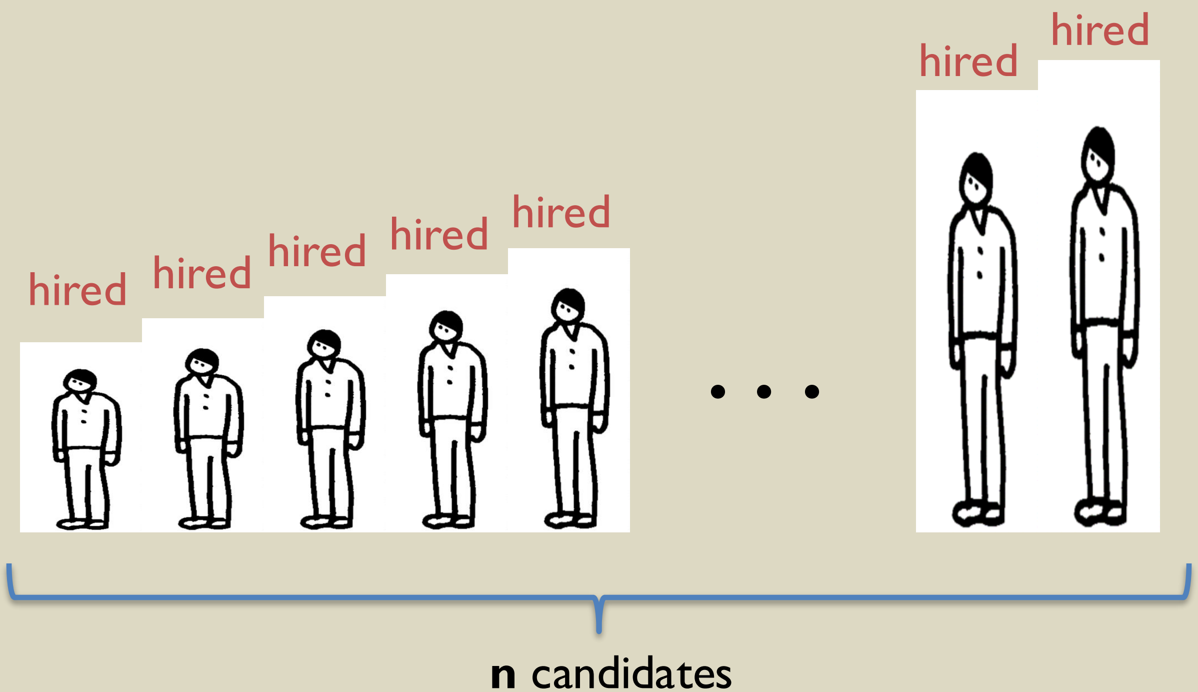


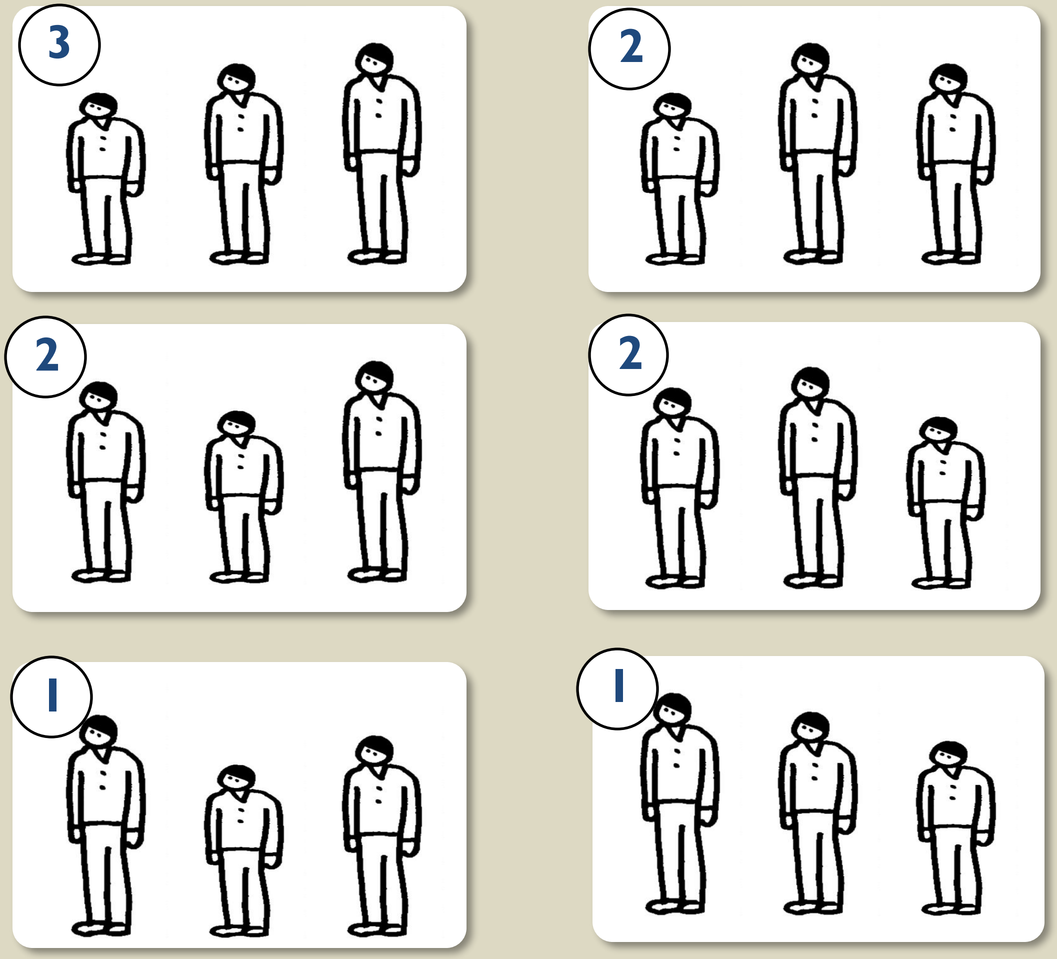
Answer: in the worst case we hire all **n** candidates

Worst-case unlikely to happen

- We only hire all candidates if they arrive in a specific order
- They are likely to arrive in a random order
- More interesting question (probabilistic analysis):

What is the expected number of hires we make over all the permutations of the candidates?

Example



Expected number of hires=

$$\frac{3+2+2+2+1+1}{6}$$

which equals 1+5/6

Calculating the expectation in general 1st trial

- **n!** permutations each equally likely
- Expectation = sum of hires in each permutation divided by **n!**

$$\frac{A_1 + A_2 + \cdots + A_{n!}}{n!}$$

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NEED A MORE CLEVER METHOD

Indicator Random Variables

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DEFINITION: Given a sample space and an event **A**, we define the **indicator random variable**

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PROOF: $\mathbf{E[X_A] = 1 * Pr\{A\} + 0 * Pr\{\overline{A}\} = Pr\{A\}}$

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- Since **Pr{H} = 1/2**, previous lemma says that $\mathbf{E[X_H] = 1/2}$

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- Could calculate

$$E[X] = \sum_{k=0}^{n} k \cdot \Pr\{X = k\}$$

- … but cumbersome

- Instead use indicator variables

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- By linearity of expectation i.e., that **E[aX + bY] = aE[X] + bE[Y]**

  holds even if ***X*** *and* ***Y*** *are dependent*

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$$E[X] = E[X_1 + X_2 + \ldots + X_n] = \underbrace{E[X_1]}_{\text{By Lemma equals } Pr\{H\} = 1/2} + E[X_2] + \ldots + E[X_n]$$

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$$= n/2$$

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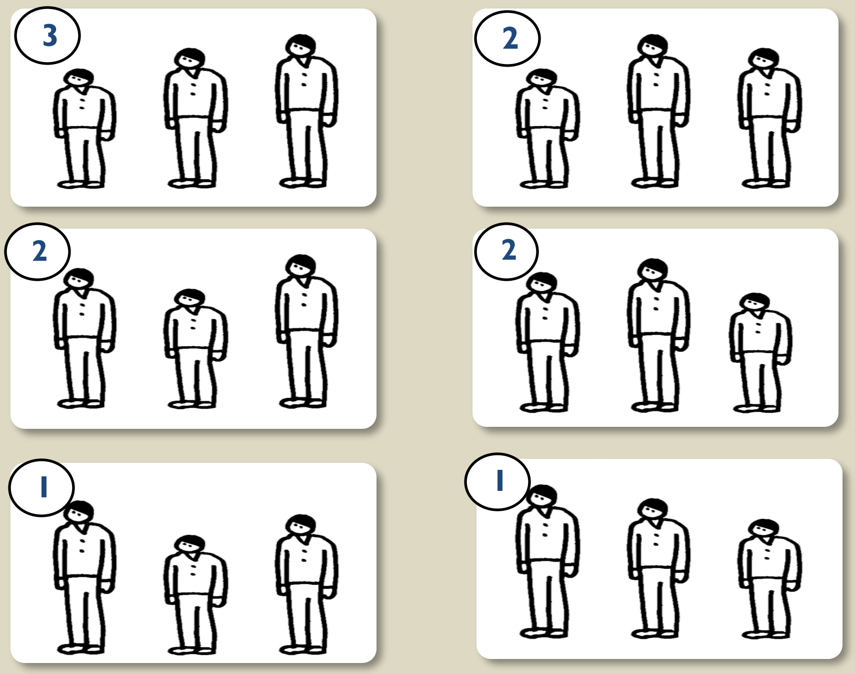

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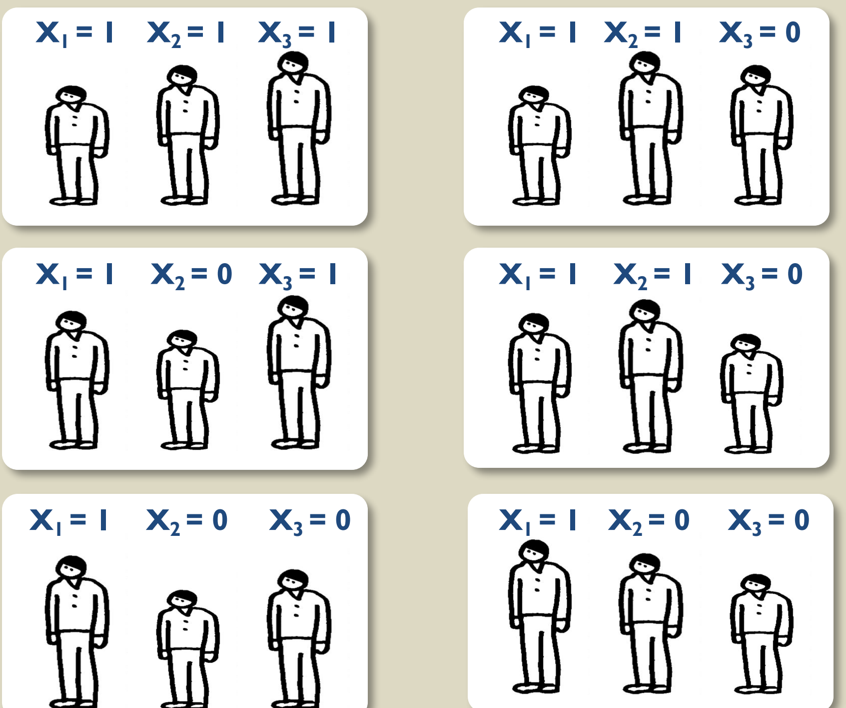

$$\mathbf{E[X] = E[X_1+X_2+X_3]}$$

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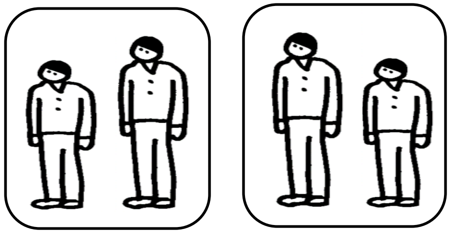

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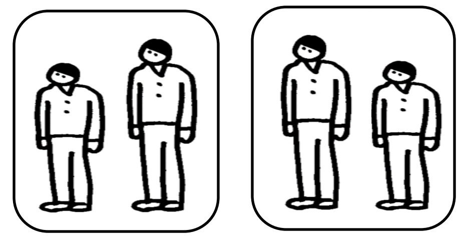
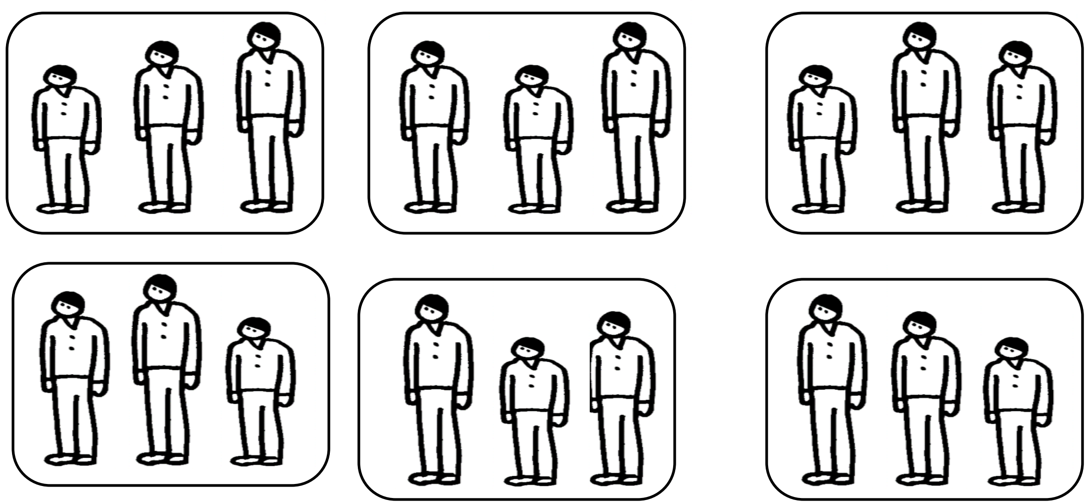


Pr{candidate **3** is hired**} = 1/3**



Probability of Hiring i'th Candidate

- i'th candidate hired iff he is tallest among the first i candidates
- Since they arrive in random order, any one of these first i candidates are equally likely to be the tallest =>

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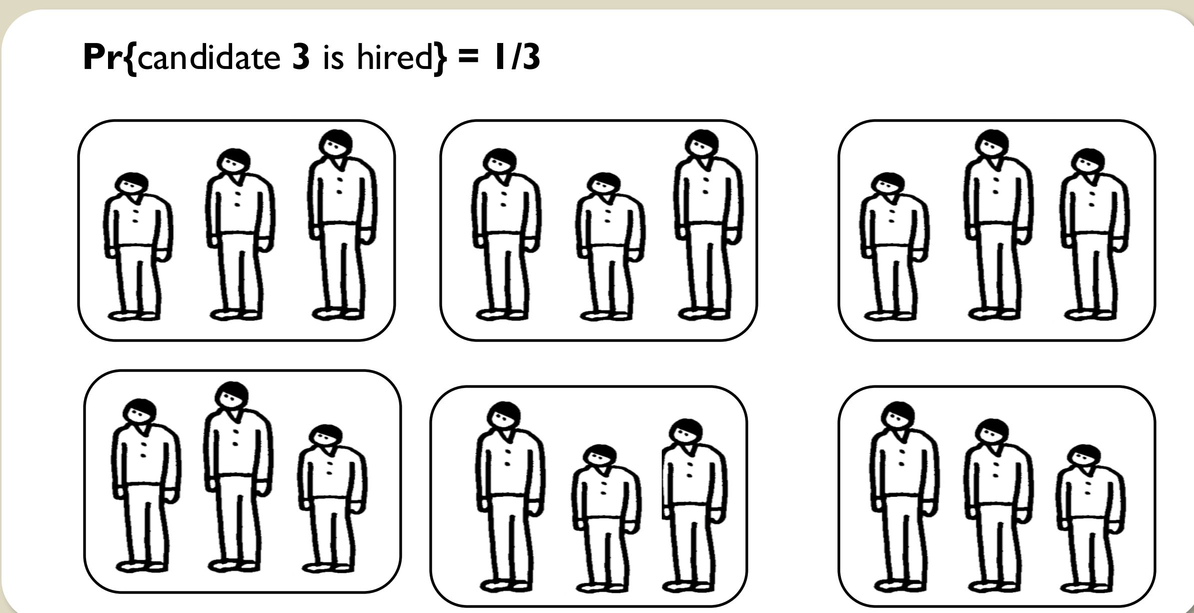

Expected Number of Hires

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Examples:

- Expected number of hires for **n=6** is **2.45**
- Expected number of hires for **n=100** is **5.1874**
- Expected number of hires for **n=10000** is **9.7876**

Questions

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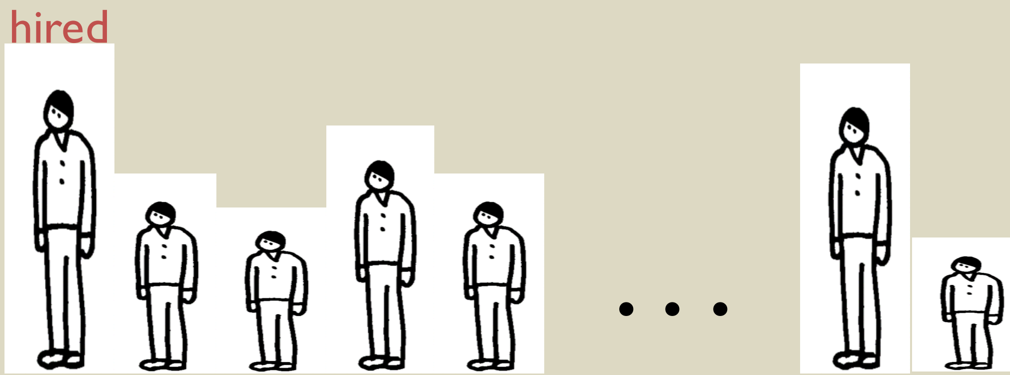

- What is the probability that we hire **n** candidates? **1/n!** (worst case order)

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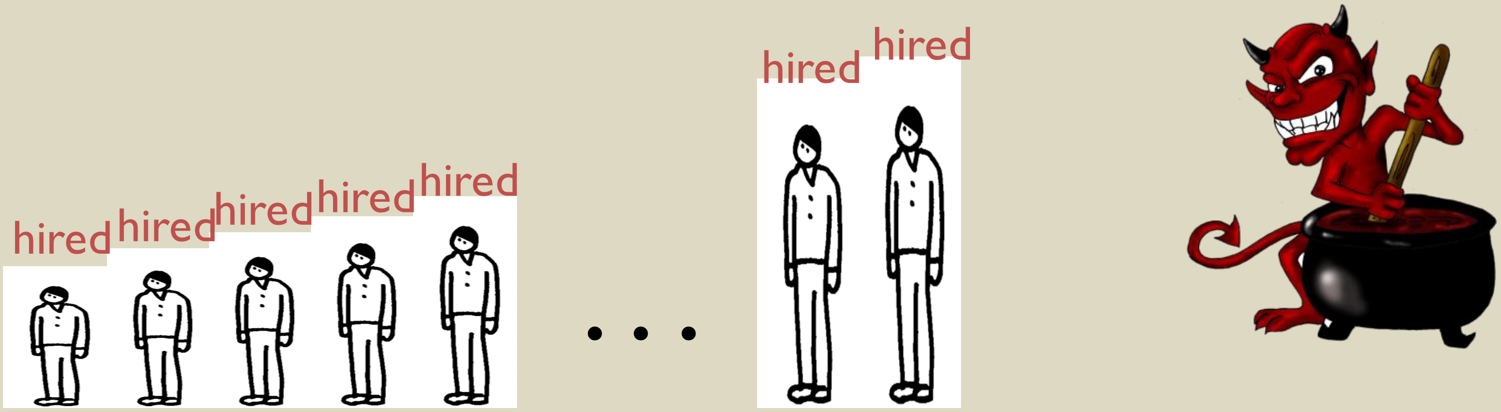
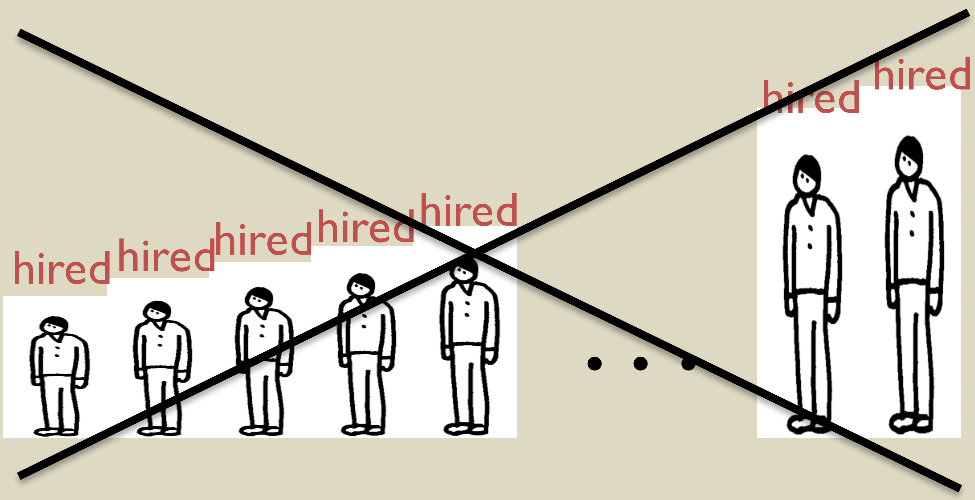


Randomized Algorithm

- Instead of assuming that the candidates arrive in random order
- **We/the algorithm** pick a random order and call the candidates in this order
- **In this way we can foul malicious users**



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- How to use **RANDOM** for generating an unbiased bit?

- Pick a pair (a,b) of random numbers: a = **RANDOM** and b = **RANDOM**
  - If $a \neq b$ return a
  - Otherwise pick a new pair