Algorithms: SHORTEST PATHS AND PROBLEM SOLVING

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EPFL School of Computer and Communication Sciences

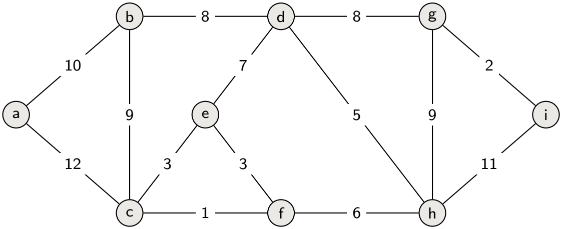
Lecture 21, 6.05.2025

Kruskal's algorithm

Start from empty forest T

Greedily maintain forest T which will become MST at the end:

at each step add cheapest edge that does not create a cycle



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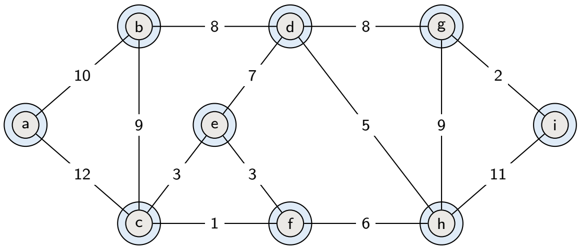

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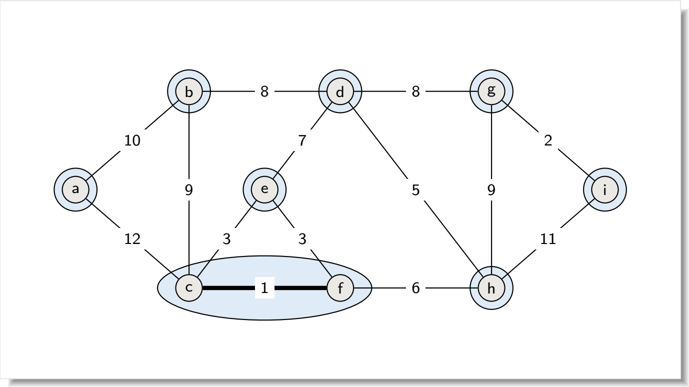

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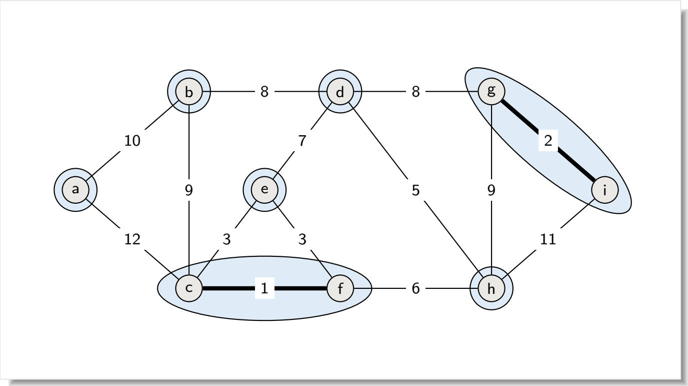

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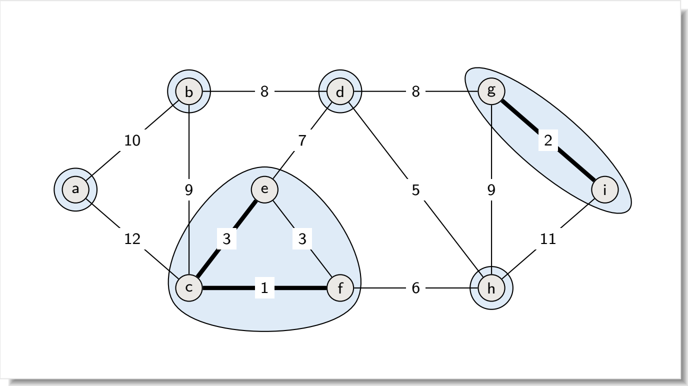

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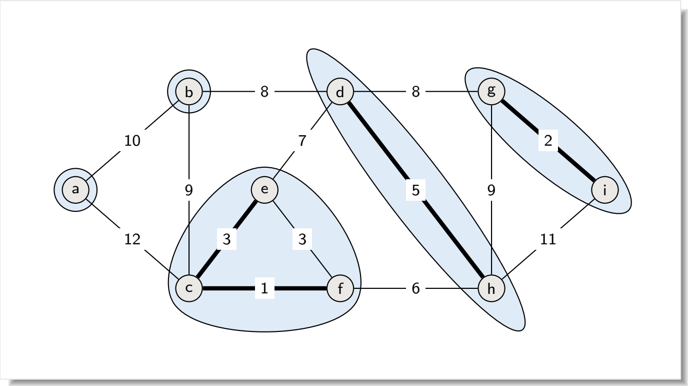

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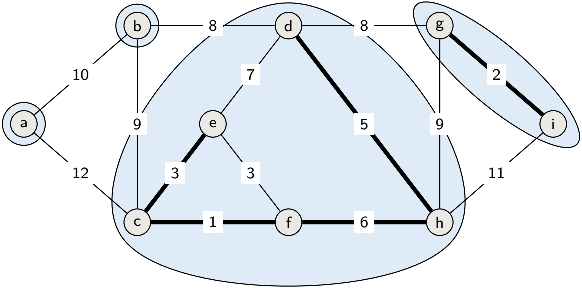

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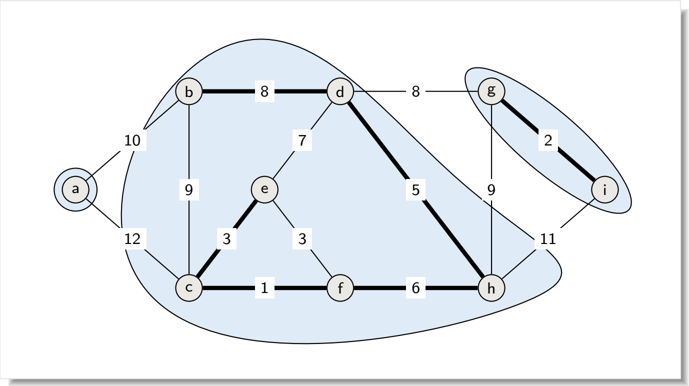

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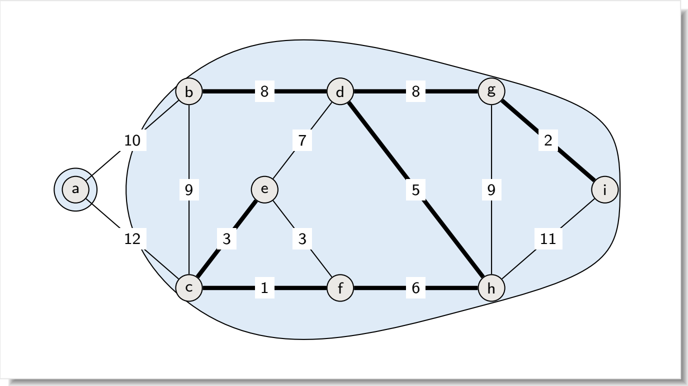

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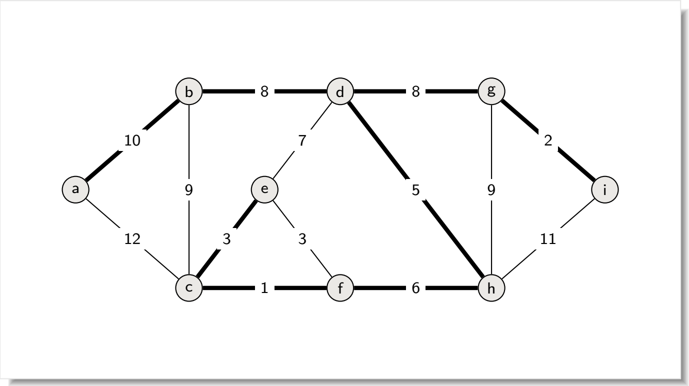

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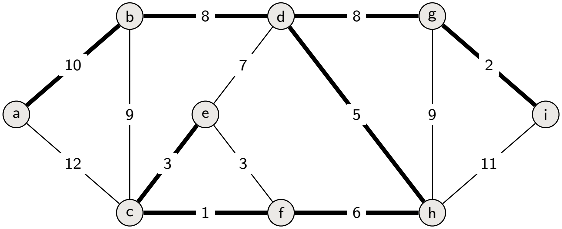

Minimum spanning tree of weight $1 + 2 + 3 + 5 + 6 + 8 + 8 + 10 = 43$

Why does it work?

Claim: T is always a sub-forest of a MST

Proof by induction on the number of components/edges in T

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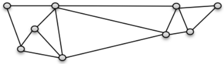
Base case: trivial

T is a union of singleton vertices



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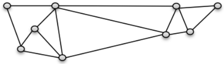
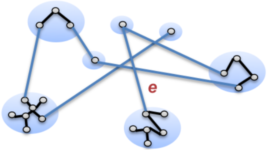


Inductive step:

1. By hypothesis, current ***T*** is a sub-forest of a MST,



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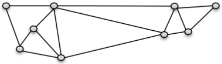

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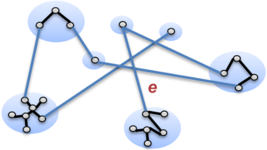
2. Edge ***e*** is an edge of minimum weight that doesn't create a cycle



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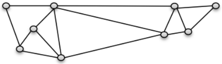

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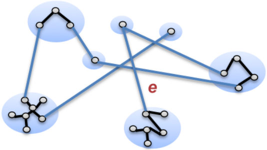

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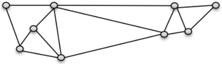

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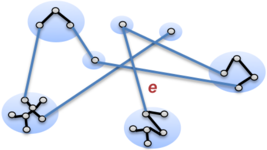
2. Edge ***e*** is an edge of minimum weight that doesn't create a cycle
3. *Suppose* ***e*** creates a cycle with MST
4. Replace an edge (with larger weight) along this cycle by ***e***



An MST since weight did not increase!

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- When edge (u, v) is added to T, make union of the two connected components/sets

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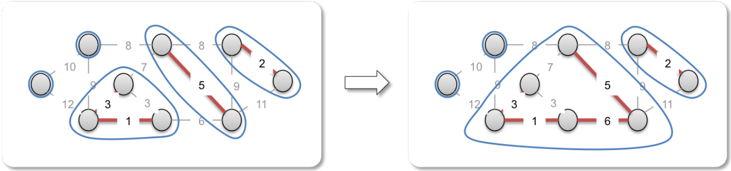

Implementation and Analysis

```
KRUSKAL(G, w)
  A = ∅
  for each vertex v ∈ G.V
      MAKE-SET(v)
  sort the edges of G.E into nondecreasing order by weight w
  for each (u, v) taken from the sorted list
      if FIND-SET(u) ≠ FIND-SET(v)
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- Second **for** loop:

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- Initialize A: $O(1)$
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  If edges already sorted time is $O(E\alpha(V))$ which is almost linear

Problem solving

A feedback edge set of a graph G is a subset F of the edges such that every cycle in G contains at least one edge in F. In other words, removing every edge in F makes the graph G acyclic. Describe and analyze a fast algorithm to compute the minimum weight feedback edge set of a given edge-weighted graph.

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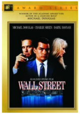
1. Prove that for any partition of the vertices V into two subsets, the minimum-weight edge with one endpoint in each subset is in the minimum spanning tree of G.
2. Prove that the maximum-weight edge in any cycle of G is not in the minimum spanning tree of G.

Summary

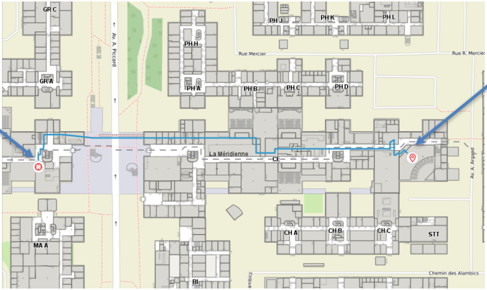
- Greedy is good (sometimes)
- Prim's algorithm

  Min-priority queue for implementation
- Kruskal's algortihm

  Union-Find for implementation
- Many applications



Satellite

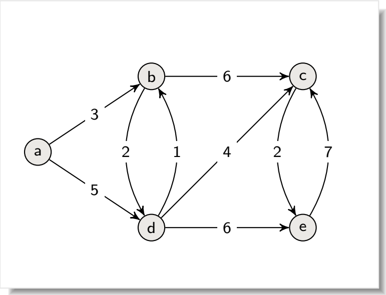


this class room

THE SHORTEST PATH PROBLEM

Shortest paths

Input: directed graph $G = (V, E)$, edge-weights $w(u, v)$ for $(u, v) \in E$



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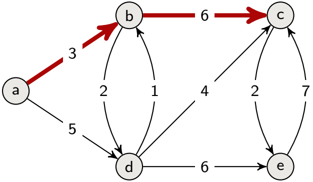
Weight of path $\langle v_0, v_1, \ldots, v_k \rangle$: $\sum_{i=1}^{k} w(v_{i-1}, v_i)$

Path of weight $3 + 6 = 9$



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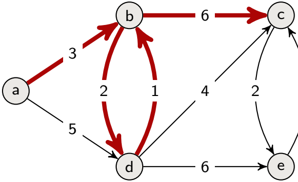
Path of weight $3 + 2 + 1 + 6 = 12$



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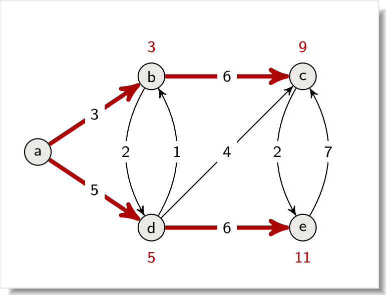

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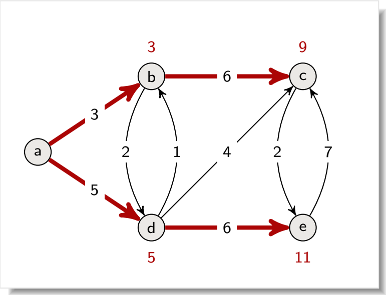

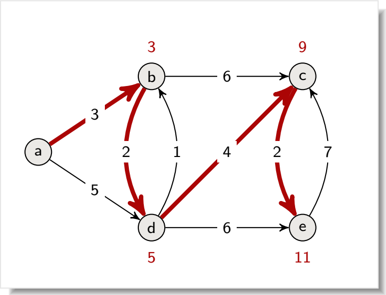

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Single-source: Find shortest paths from **source** vertex to every vertex

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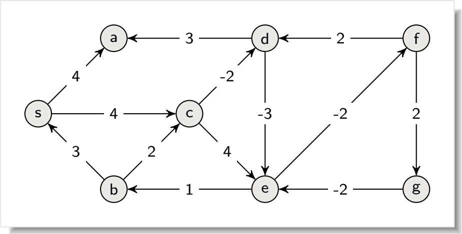
All-pairs: Find shortest path from u to v for all pairs u, v of vertices

Can be solved by solving single-source for each vertex. Better algorithms known

NEGATIVE WEIGHTS AND APPLICATIONS

Negative-weight edges

We will allow **negative weights**



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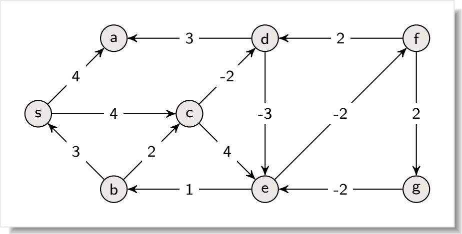
- Then we can just keep going around it to have paths of length $-\infty$



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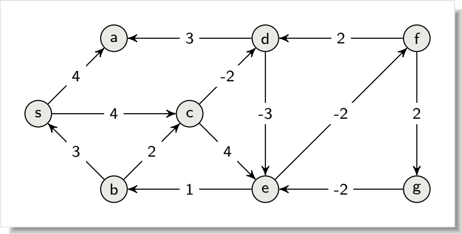
Some algorithms only work with positive weights (Dijkstra's algorithm)



Why negative weights?

Example: Buying and selling currency

Input: table of exchange rates

	Euro	Dollar	RMB	CHF
Euro	1.0	1.3	8	1.2
Dollar	0.77	1.0	6.2	0.93
RMB	0.14	0.16	1.0	0.15
CHF	0.83	1.07	6.67	1.0

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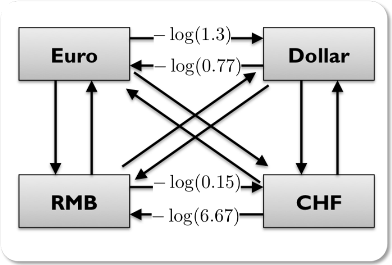
Input: table of exchange rates

	Euro	Dollar	RMB	CHF
Euro	1.0	1.3	8	1.2
Dollar	0.77	1.0	6.2	0.93
RMB	0.14	0.16	1.0	0.15
CHF	0.83	1.07	6.67	1.0

Graph:

- a vertex for each currency
- An edge of weight between two currencies of weight –log(rate)



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Example: Buying and selling currency

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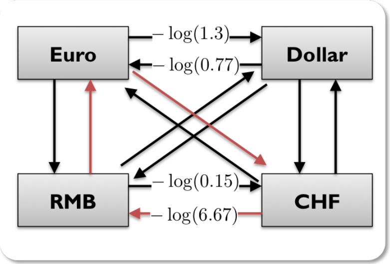

Check for negative cycle

$$-\log(6.67) - \log(0.14) - \log(1.2) \approx -0.11$$

Why negative weights?

Example: Buying and selling currency

Strategy from negative cycle:

Start with 10 000 CHF

Convert CHF to RMB

Convert RMB to Euro

Convert Euro to CHF

Amount of CHF obtained:

$$10^{\log(6.67)+\log(0.14)+\log(1.2)} \cdot 10000$$

$\approx$ 13000

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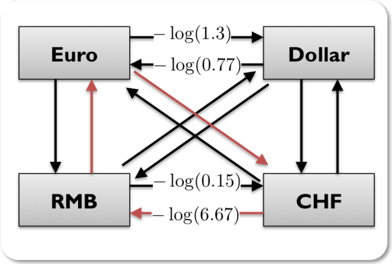

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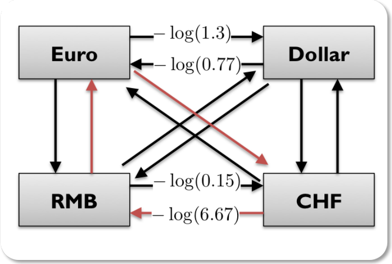

BELLMAN-FORD ALGORITHM

Algorithm

Input: directed graph with edge weights, a source s, no negative cycles

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For each vertex v keep track of

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Start by trivial initialization:

```
INIT-SINGLE-SOURCE(G, s)
  for each v ∈ G.V
      v.d = ∞
      v.π = NIL
  s.d = 0
```

Improving the shortest-path estimate

Can we improve the shortest path estimate for v by going through u and taking (u, v)?

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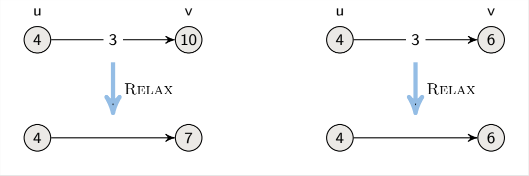
$\textsc{Relax}(u, v, w)$

if $v.d > u.d + w(u, v)$
 $v.d = u.d + w(u, v)$
 $v.\pi = u$



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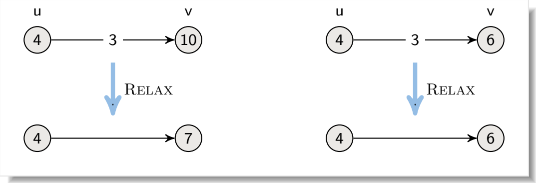

Bellman-Ford updates shortest-path estimates iteratively by using RELAX

Bellman-Ford Algorithm

BELLMAN-FORD(G, w, s)
  INIT-SINGLE-SOURCE(G, s)
  for $i = 1$ to $|G.V| - 1$
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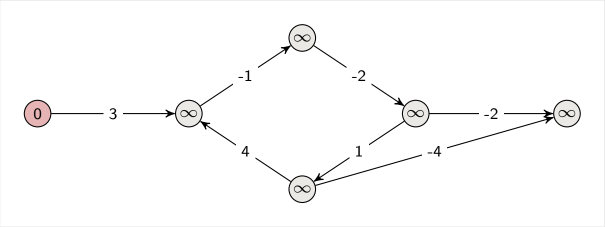

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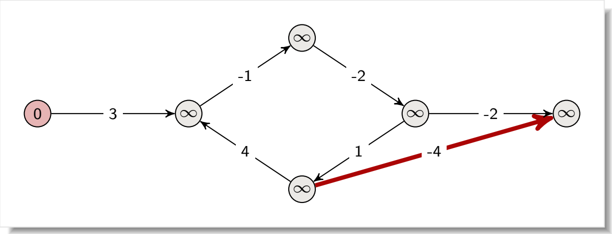

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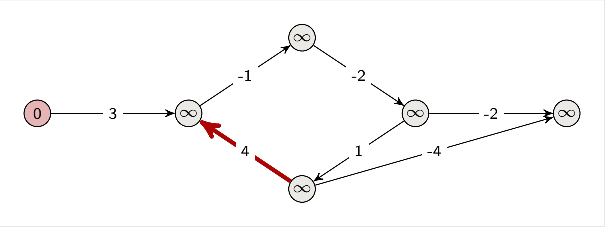

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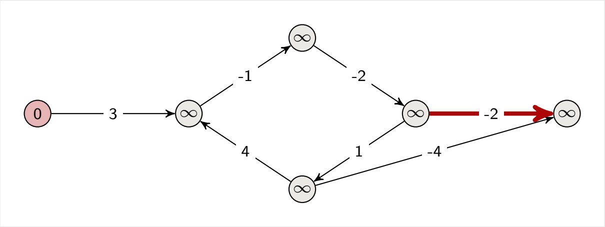

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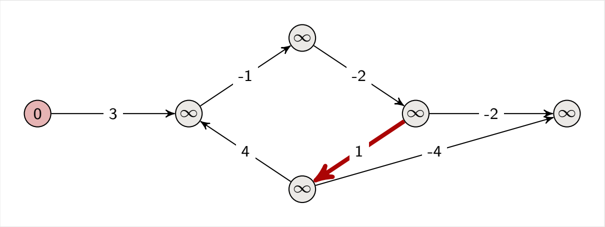

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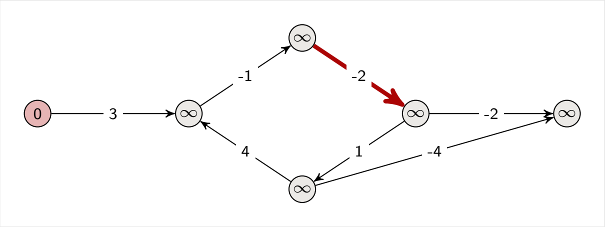

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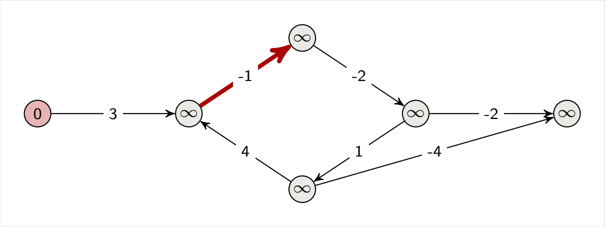

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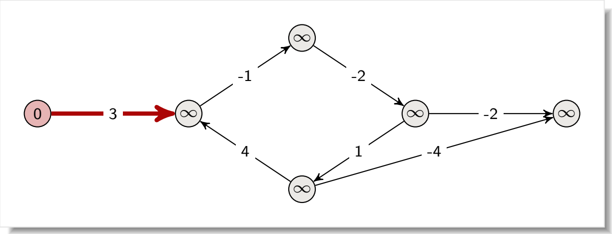

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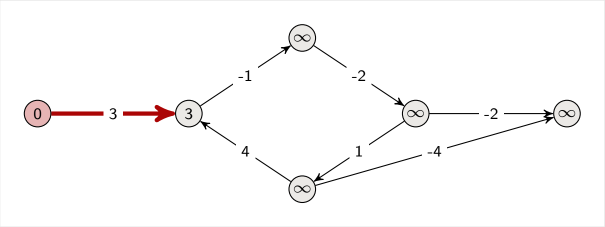

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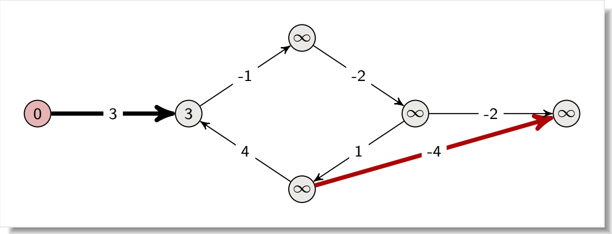

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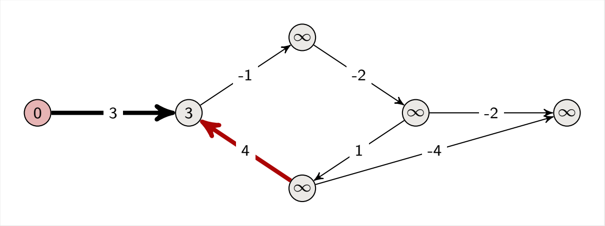

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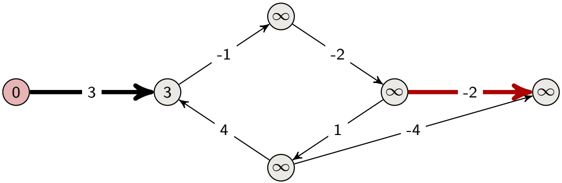

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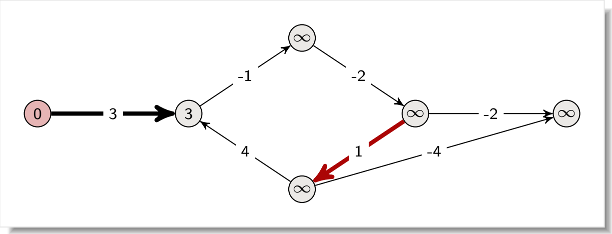

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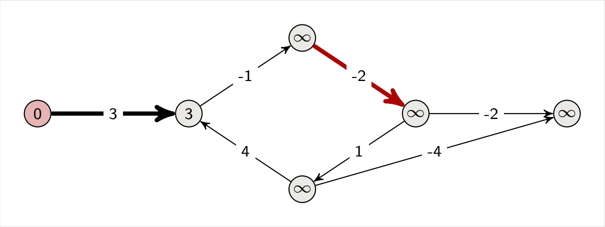

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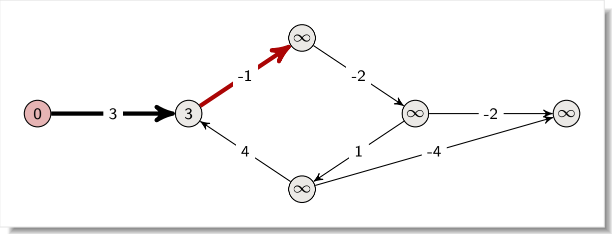

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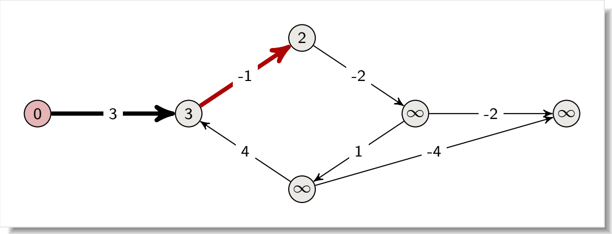

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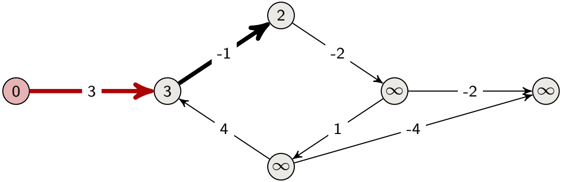

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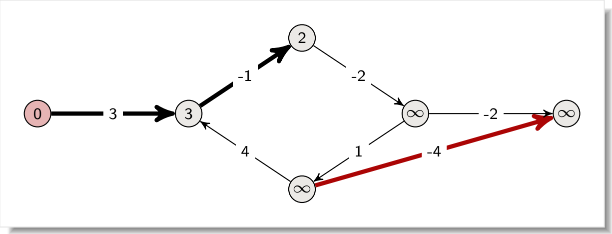

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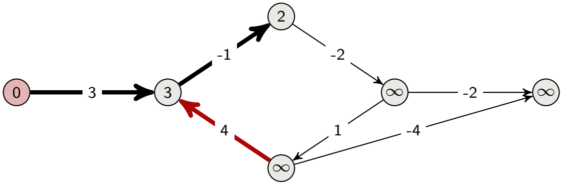

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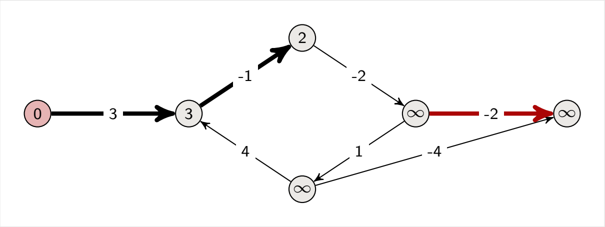

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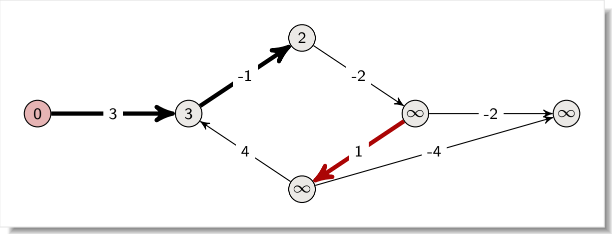

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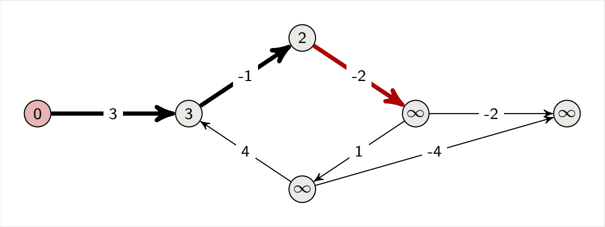

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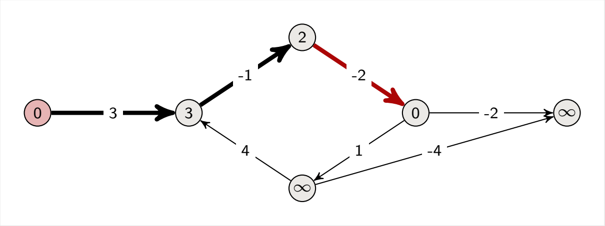

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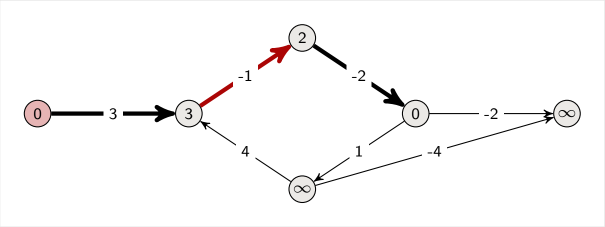

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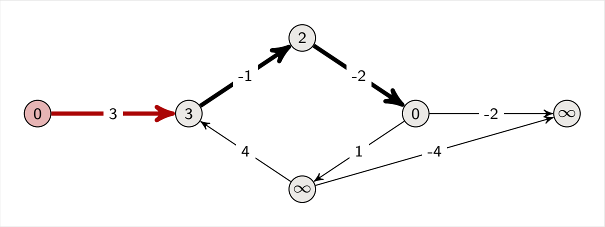

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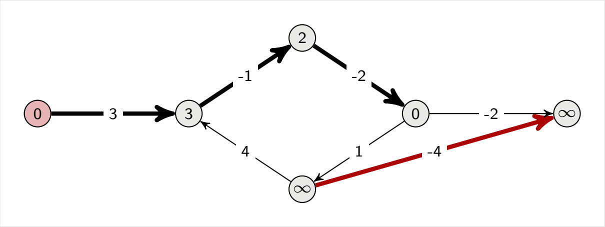

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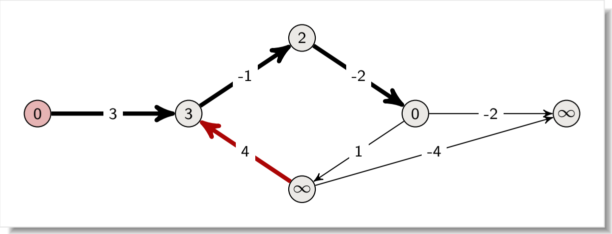

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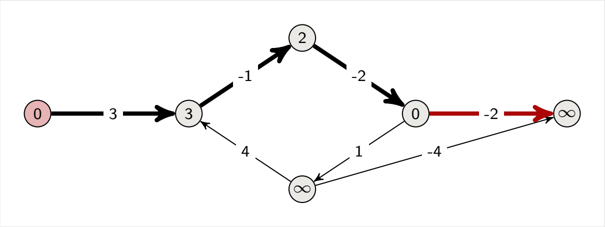

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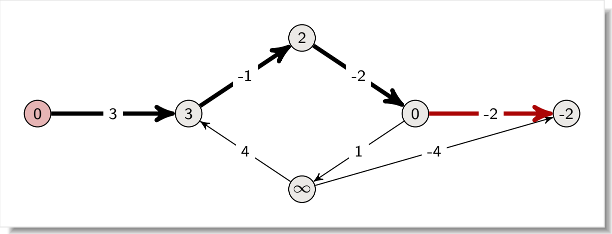

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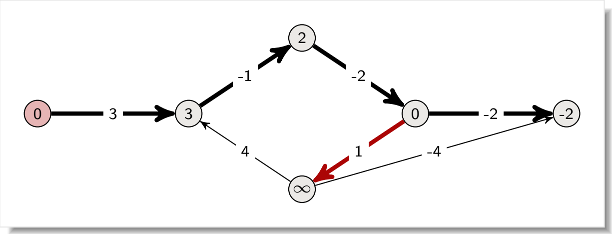

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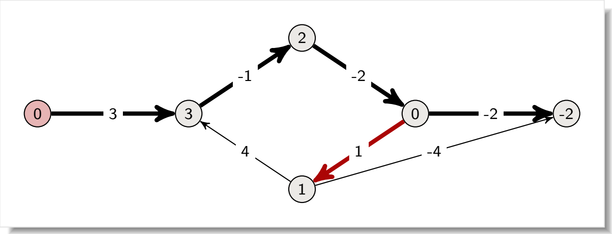

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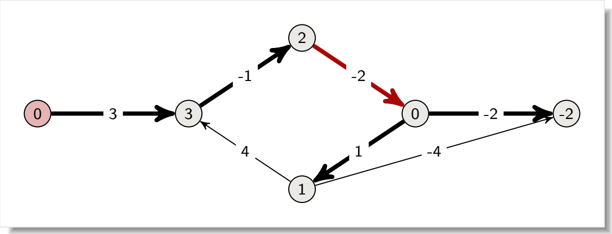

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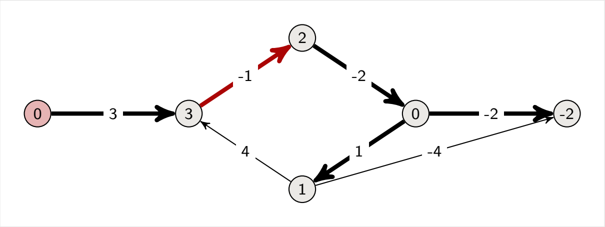

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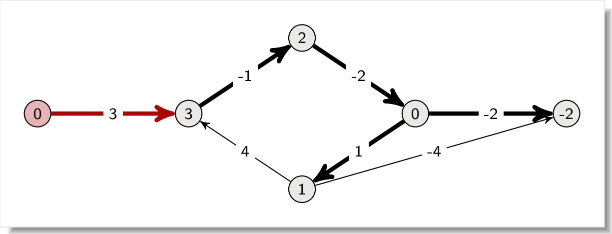

Bellman-Ford Algorithm

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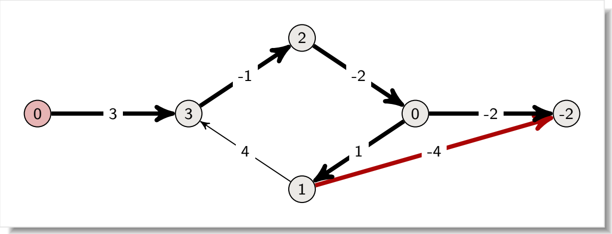

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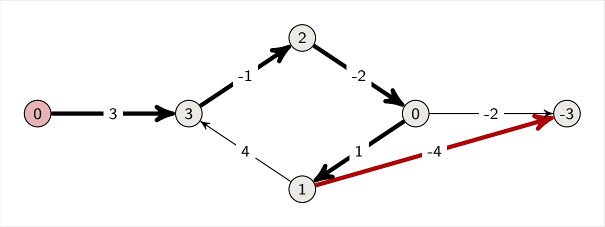

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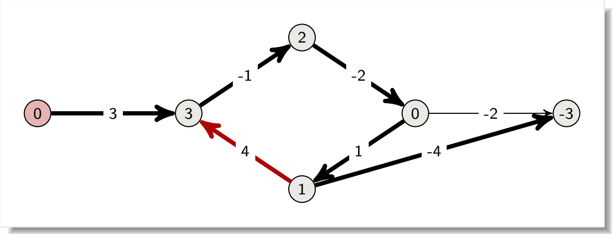

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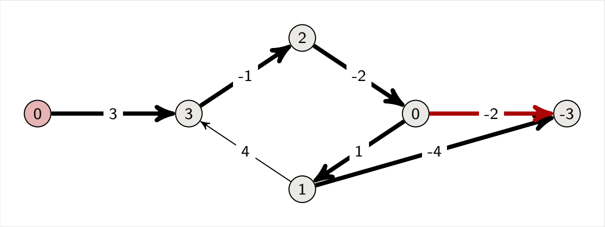

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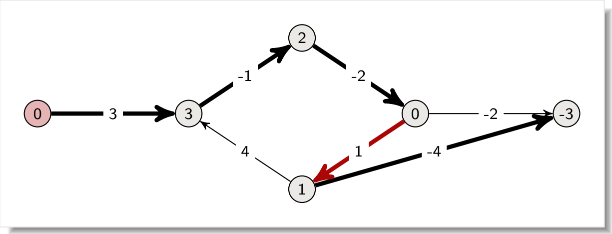

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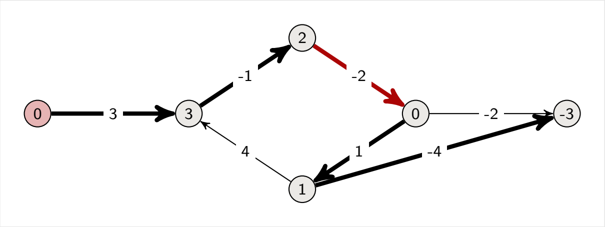

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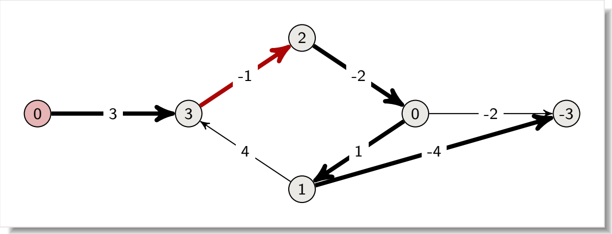

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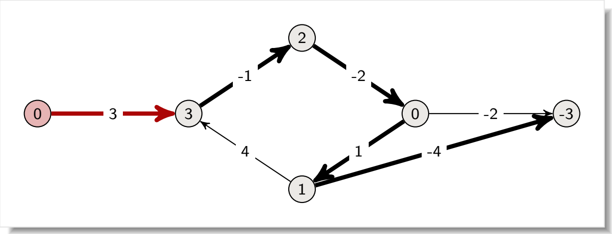

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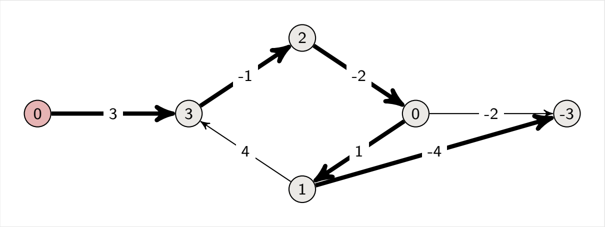

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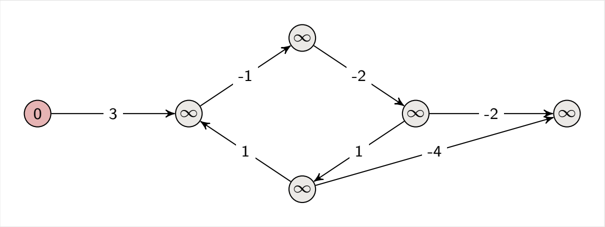

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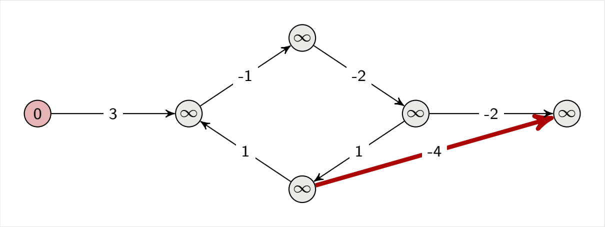

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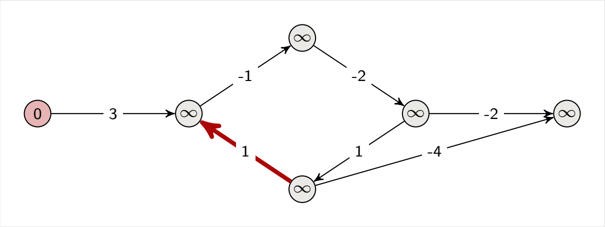

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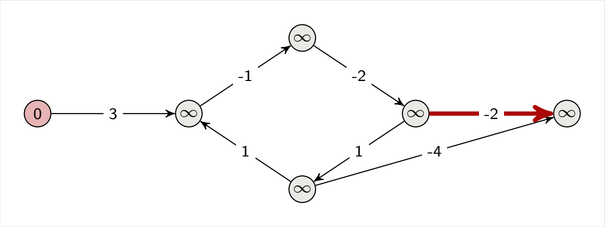

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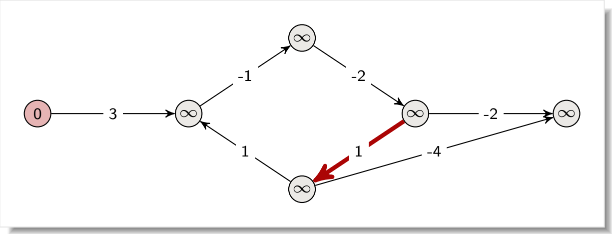

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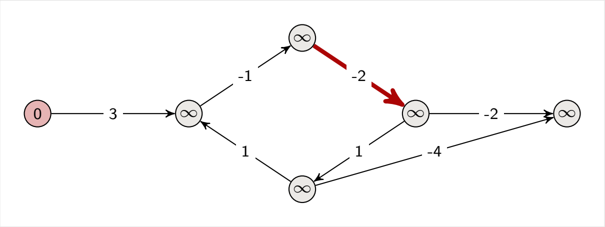

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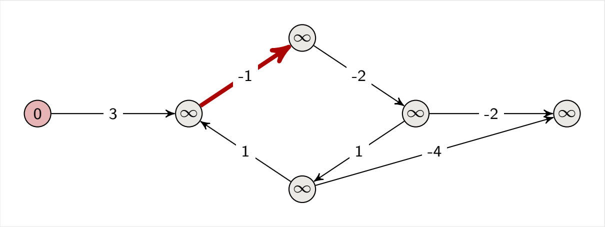

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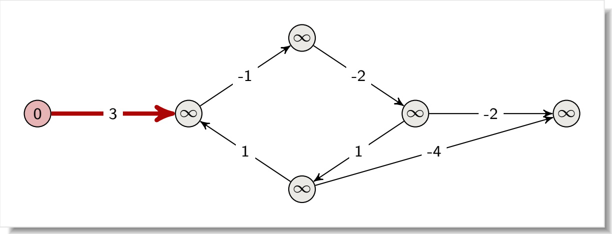

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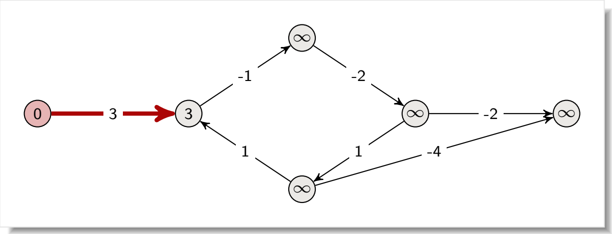

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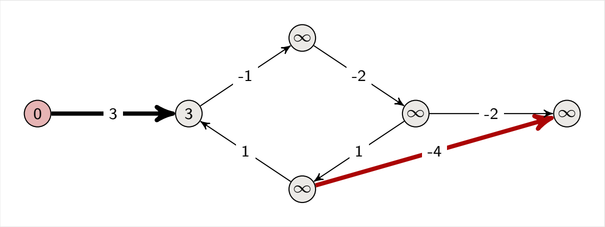

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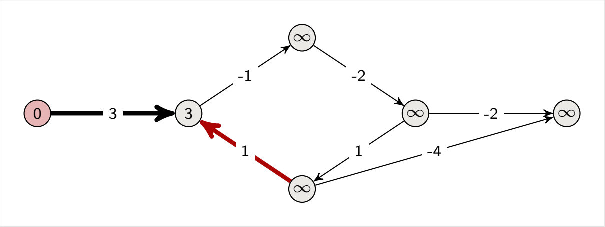

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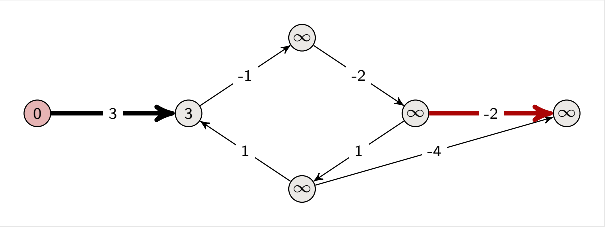

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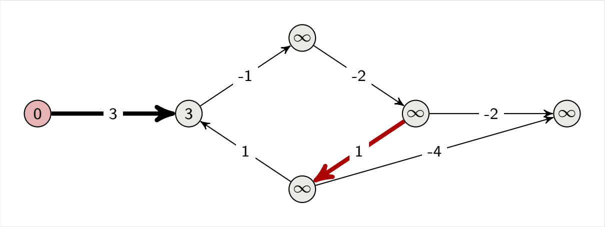

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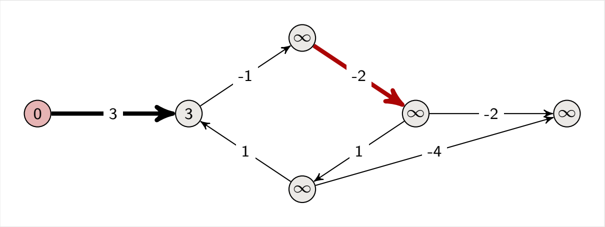

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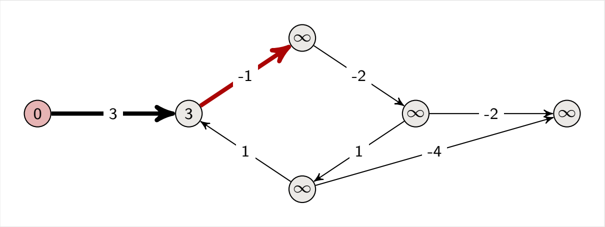

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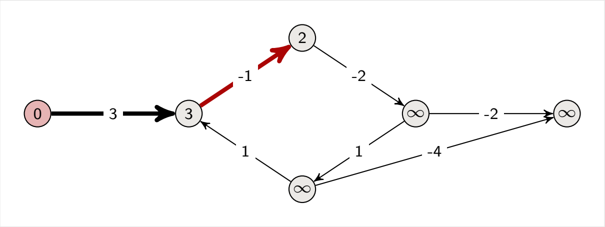

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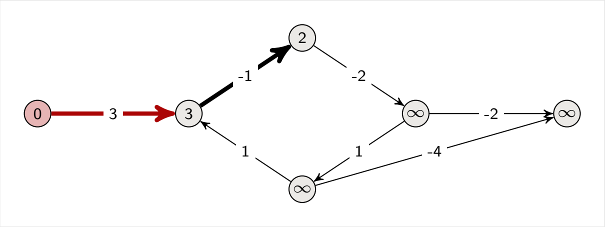

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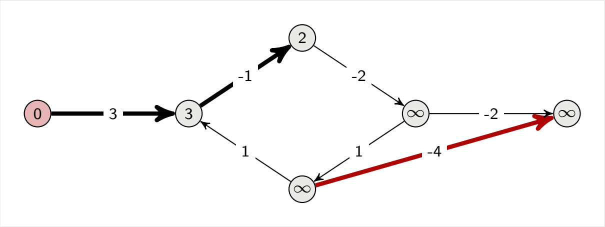

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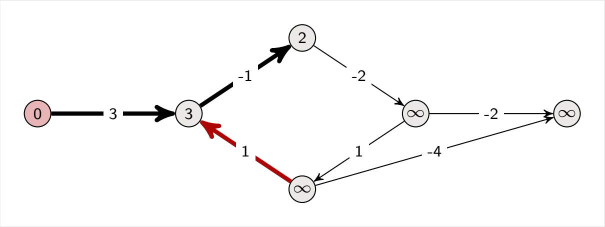

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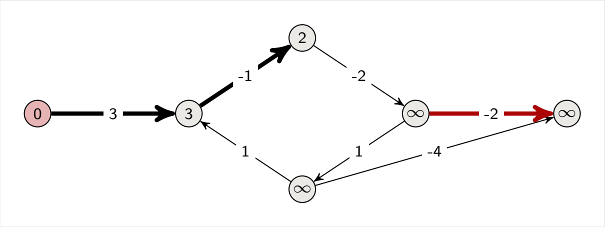

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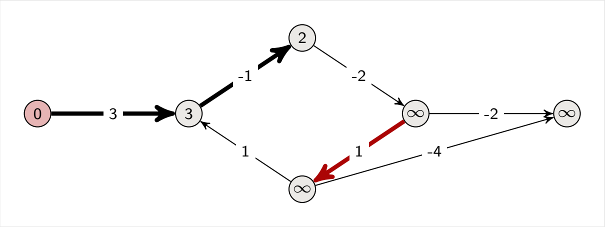

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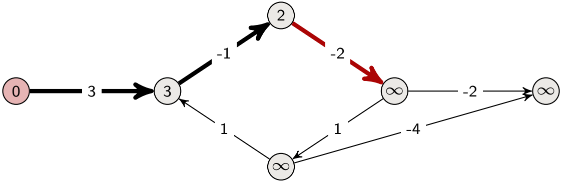

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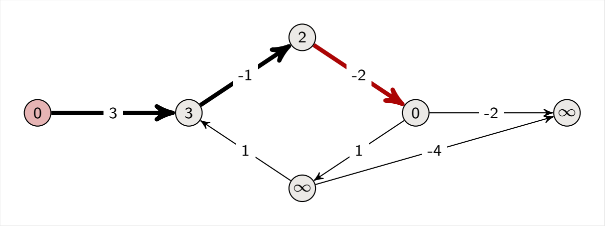

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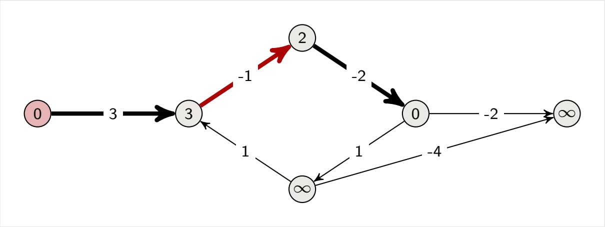

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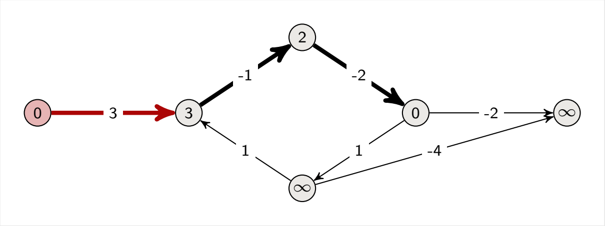

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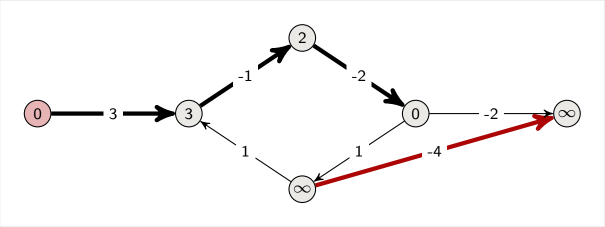

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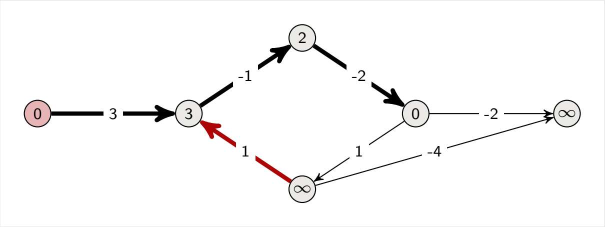

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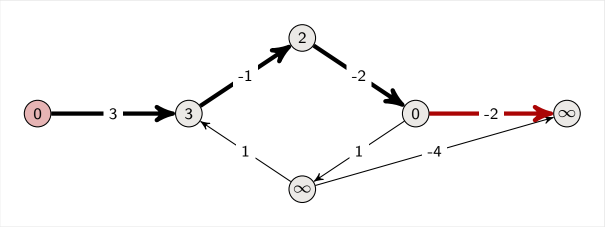

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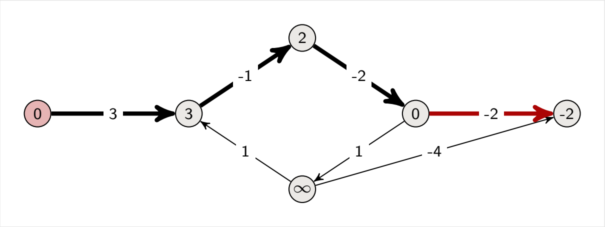

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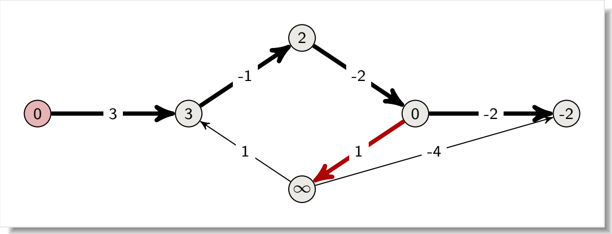

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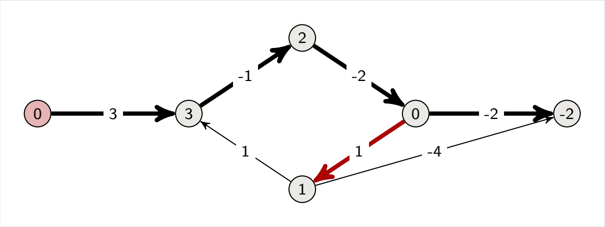

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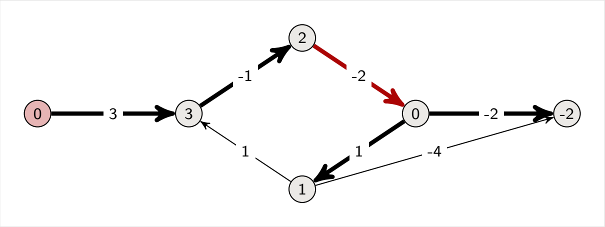

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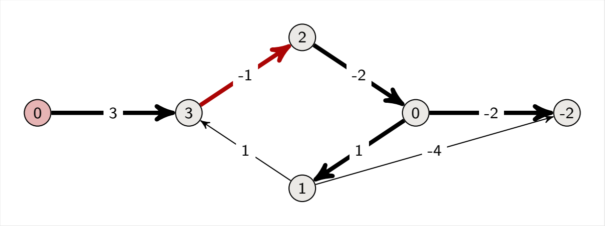

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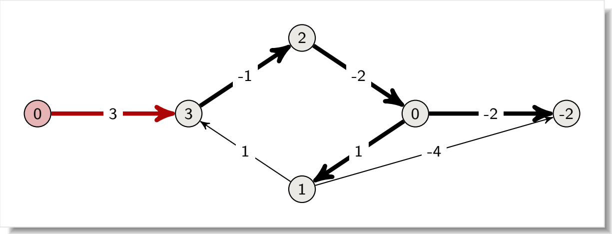

Bellman-Ford Algorithm

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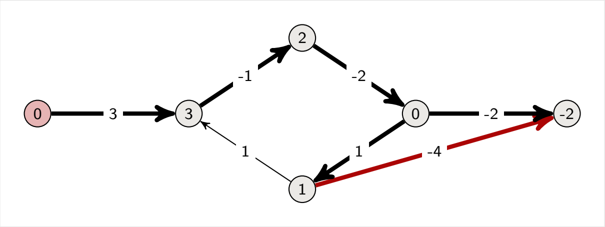

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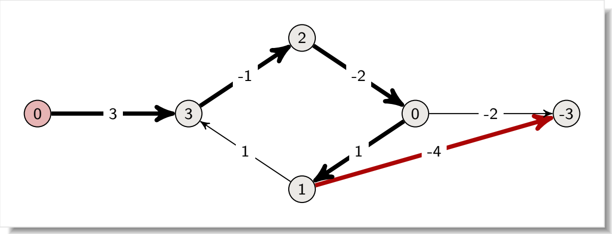

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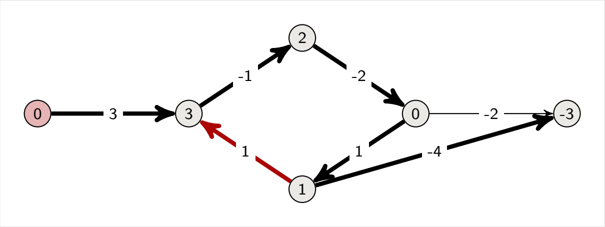

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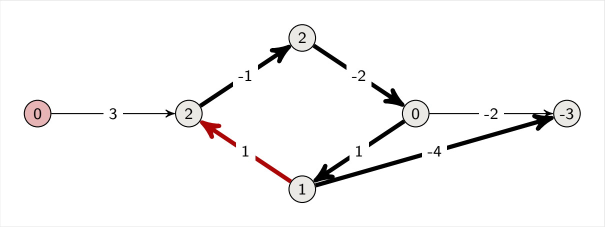

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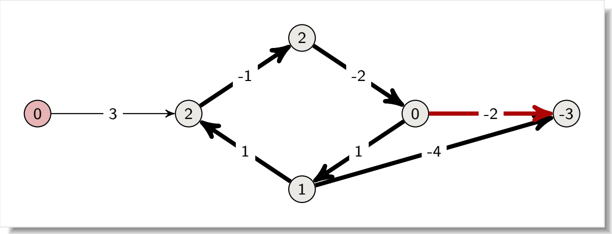

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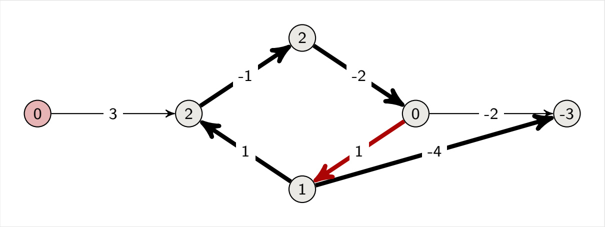

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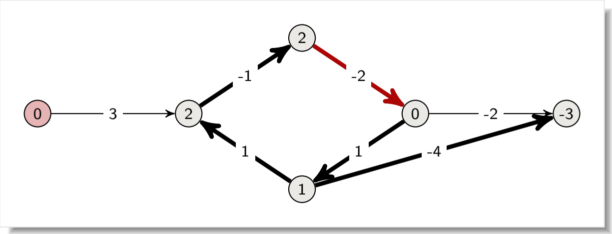

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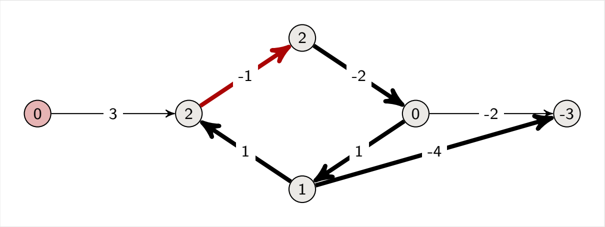

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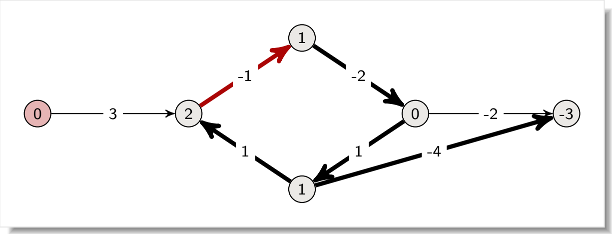

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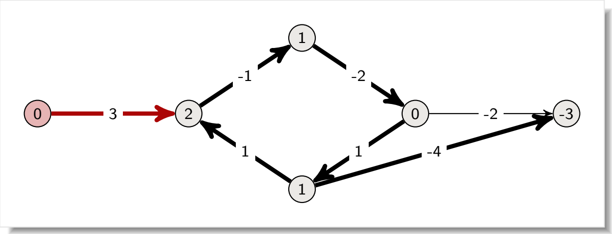

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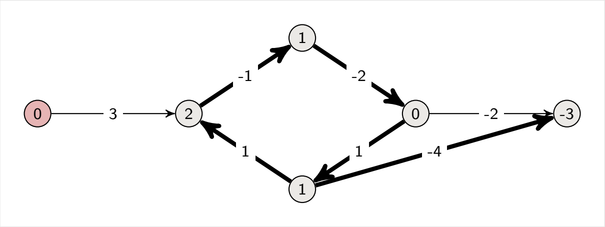


Correctness

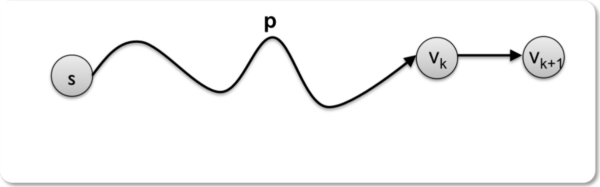
Only guaranteed to work if no negative cycles!

As we shall see later, it can also be used to detect negative cycles

Optimal substructure

If $\langle s, v_1, \ldots, v_k, v_{k+1} \rangle$ is a shortest path from s to v_{k+1}

Then $\langle s, v_1, \ldots, v_k \rangle$ is a shortest path from s to v_k



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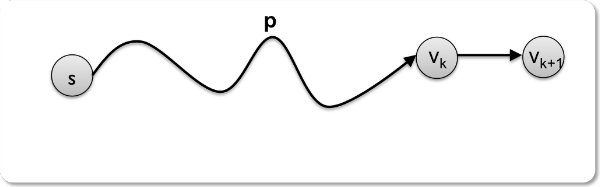

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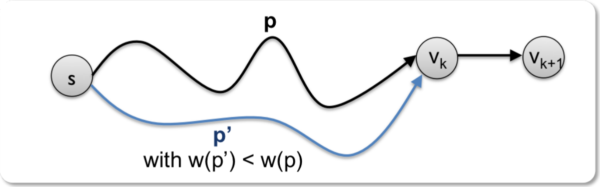
- Suppose toward contradiction: there exists shorter path p' from s to v_k



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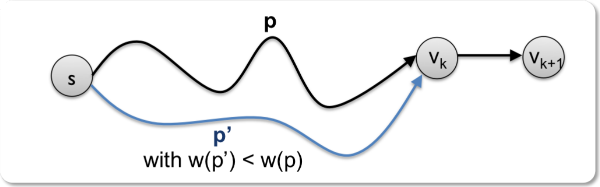
Proof:

- Suppose toward contradiction: there exists shorter path p' from s to v_k
- Then weight of $p' + (v_k, v_{k+1})$ is smaller than $p + (v_k, v_{k+1})$ which contradicts that $p + (v_k, v_{k+1})$ was a shortest path from s to v_{k+1}



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Invariant: $\ell(v)$ is at most the length of the shortest path from s to v using at most i edges after the i'th iteration

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Inductive step: consider any shortest path from s to v_{k+1} using at most i edges

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- By induction hypothesis $\ell(v_k) \leq w(p)$ after previous iteration
- Hence, $\ell(v_{k+1}) \leq \ell(v_k) + w(v_k, v_{k+1}) \leq$ "length of shortest path from s to v_{k+1} using at most i edges" in the i-th iteration

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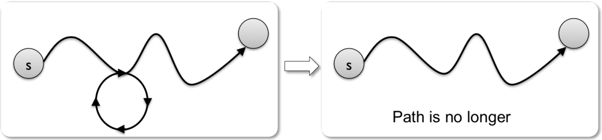
Proof: If there is a path with n or more edges, then there is a cycle and since its weight is non-negative it can be removed



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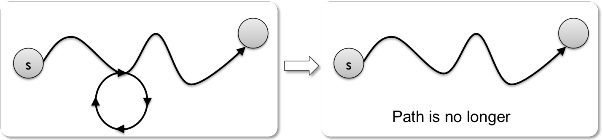

Therefore, Bellman-Ford will return correct answer if no negative cycles after $n-1$ iterations

Detecting negative cycles

There is a negative cycle reachable from the source if and only if the ℓ-value of at least one node changes if we run one more (n:th) iteration of Bellman-Ford

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We need to prove: If the ℓ-value of the vertices don't change in the n:th iteration, then there is no negative cycle that is reachable from the source

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So for a cycle $v_0 - v_1 - \cdots - v_{t-1} - v_t = v_0$,

$$\sum_{i=1}^{t} \ell(v_i) \leq \sum_{i=1}^{t} (\ell(v_{i-1}) + w(v_{i-1}, v_i)) = \sum_{i=1}^{t} \ell(v_{i-1}) + \sum_{i=1}^{t} w(v_{i-1}, v_i)$$

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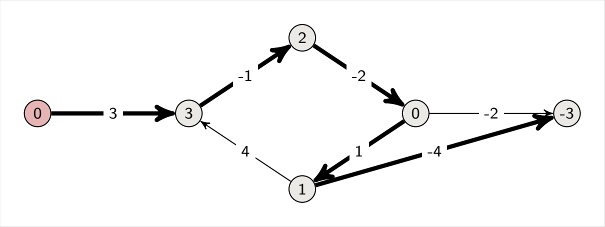
Red sums are the same, hence the cycle is non-negative $0 \leq \sum_{i=1}^{t} w(v_{i-1}, v_i)$

Example 1

BELLMAN-FORD(G, w, s)

INIT-SINGLE-SOURCE(G, s)
for $i = 1$ to $|G.V| - 1$
  for each edge $(u, v) \in G.E$
    RELAX(u, v, w)
for each edge $(u, v) \in G.E$
  if $v.d > u.d + w(u, v)$
    return FALSE
return TRUE



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$$
\begin{array}{l}
\textsc{Bellman-Ford}(G, w, s) \\
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\qquad \textbf{for} \text{ each edge } (u, v) \in G.E \\
\qquad\quad \textsc{Relax}(u, v, w) \\
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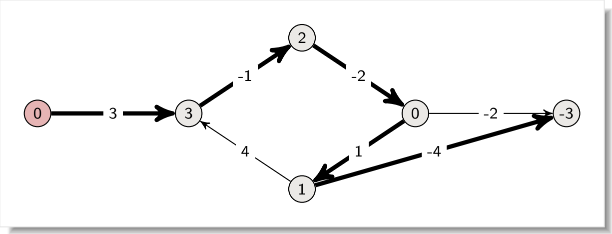
\qquad \textbf{if } v.d > u.d + w(u, v) \\
\qquad\quad \textbf{return } \text{FALSE} \\
\quad \textbf{return } \text{TRUE}
\end{array}
$$



No negative cycles, BF returns TRUE

Example 2

```
BELLMAN-FORD(G, w, s)
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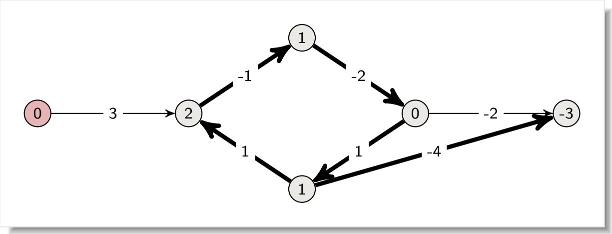

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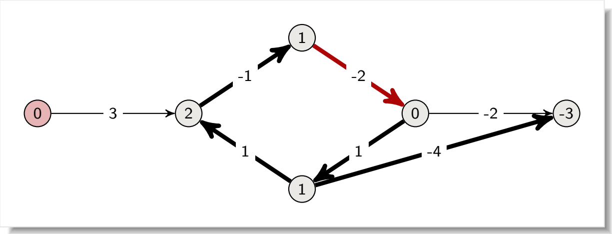

Negative cycles, BF returns FALSE

Runtime analysis

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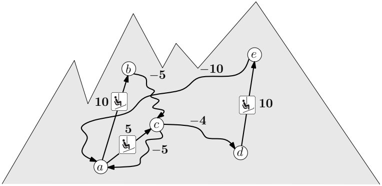
Total time: $\Theta(E \cdot V)$

Final comments on Bellman-Ford

- Can be used to find negative cycles
  - Run for n-iterations and detect cycles in "shortest path tree" these will correspond to negative cycles
- Easy to implement in distributed settings: each vertex repeatedly ask their neighbors for the best path
  - Good for routing and dynamic networks

Problem solving (20 pts * exam question last year)

The famous alpine ski racer Lindsey Vonn has sent her assistant to measure the altitude differences of the ski lifts and slopes of a famous ski resort in Switzerland. The assistant returns after several days of hard work with a map of all ski lifts and all slopes together with the altitude difference between the start station and the end station of each lift and the altitude difference between the start station and end station of each slope. A slope starts from the end station of a lift and ends at the start station of a potentially different lift (see the figure below).
Lindsey wants to verify the map of her assistant by performing the following sanity check: starting from any point A, no matter how we ski (using lifts and slopes), the altitude change when we return to A should be 0 (i.e., neither strictly negative nor strictly positive). Design an algorithm to perform this sanity check and analyze its running time in terms of the number of slopes and ski lifts ($|E|$) and the number of start and end stations ($|V|$). Your algorithm should run in polynomial time (in $|V|$ and $|E|$).



DIJKSTRA'S ALGORITHM

Dijkstra's algorithm

- Only works when all weights are nonnegative
- Greedy and faster than Bellman-Ford
- Similar idea to Prim's algorithm (essentially weighted version of BFS)

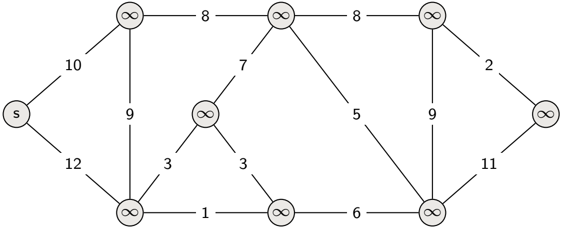
Dijkstra's algorithm

Start with source $S = \{s\}$

Greedily grow S:

at each step add to S the vertex that is closest to s

(minimum over $v \notin S$ of minimum over $u \in S, u.d + w(u, v)$)



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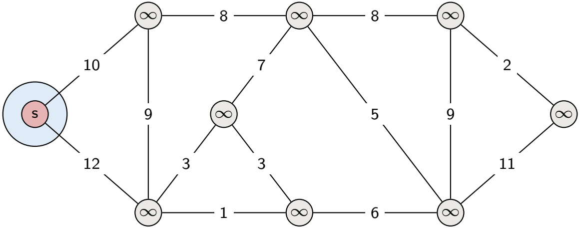

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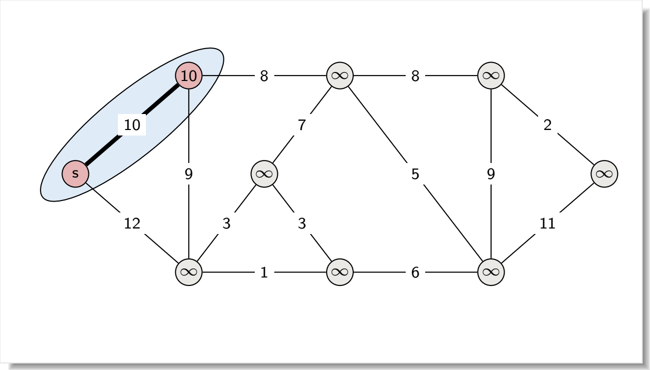

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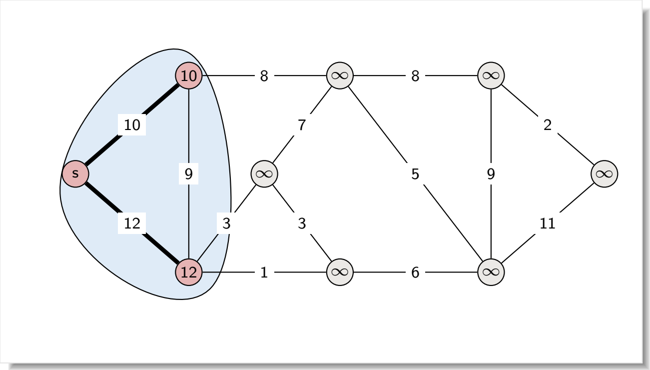

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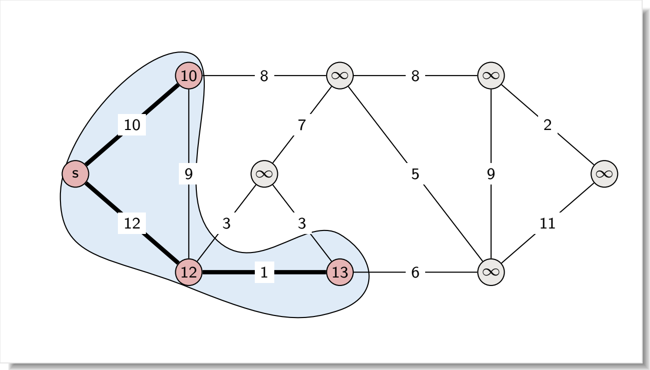

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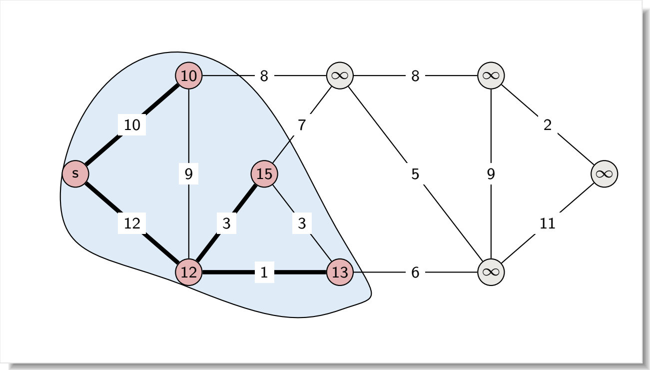

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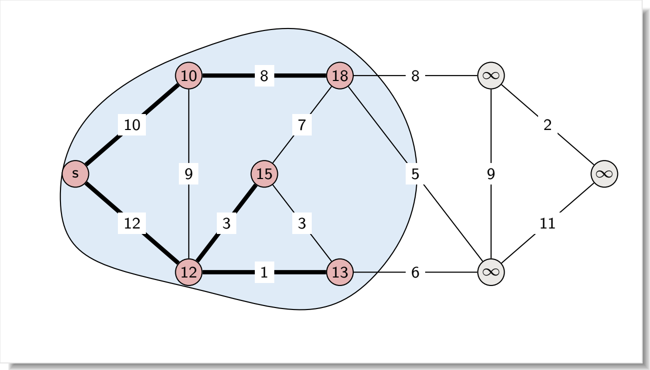

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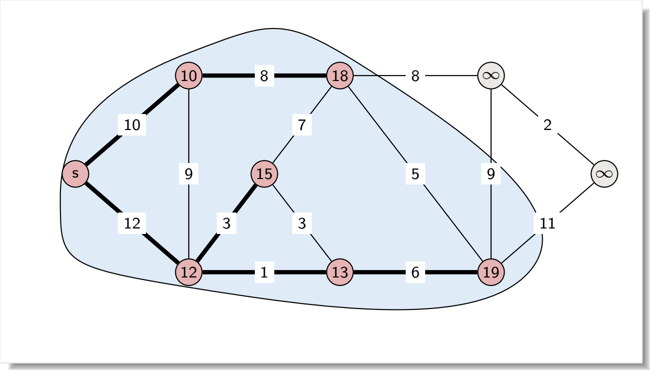

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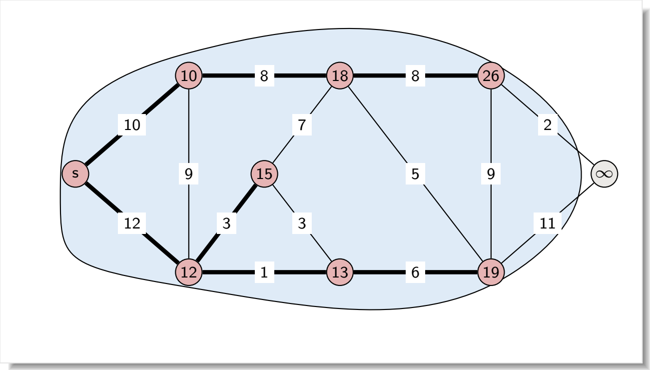

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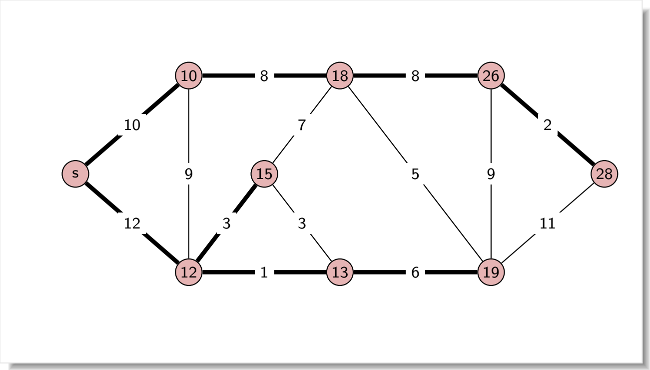

Implementation and Running Time

Implementation with priority-queue as Prim's algorithm with shortest path keys:

```
DIJKSTRA(G, w, s)
  INIT-SINGLE-SOURCE(G, s)
  S = ∅
  Q = G.V          // i.e., insert all vertices into Q
  while Q ≠ ∅
      u = EXTRACT-MIN(Q)
      S = S ∪ {u}
      for each vertex v ∈ G.Adj[u]
          RELAX(u, v, w)
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- More careful implementation time is $O(V \lg V + E)$

Problem Solving

Show that Dijkstra's Algorithm is correct by proving the following loop invariant:

"At the start of each iteration, we have for all $v \in S$ that the distance $v.d$ from s to v is equal to the shortest path from s to v"