Algorithms: Ford-Fulkerson Method

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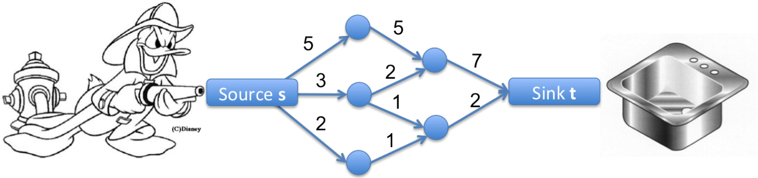
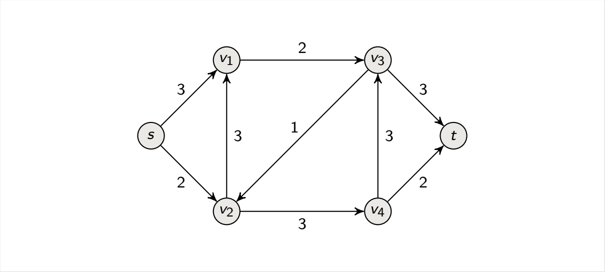
EPFL School of Computer and Communication Sciences

Lecture 18, 16.04.2025



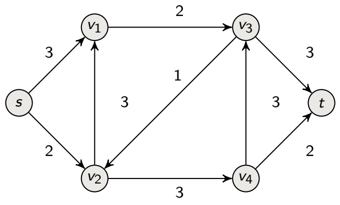
FLOW NETWORKS

Flow Network



- Directed graph $G = (V, E)$
- Each edge (u, v) has a capacity $c(u, v) \geq 0$ ($c(u, v) = 0$ if $(u, v) \notin E$)
- Source s and sink t (flow goes from s to t)
- No antiparallel edges (assumed w.l.o.g. for simplicity)

Definition of a flow



A flow is a function $f : V \times V \to \mathbb{R}$ satisfying:

Capacity constraint: For all $u, v \in V : 0 \leq f(u, v) \leq c(u, v)$

Flow conservation: For all $u \in V \setminus \{s, t\}$,

$$\underbrace{\sum_{v \in V} f(v, u)}_{\text{flow into } u} = \underbrace{\sum_{v \in V} f(u, v)}_{\text{flow out of } u}$$

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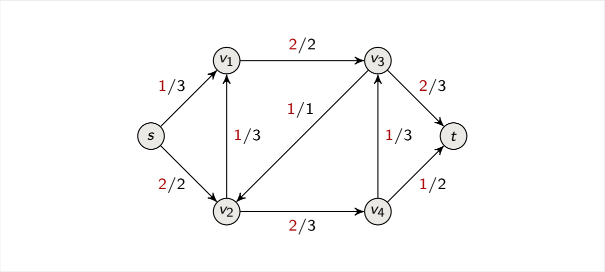

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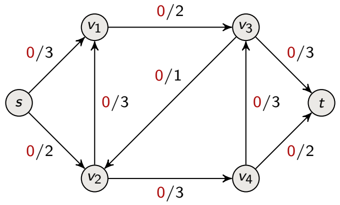

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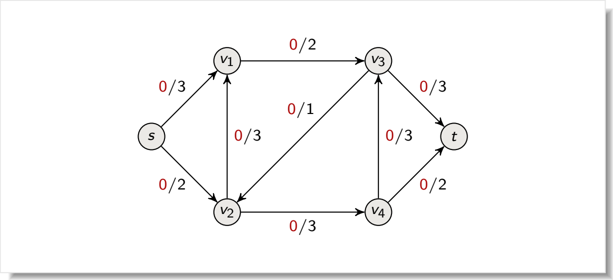
Value of a flow



Value of a flow $f = |f|$

$$= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

$=$ flow out of source $-$ flow into source

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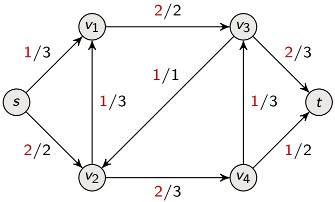

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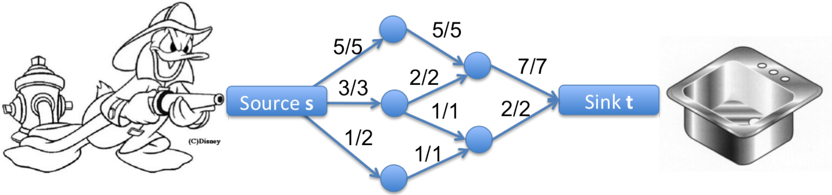

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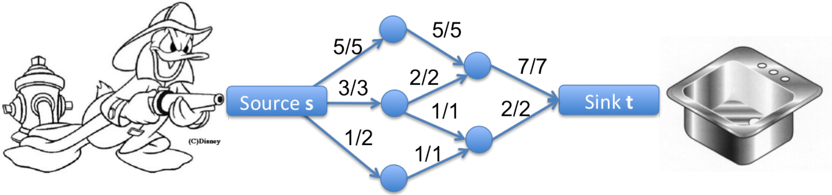
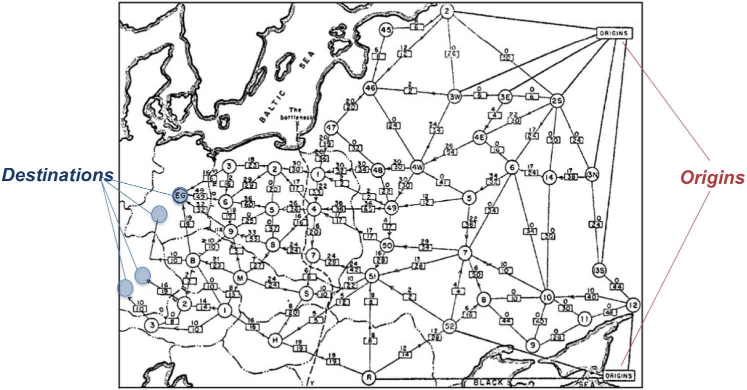
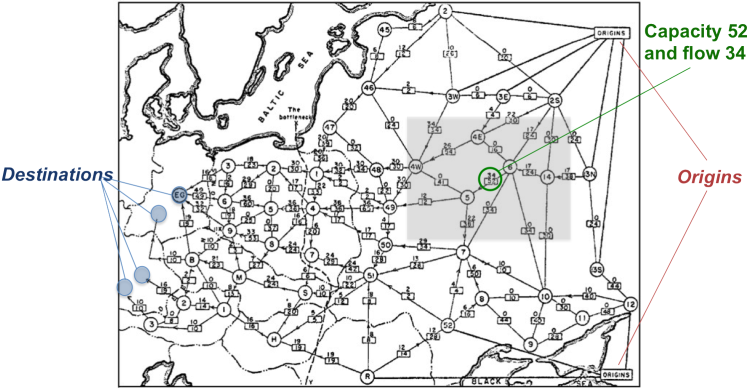




- Schematic diagram of the railway network of the western Soviet union and easter European countries, from Harris & Ross (1955), declassified by pentagon in 1999.



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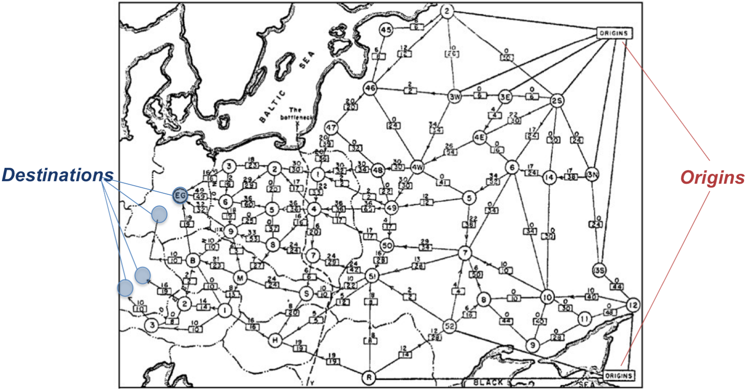
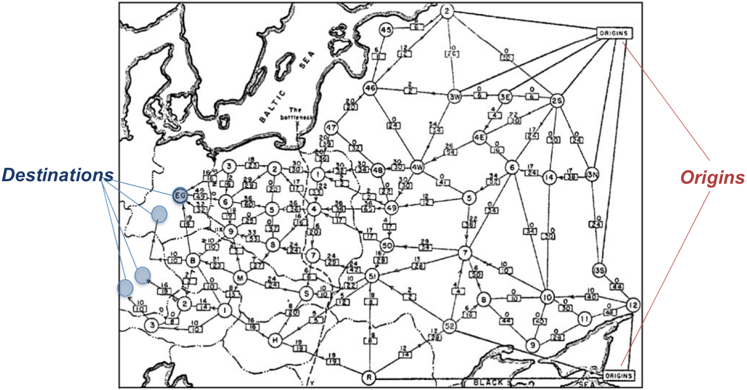
Goal of Soviet union



Goal of Soviet union

Maximize throughput from the “origins” to the destinations



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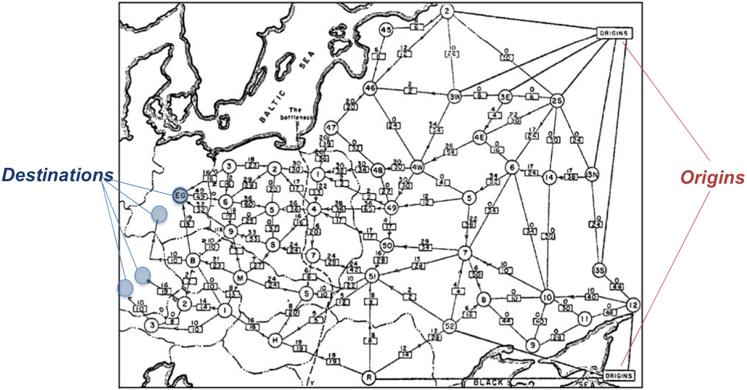
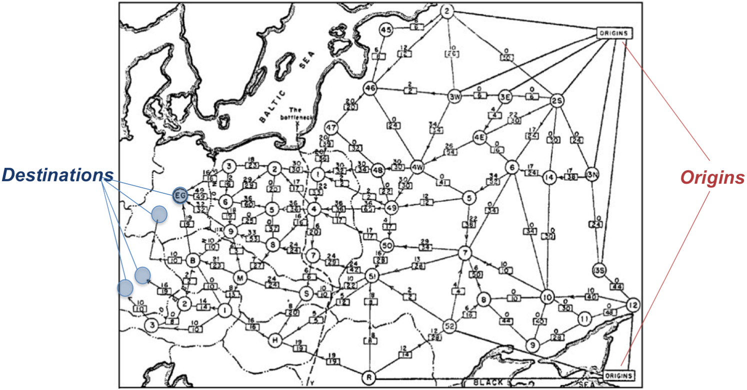
Maximize throughput from the "origins" to the destinations

Ford-Fulkerson method solves it



Goal of US Air Force (1950's)



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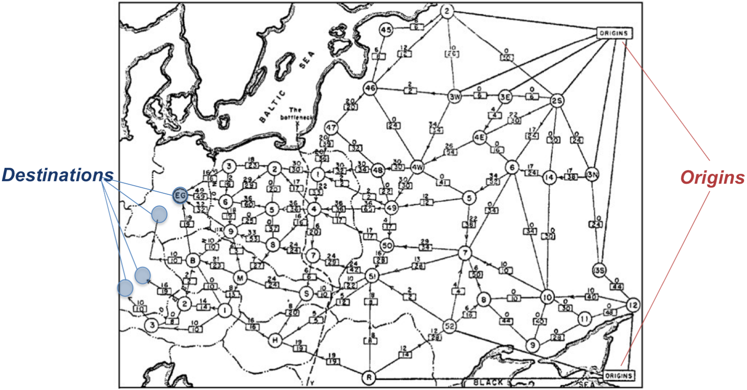
Disrupt flow of goods into satellite countries in the best possible way



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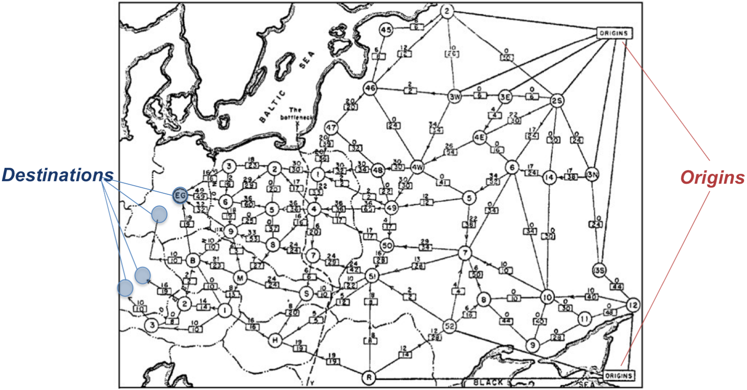

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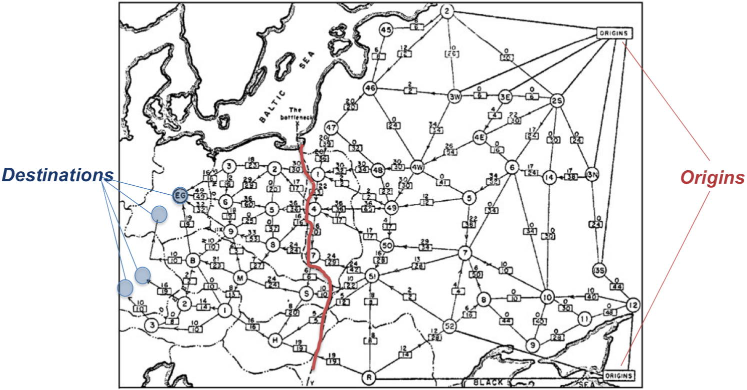
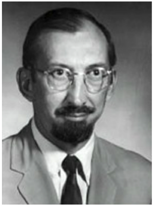


L. R. Ford, Jr. (1927-)



D, R, Fulkerson (1924-1976)

MAXIMUM-FLOW PROBLEM

Ford-Fulkerson Method

The Ford-Fulkerson Method'54

FORD-FULKERSON-METHOD(G, s, t):

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Basic idea:

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- As long as there is a path from source to sink, with available capacity on all edges in the path

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Basic idea:

- As long as there is a path from source to sink, with available capacity on all edges in the path
- send flow along one of these paths and then we find another path and so on

Residual network

- Given a flow f and a network $G = (V, E)$
- the residual network consists of edges with capacities that represent how we can change the flow on the edges

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Residual capacity:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$

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Amount of capacity left (for the case $(u,v) \in E$)

Amount of flow that can be reversed (for the case $(v,u) \in E$)

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Amount of capacity left (first case)

Amount of flow that can be reversed (second case)

Residual network:

$G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$

Examples

Residual network: $G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$ and

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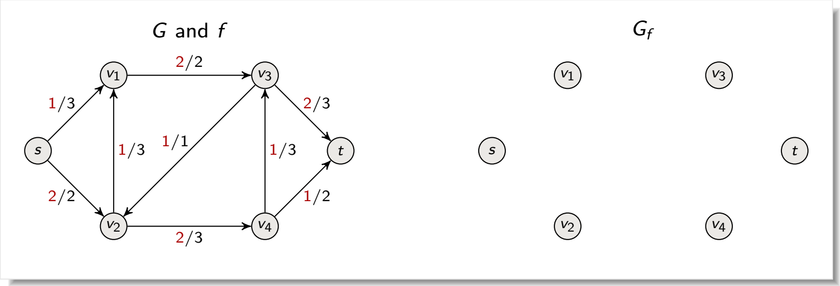

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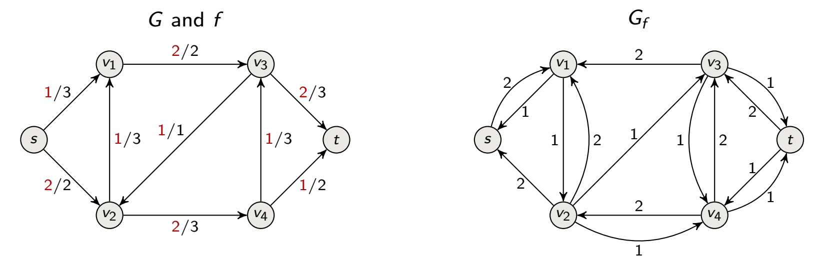

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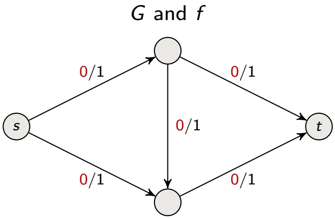

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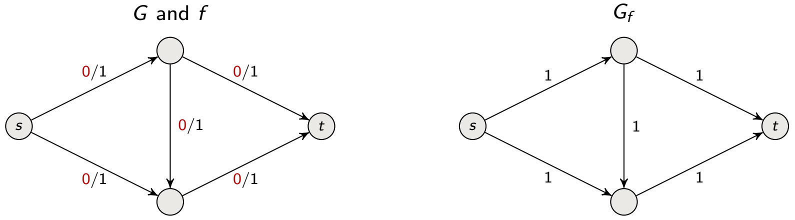
Augmenting path = simple path from s to t



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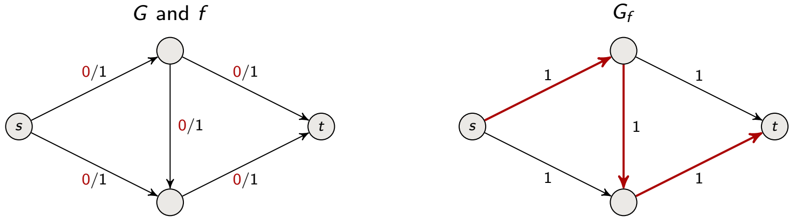
Exists augmenting path **p**



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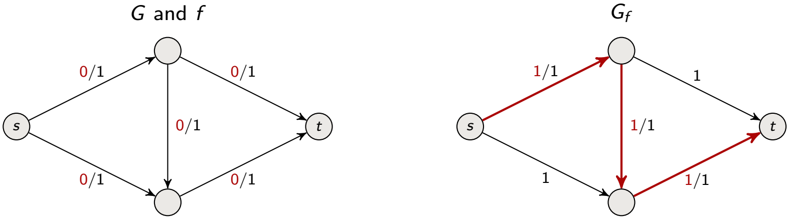
Exists augmenting path **p** with flow $\mathbf{f_p}$ of value = min capacity on **p**



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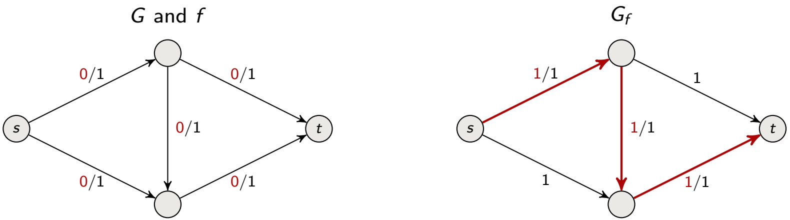

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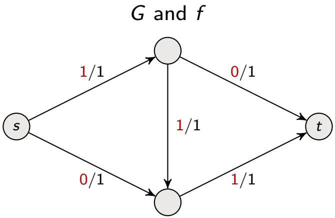

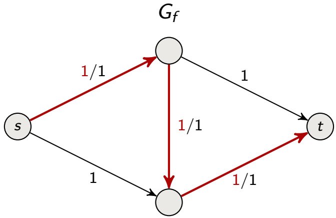

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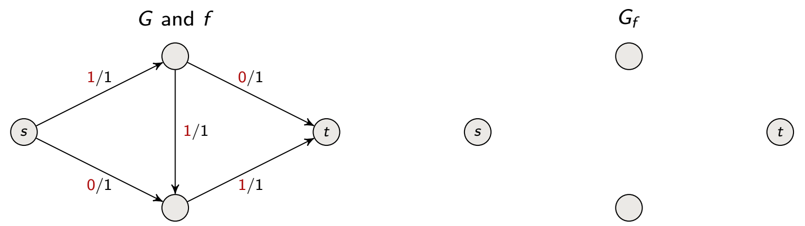

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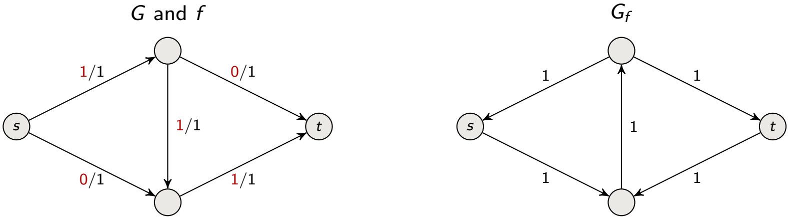

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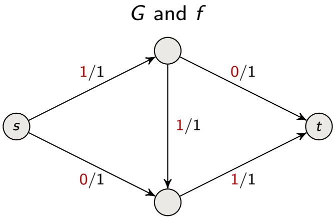

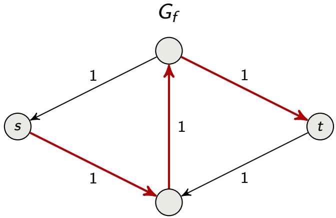

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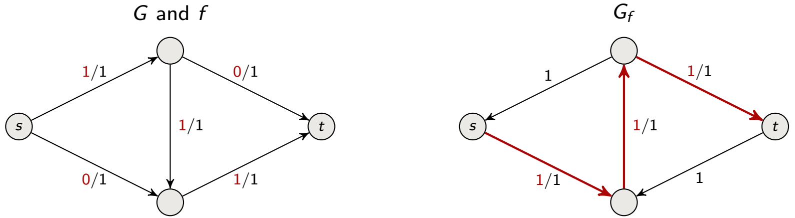

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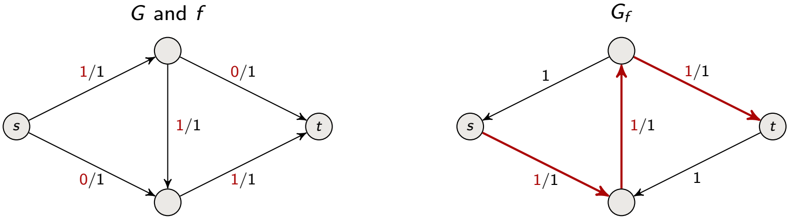

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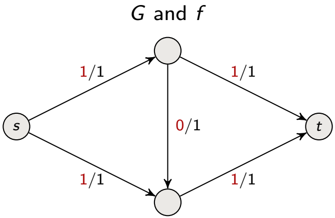

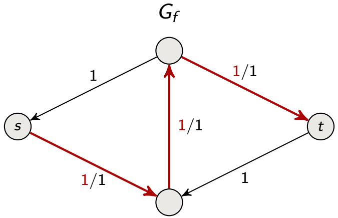

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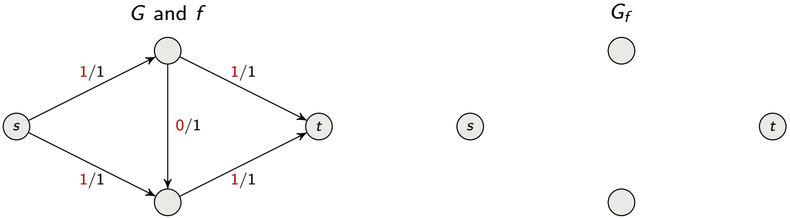

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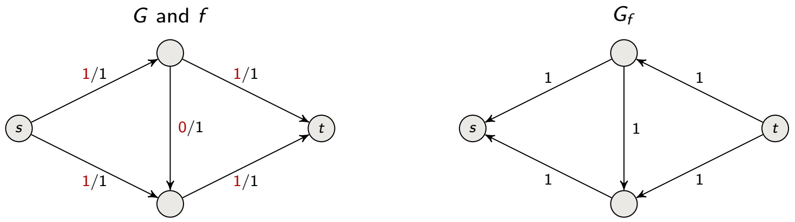

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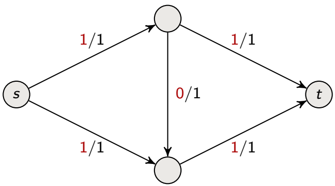
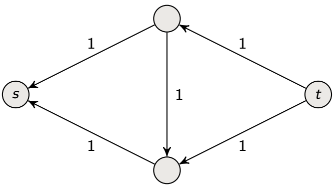
No augmenting path and flow of value 2 is optimal

G and f



G_f



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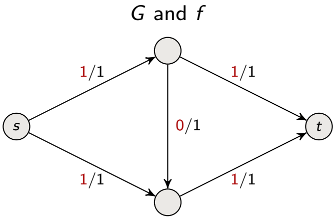

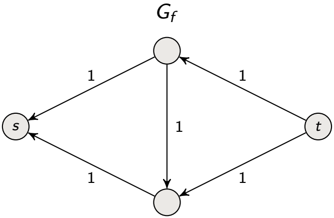

The Ford-Fulkerson Method

Max-flow

Start with 0-flow
while there is an augmenting path from s to t in residual network **do**

- Find augmenting path
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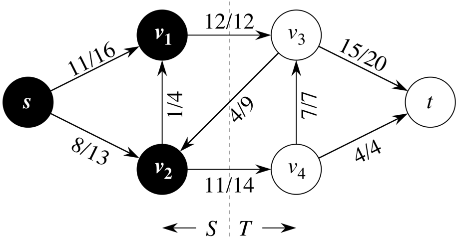
Gives a way to verify that the step-by-step calculations of the flow are correct!

WHY IS RETURNED FLOW OPTIMAL? (MIN-CUTS)

Cuts in flow networks

A cut of flow network $G(V, E)$ is

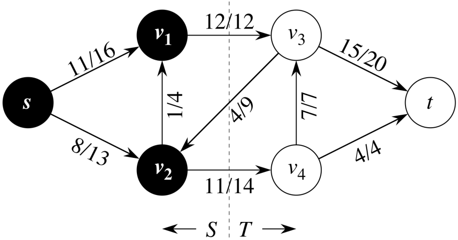
- a partition of V into S and $T = V \setminus S$
- such that $s \in S$ and $t \in T$



Net flow across a cut

The net flow across cut (S, T) is

$$f(S,T) = \underbrace{\sum_{u \in S, v \in T} f(u,v)}_{\text{flow leaving } S} - \underbrace{\sum_{u \in S, v \in T} f(v,u)}_{\text{flow entering } S}$$



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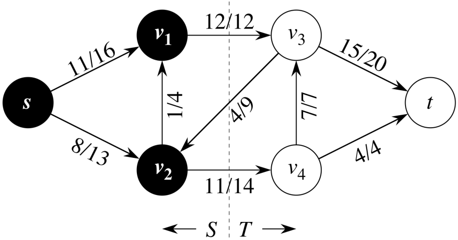
What is the net flow of this cut?



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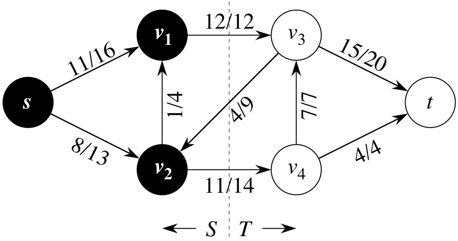
What is the net flow of this cut? $12 + 11 - 4 = 19$



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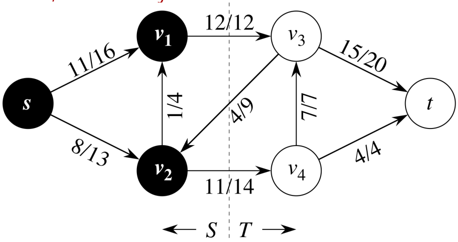
What is the net flow of this cut? $12 + 11 - 4 = 19$ Note that this equals the value of the flow; it's always the case!



Net flow equals flow value for any cut

Theorem

For any cut (S, T), $|f| = f(S, T)$.

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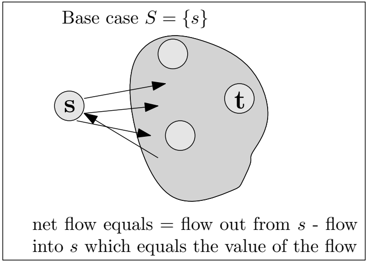

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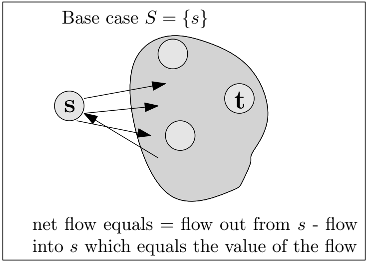

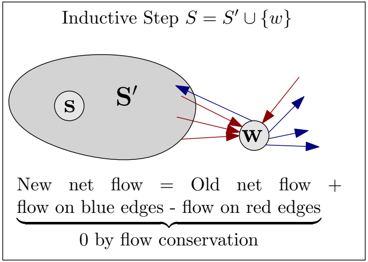

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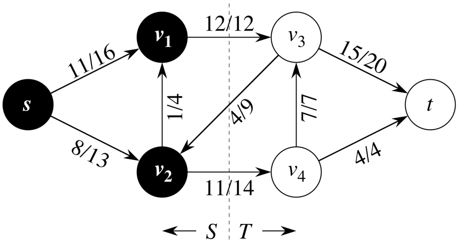

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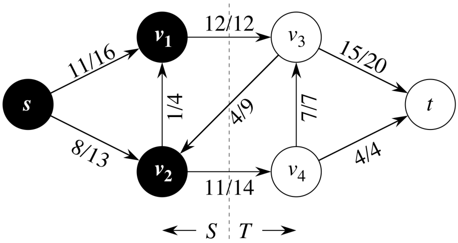

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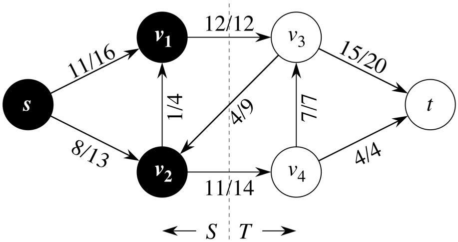
What is the capacity of this cut? $12 + 14 = 26$



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Max-flow is at most capacity of a cut

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Therefore: **max-flow** $\leq$ **min-cut**

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We shall prove

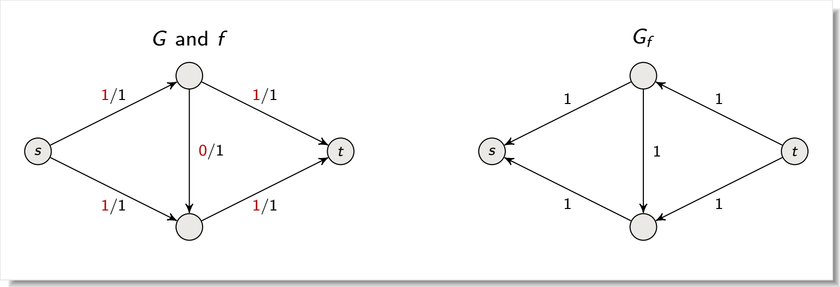
Theorem (max-flow min-cut theorem)

max-flow = min-cut

Examples

Consider f obtained by running Ford-Fulkerson and let

$S = \{v \in V : \text{there is a path from } s \text{ to } v \text{ in } G_f\}$ and $T = V \setminus S$



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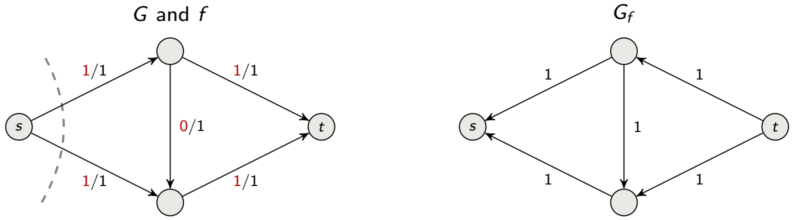

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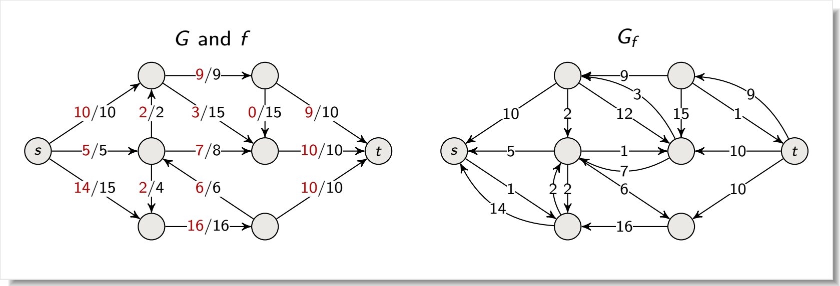

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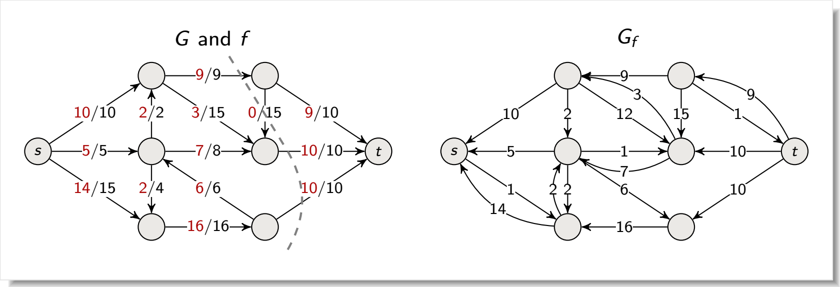

Max-flow min-cut theorem

Let $G = (V, E)$ be a flow network with source s and sink t and capacities c and a flow f.

The following are equivalent:

1. f is a maximum flow
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Proof. (1) $\Rightarrow$ (2): Suppose toward contradiction that G_f has an augmenting path p.

However, then Ford-Fulkerson method would augment f by p to obtain a flow if increased value which contradicts that f is a maximum flow

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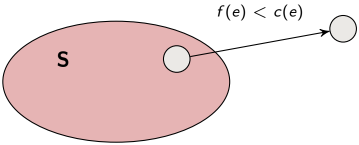
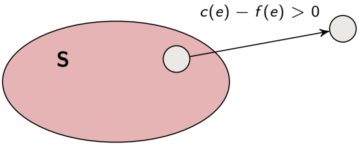
Every edge flowing out of S in G must be at capacity, otherwise we can reach a node outside S in the residual network.



Original graph



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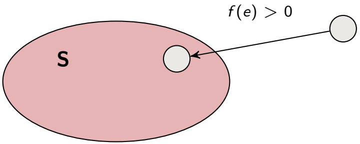
Every edge flowing into S in G must have flow 0, otherwise we can reach a node outside S in the residual network.



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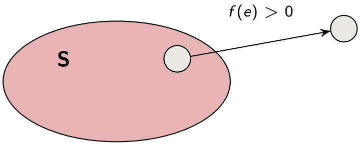

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Therefore

$$|f| = f(S, T)$$

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So f is a maximum flow

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TIME FOR FINDING MAX-FLOW (OR MIN-CUT)

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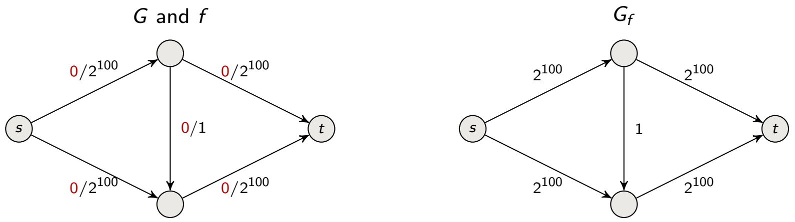
- Running time is $O(E \cdot |f_{\max}|)$ where $|f_{\max}|$ denotes the value of a maximum flow

Problematic case



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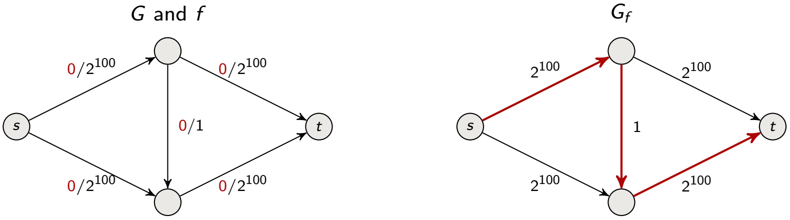

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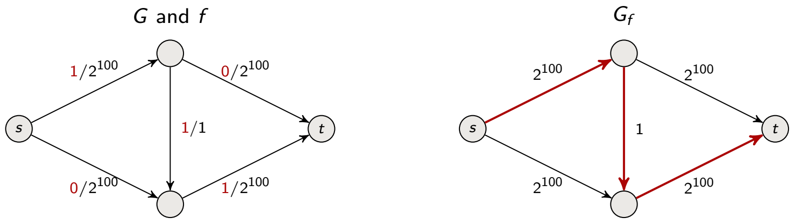

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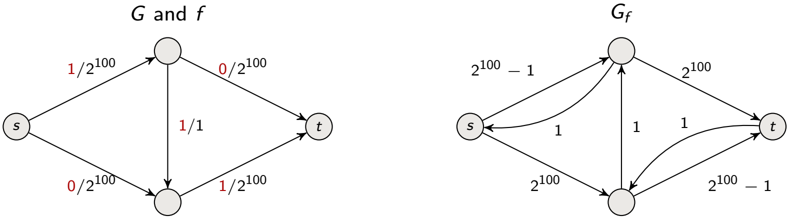

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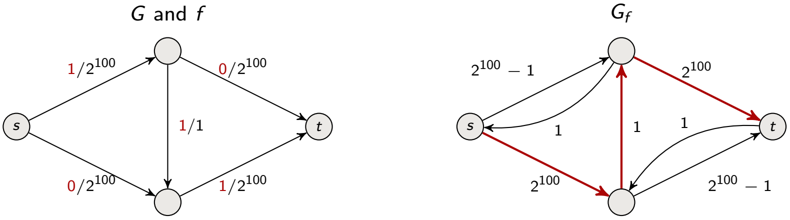

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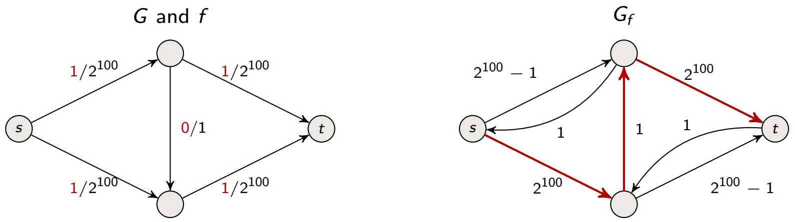

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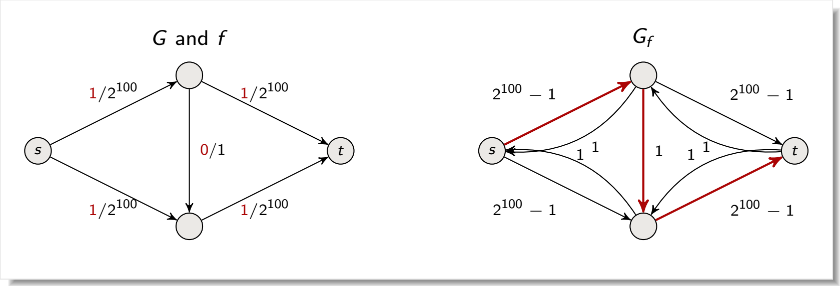

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- The sun stops to shine
-
- Something happens to the universe
-
-
-
-
-

Problematic case

- you graduate
- I retire
-
- The sun stops to shine
-
- Something happens to the universe
-
-
-
-
-
- Our algorithm returns a max-flow

Even more bad news

If capacities are irrational then the Ford-Fulkerson method might not terminate

Good news

If we either take the shortest path or the fattest path then this will not happen if the capacities are integers without proof

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BFS shortest path	$\leq \frac{1}{2} E \cdot V$
Fattest path	$\leq E \cdot \log(E \cdot U)$

Good news

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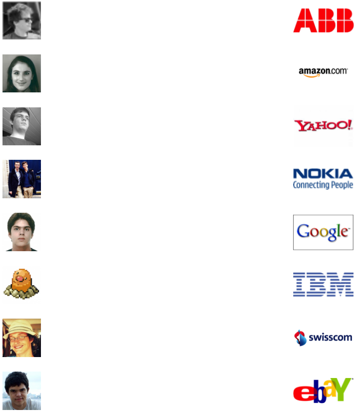
BFS shortest path	$\leq \frac{1}{2} E \cdot V$
Fattest path	$\leq E \cdot \log(E \cdot U)$

- U is the maximum flow value
- Fattest path: choose augmenting path with largest minimum capacity (bottleneck)

APPLICATIONS OF MAX-FLOW

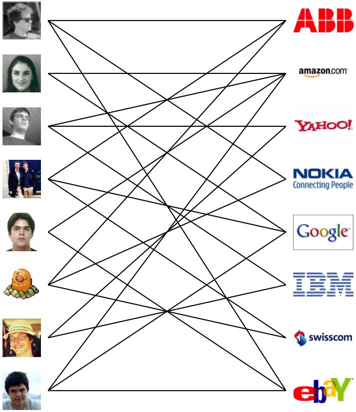
Bipartite matching

- N students apply for M jobs



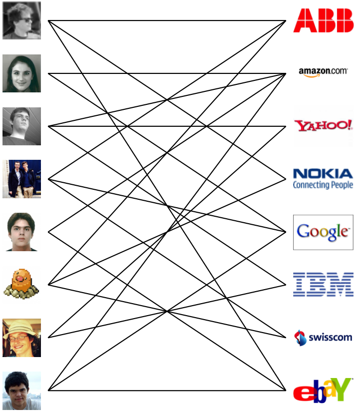
Bipartite matching

- N students apply for M jobs
- Each get several offers

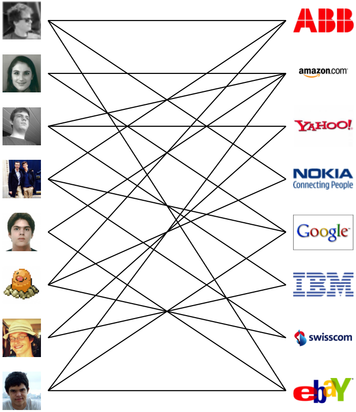


Bipartite matching

- N students apply for M jobs
- Each get several offers
- Is there a way to match all students to jobs? obviously M has to be at least equal to N

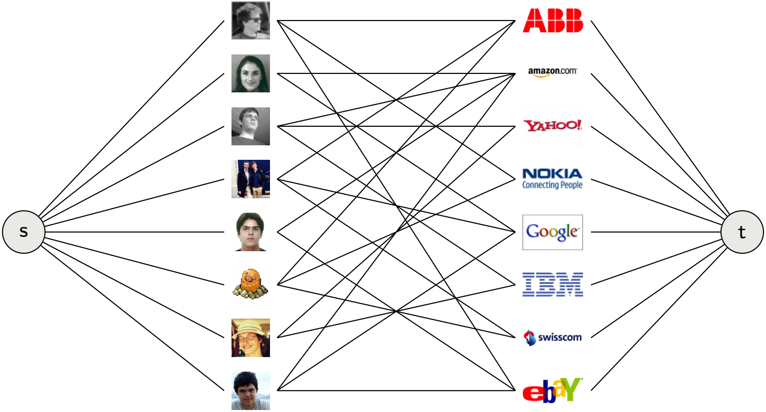


Bipartite matching as flow problem



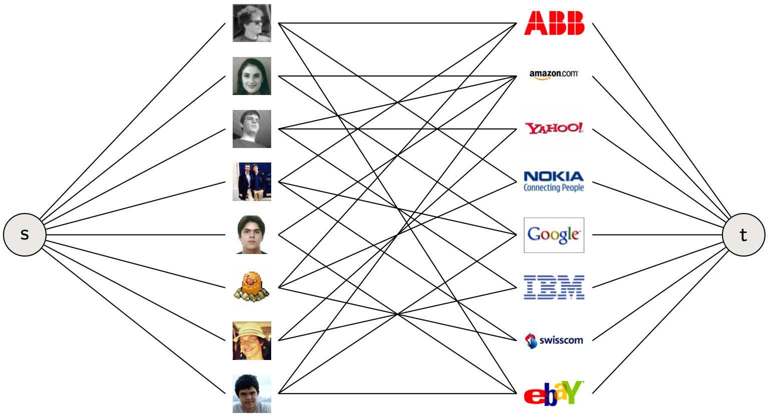
Bipartite matching as flow problem

- Add source s and sink t with edges from s to students and from jobs to t



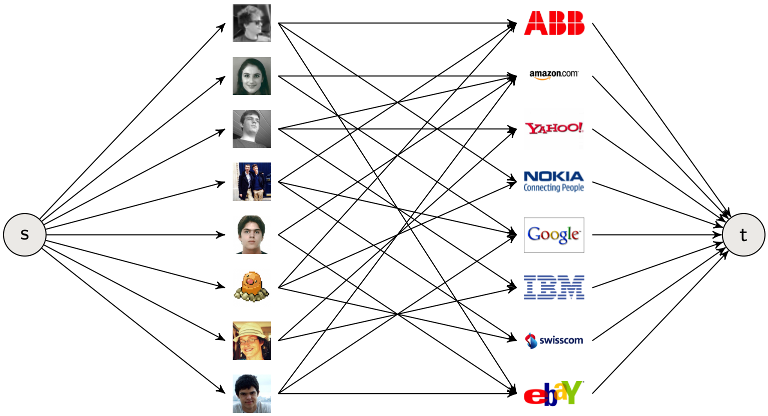
Bipartite matching as flow problem

- Add source s and sink t with edges from s to students and from jobs to t
- All edges have capacity one



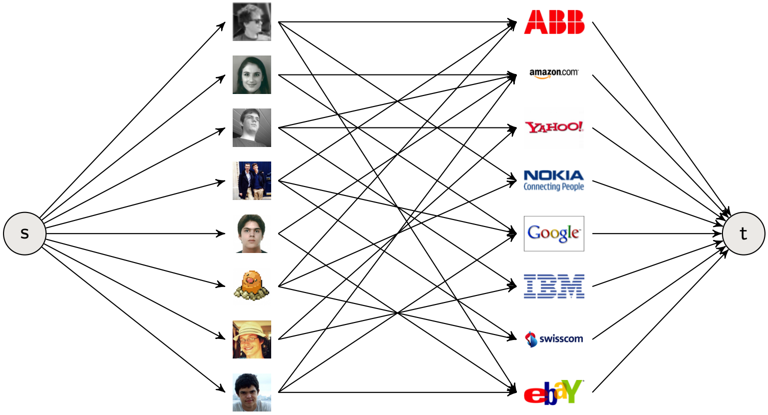
Bipartite matching as flow problem

- Add source s and sink t with edges from s to students and from jobs to t
- All edges have capacity one
- Direction is from left to right



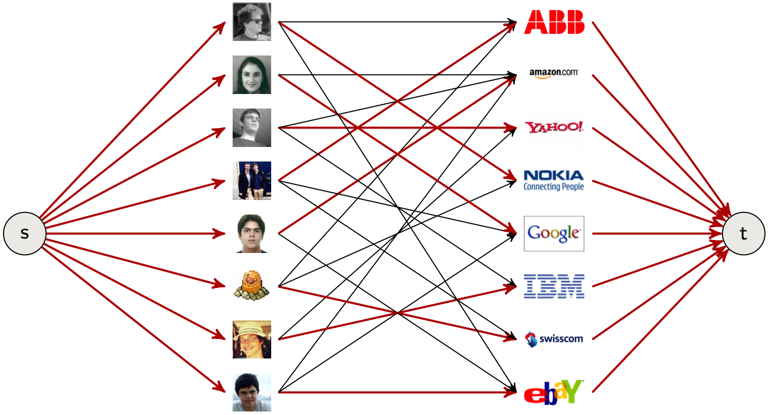
Bipartite matching as flow problem

- Run the Ford-Fulkerson method



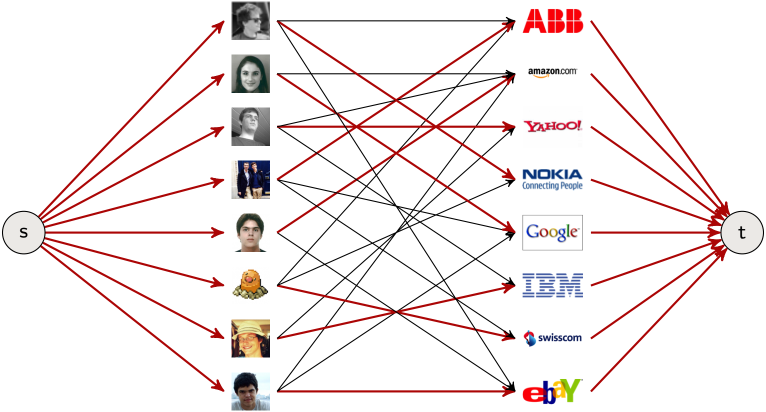
Bipartite matching as flow problem

- Run the Ford-Fulkerson method



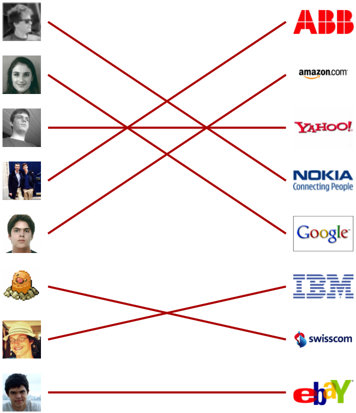
Bipartite matching as flow problem

- Run the Ford-Fulkerson method
- Matching is complete



Bipartite matching as flow problem

- Run the Ford-Fulkerson method
- Matching is complete



Why does it work?

Every matching defines a flow of value equal to the number of edges in matching

- Put flow 1 on
  - Edges of the matching
  - Edges from s to matched student nodes
  - Edges from matched job nodes to t
- Put flow 0 on all other edges

Works because flow conservation is equivalent to: no student is matched more than once, no job is matched more than once

Why does it work?

Every flow during the algorithm defines a matching of size equal to its value

- Flows obtained by Ford-Fulkerson are integer valued if capacities are integral, so value on every edge is 0 or 1
- Edges between students and jobs with flow 1 are a matching by flow conservation
  - There cannot be more than one edge with flow 1 from a student node
  - There cannot be more than one edge with flow 1 into a job node

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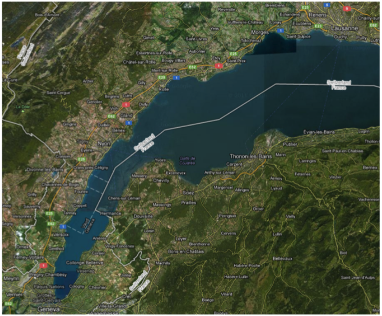
So, maximum flow is a maximum matching!

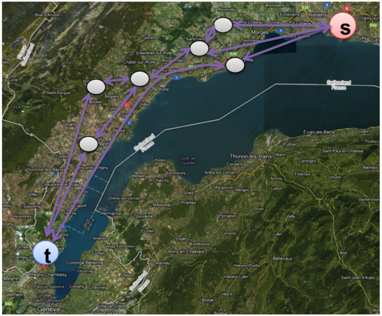
Edge-disjoint paths

- You want to travel to a nice location these winter holidays
- You need to drive from Lausanne to Geneva airport
- Winter season $\Rightarrow$ risk that roads are closed
- How many different routes can you take that does not share a common road?



Edge-disjoint paths as flow network

- s = Lausanne
- t = Geneva airport
- An edge capacity of 1 in both directions for each road
- (make anti-parallel using gadgets)



Solution

- max-flow = # edge-disjoint paths
- min-cut = min #roads to be closed so that there is no route from Lausanne to Geneva airport