Algorithms: FLOWS AND CUTS

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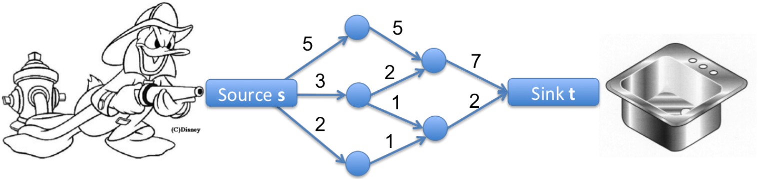
School of Computer and Communication Sciences

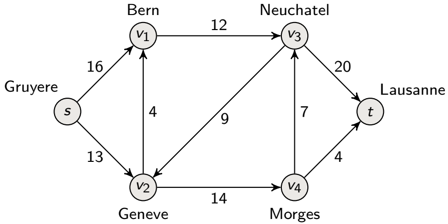
Lecture 16, 15.04.2025



FLOW NETWORKS

Flow Network

Transfer as much cheese as possible from Gruyere to Lausanne



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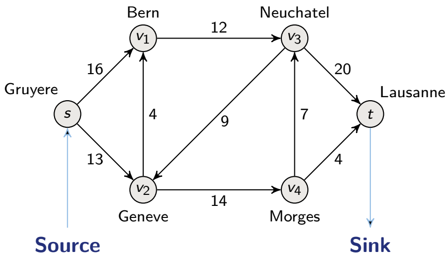

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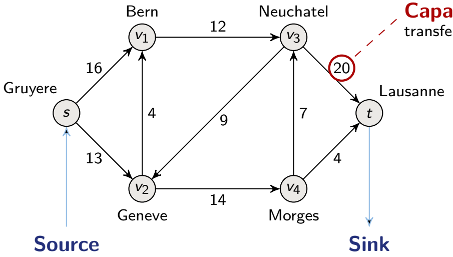

Capacity: at most 20 cheeses can be transferred from Neuchatel to Lausanne

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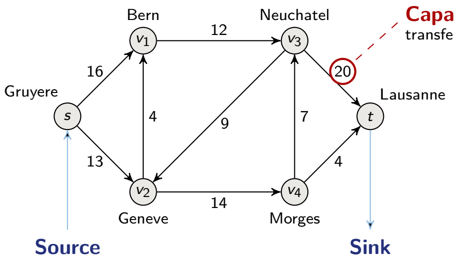

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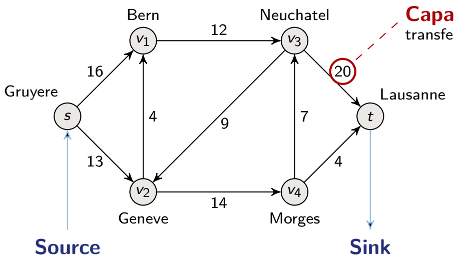

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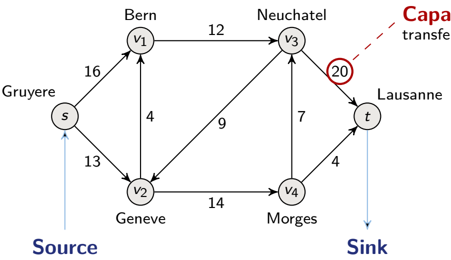
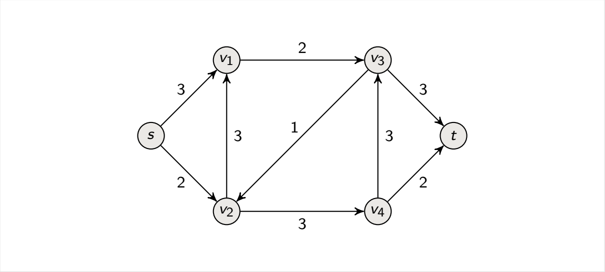


Capacity: at most 20 cheeses can be transferred from Neuchatel to Lausanne

- a graph to model flow through edges (pipes)
- each edge has a capacity an upper bound on the flow rate (pipes have different sizes)
- Want to maximize rate of flow from the source to the sink

Flow Network (formally)



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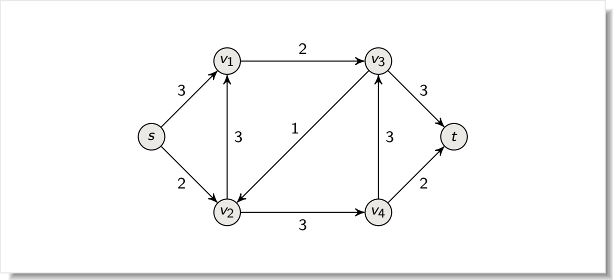

- Directed graph $G = (V, E)$

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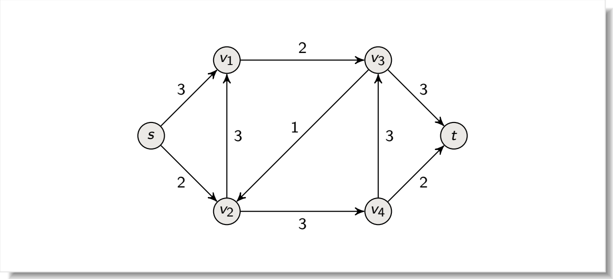

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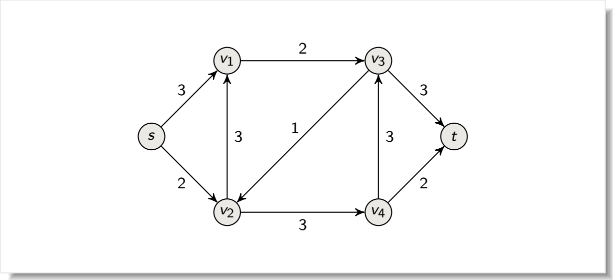

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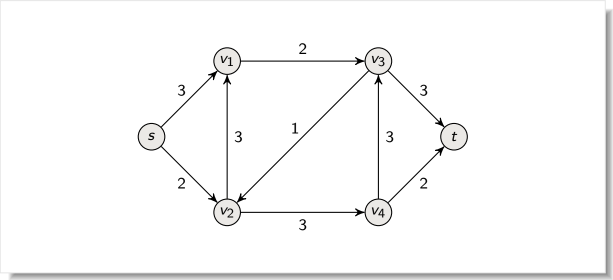

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- Source s and sink t (flow goes from s to t)
- No antiparallel edges (assumed w.l.o.g. for simplicity)

Why is “no antiparallel edges” w.l.o.g.?

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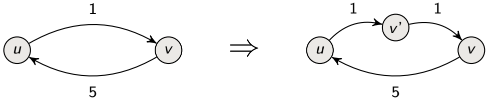

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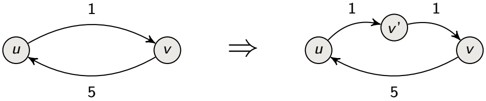

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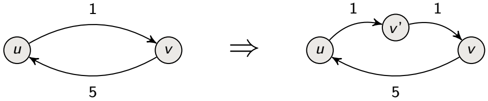

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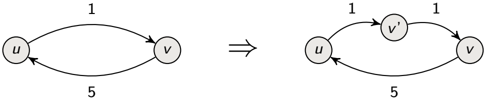

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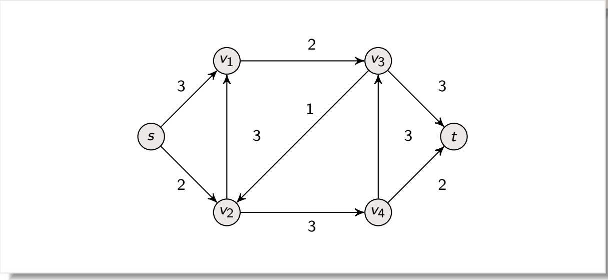
- Create a new vertex v'
- Replace (u, v) by two new edges (u, v') and (v', v) with $c(u, v') = c(v', u) = c(u, v)$
- Repeat this $O(E)$ times to get an equivalent flow network with no antiparallel edges.

Definition of a flow



A flow is a function $f : V \times V \to \mathbb{R}$ satisfying:

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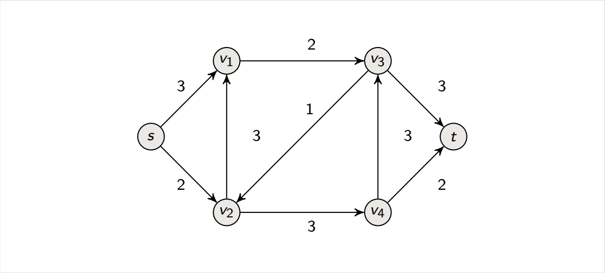

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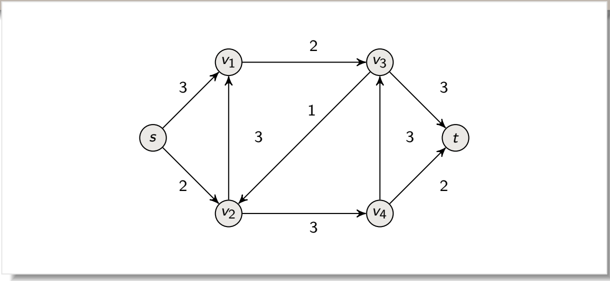

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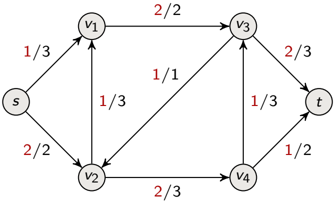

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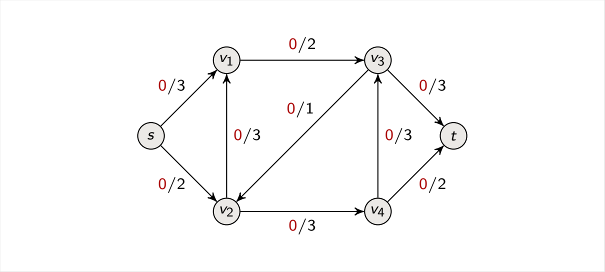

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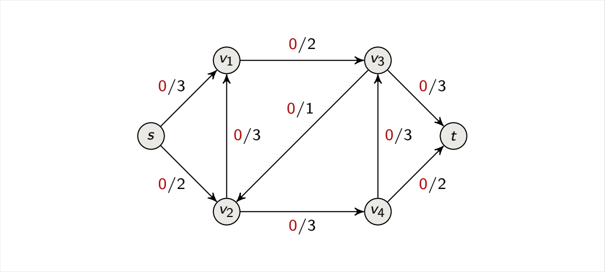
Value of a flow



Value of a flow $f = |f|$

$$= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

$=$ flow out of source $-$ flow into source

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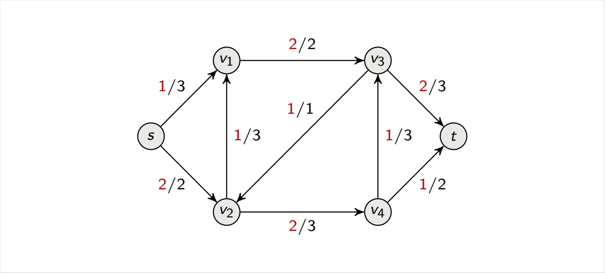

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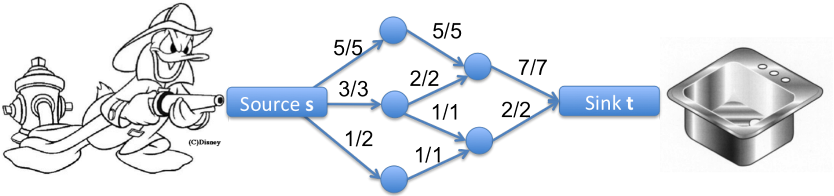
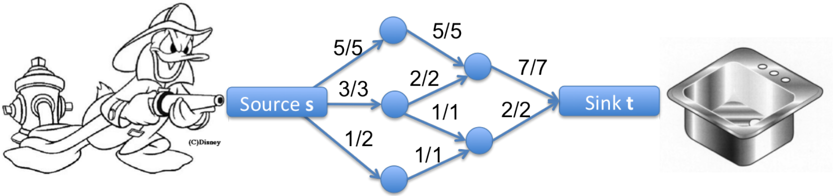
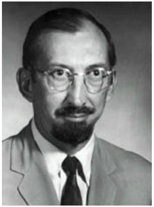


What’s the value of this flow? 9



L. R. Ford, Jr. (1927-)



D, R, Fulkerson (1924-1976)

MAXIMUM-FLOW PROBLEM

Ford-Fulkerson Method

The Ford-Fulkerson Method'54

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FORD-FULKERSON-METHOD($G, s, t$):

Initialize flow $f$ to 0
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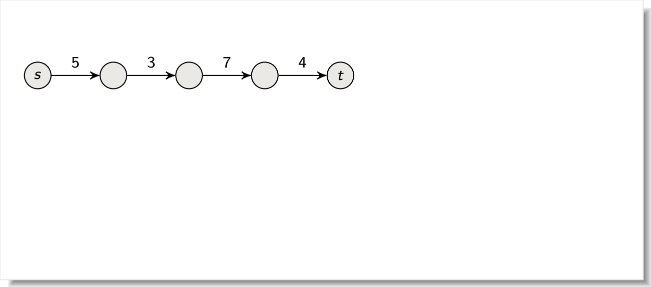

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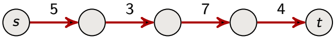

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$\Rightarrow$ Push flow on **p**

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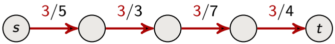

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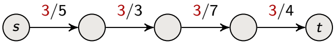

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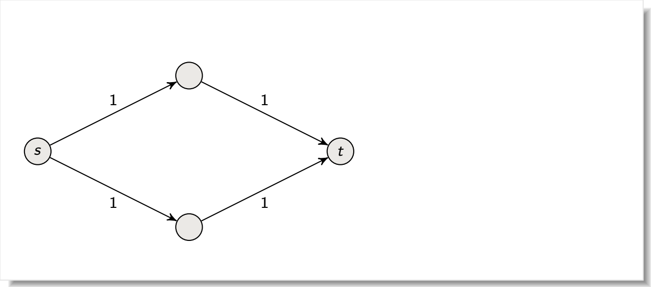

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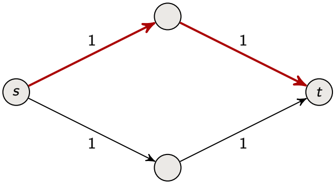

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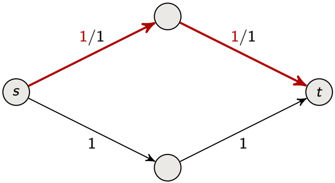

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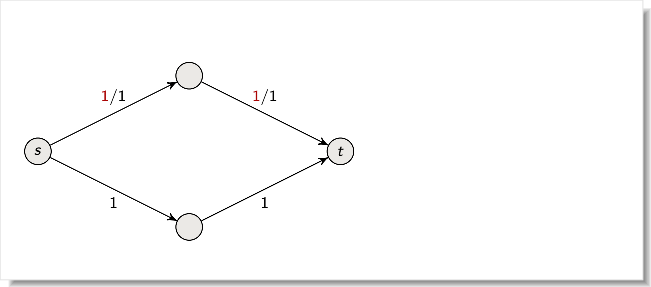

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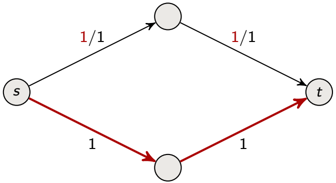

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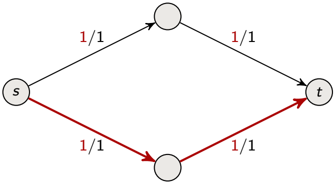

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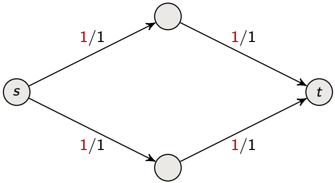

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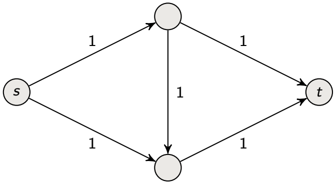

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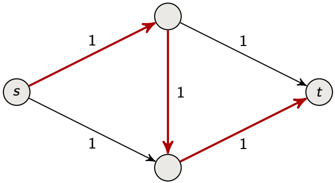

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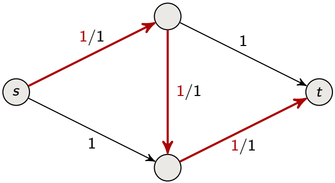

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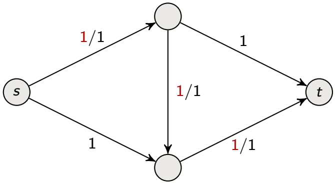

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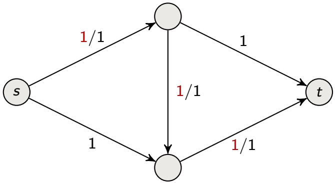

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☹

What went wrong? How can we fix it?

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Residual network

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Amount of capacity left

Amount of flow that can be reversed

Residual network

- Given a flow f and a network $G = (V, E)$
- the residual network consists of edges with capacities that represent how we can change the flow on the edges

Residual capacity:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \longleftarrow \boxed{\text{Amount of capacity left}} \\ f(v, u) & \text{if } (v, u) \in E \longleftarrow \boxed{\text{Amount of flow that can be reversed}} \\ 0 & \text{otherwise} \end{cases}$$

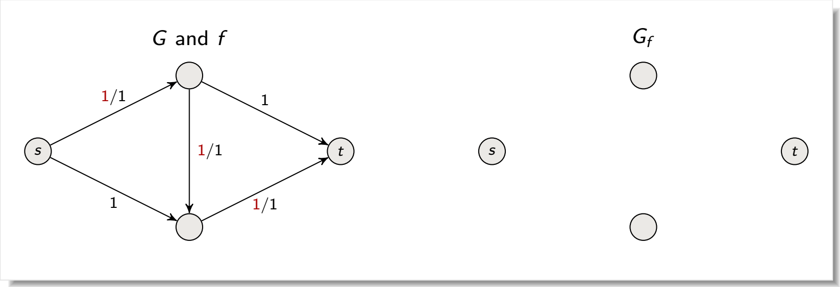
Residual network:

$G_f = (V, E_f)$ where $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$

Examples

Residual network: $G_f = (V, E_f)$ where $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$ and

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$



Examples

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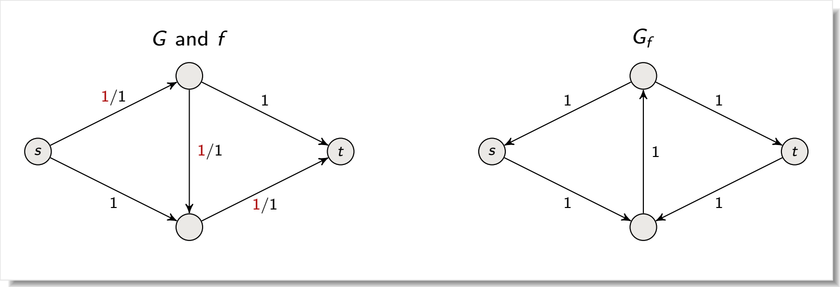

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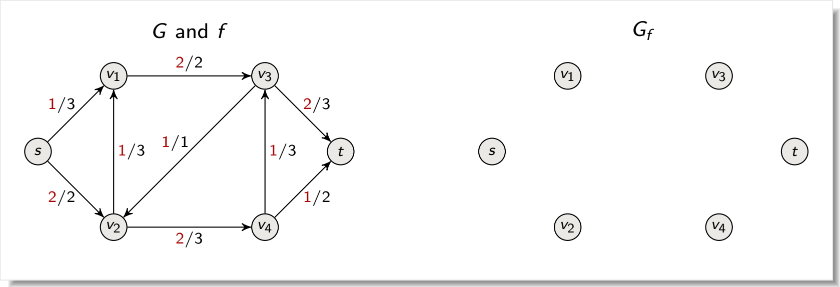

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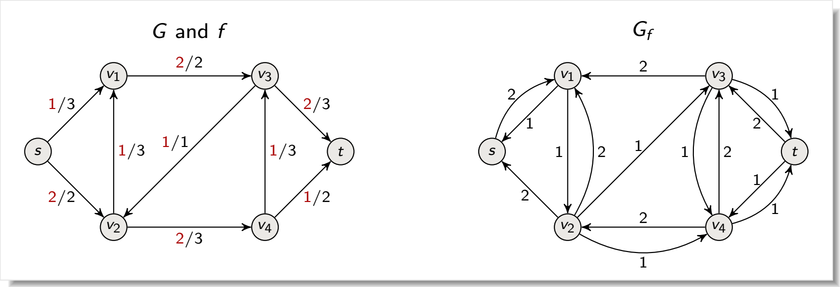

The Ford-Fulkerson Method'54

```
Ford-Fulkerson-Method($G, s, t$):

1. Initialize flow $f$ to 0
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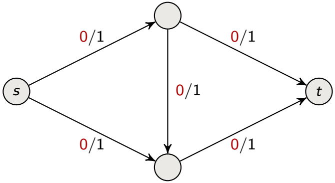
G and f



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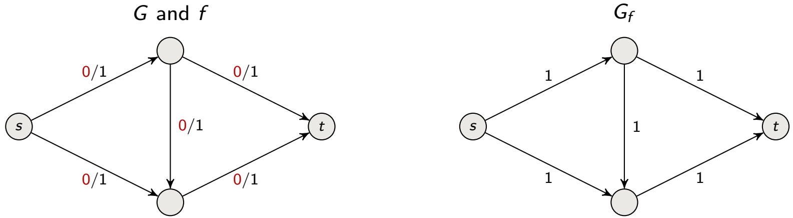
Augmenting path = simple path from s to t



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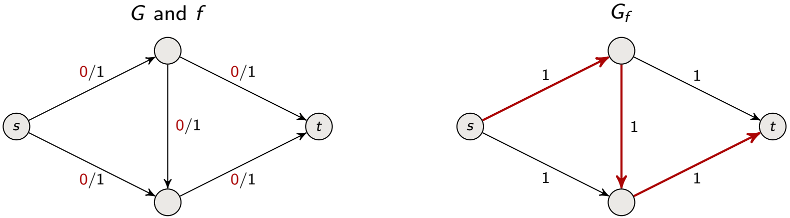
Exists augmenting path **p**



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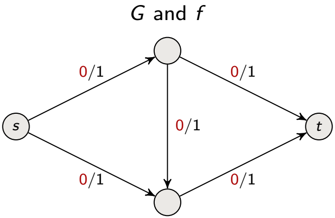
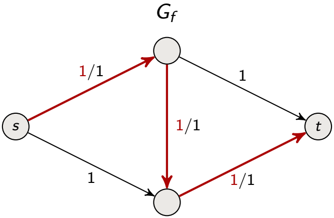
Exists augmenting path **p** with flow $\mathbf{f_p}$ of value = min capacity on **p**





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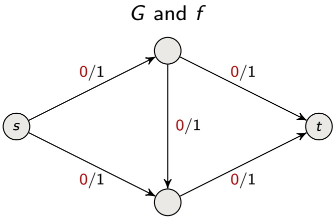

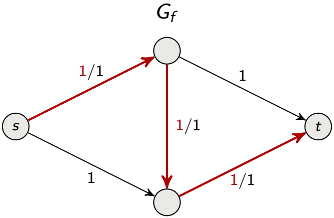

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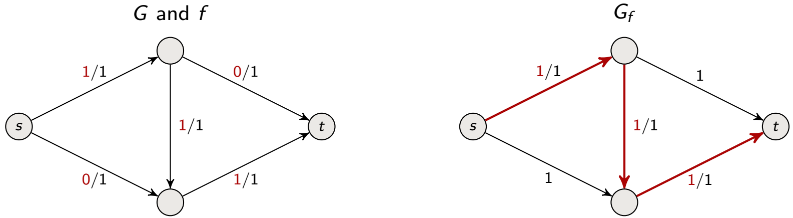

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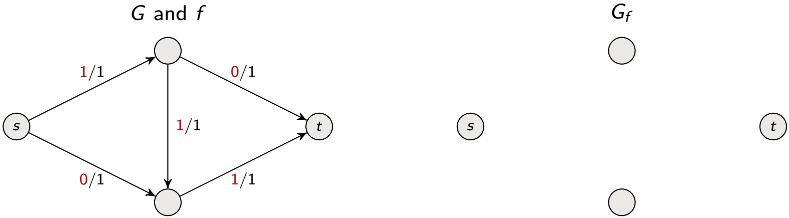

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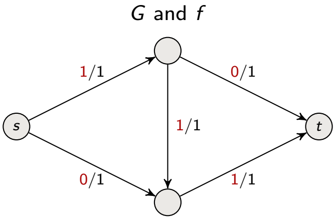

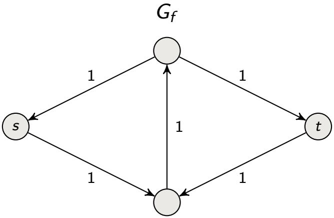

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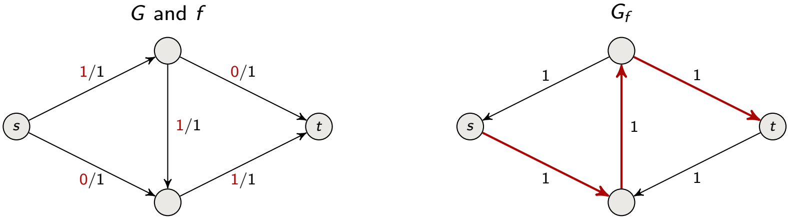

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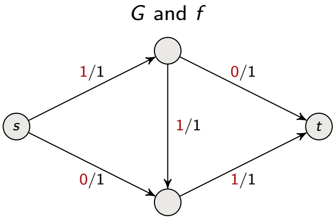

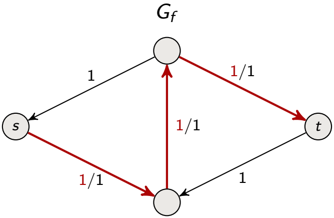

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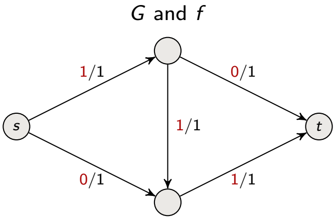

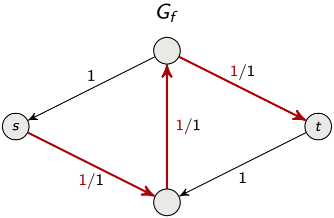

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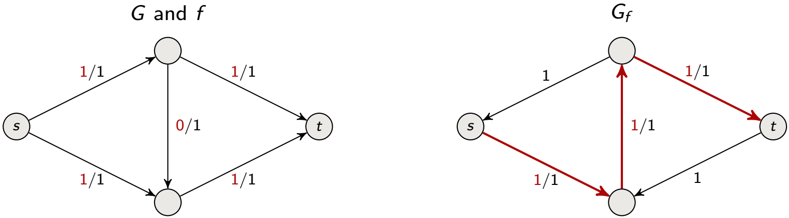

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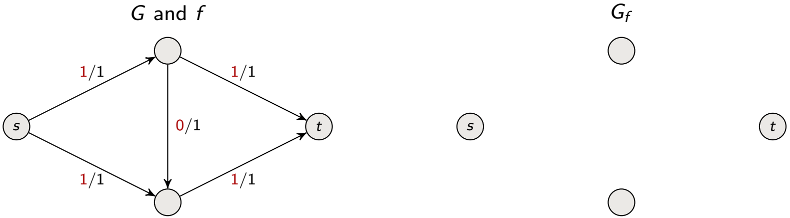

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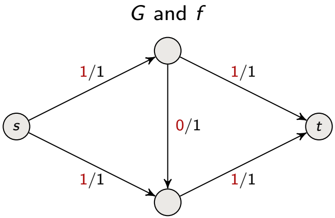

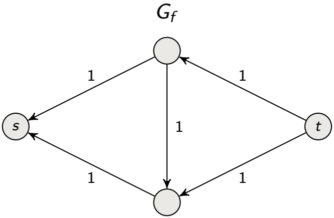

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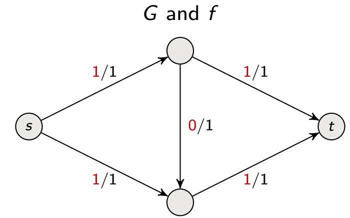
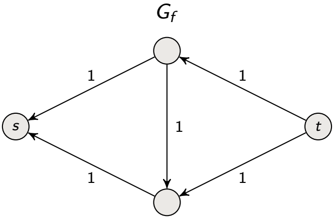
No augmenting path and flow of value 2 is optimal





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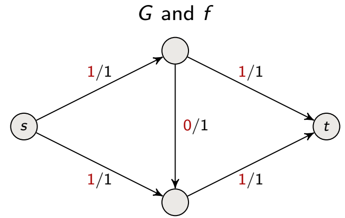

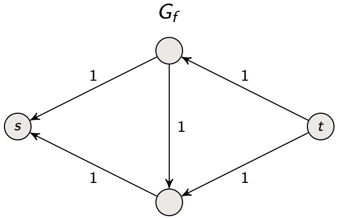

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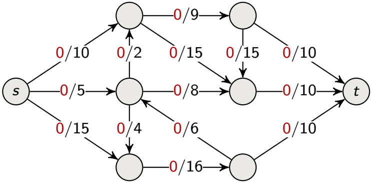
G and f



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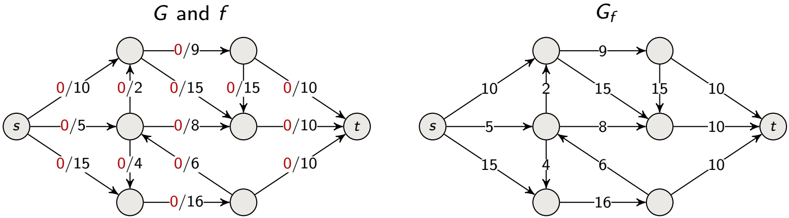

G and f

G_f

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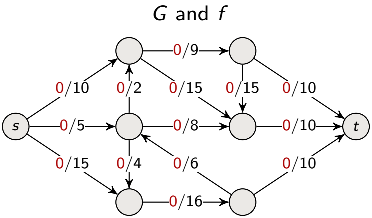

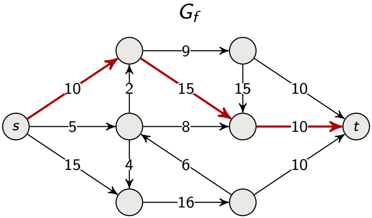

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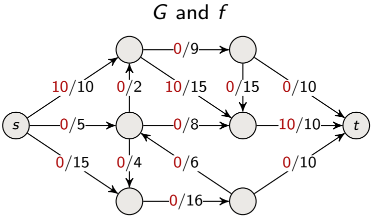

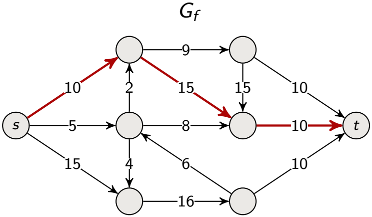

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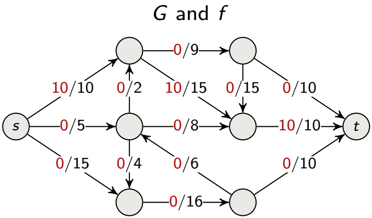

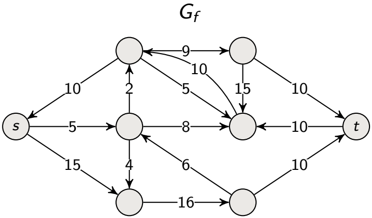

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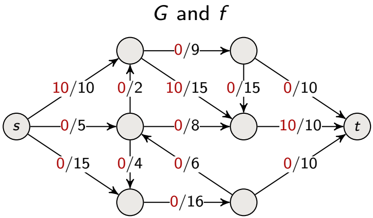

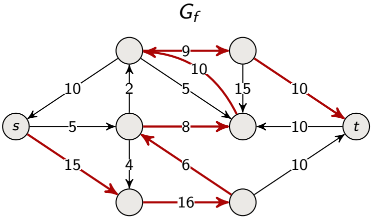

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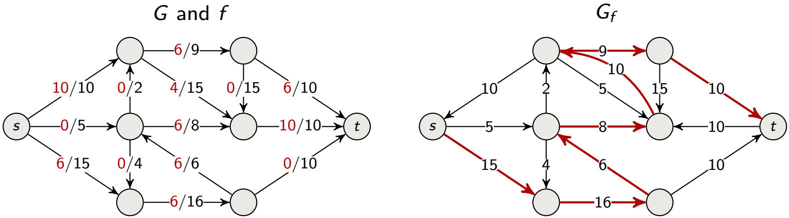

G and f

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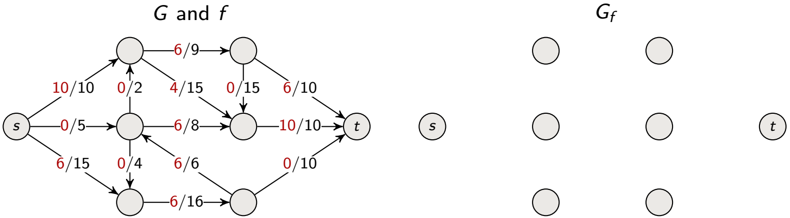

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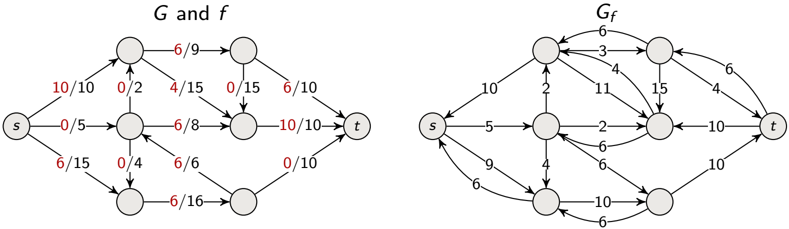

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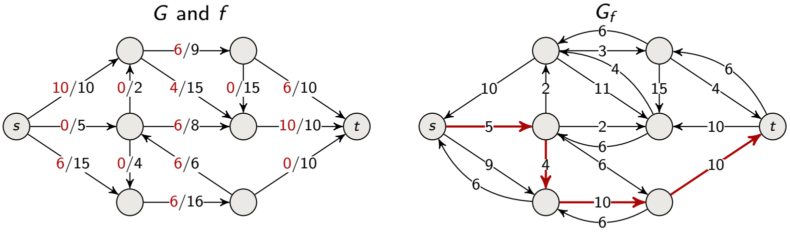

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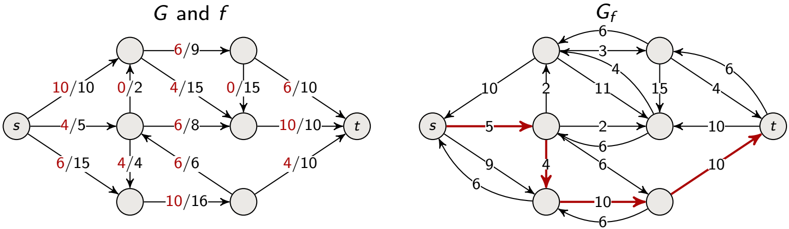

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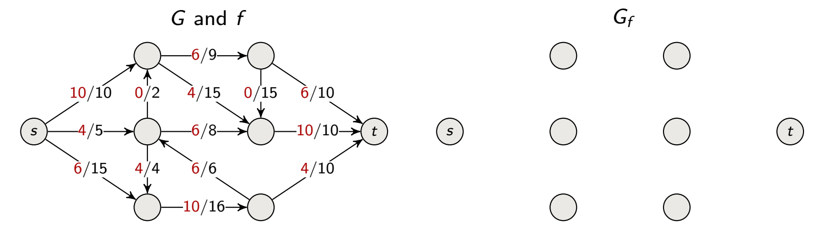

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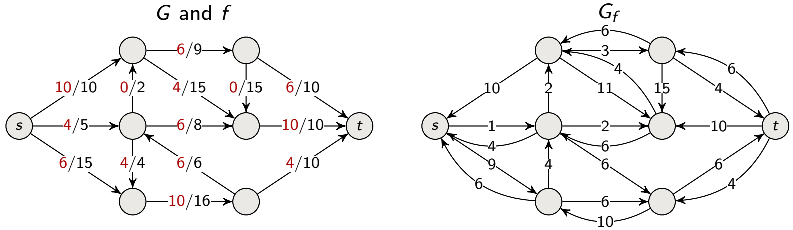

The Ford-Fulkerson Method'54

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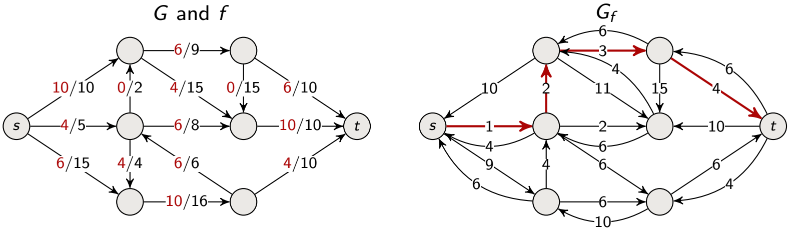

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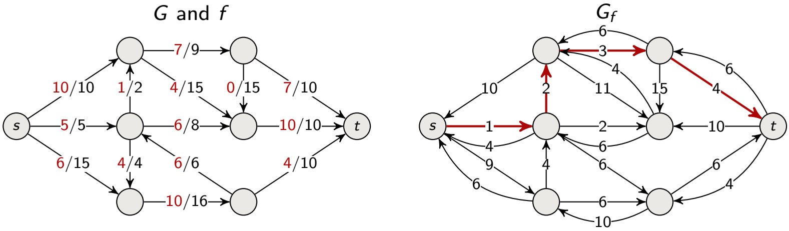

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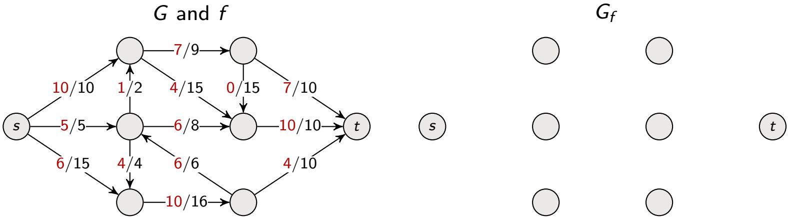

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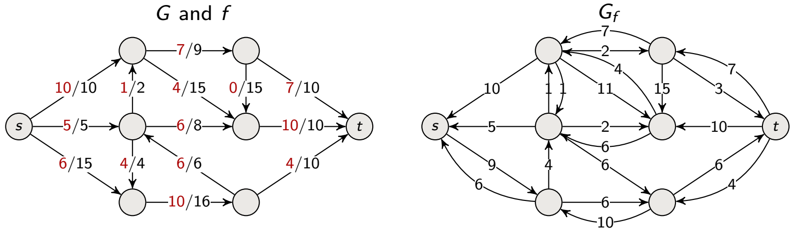

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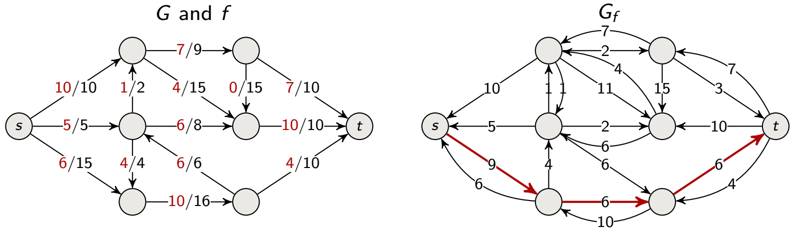

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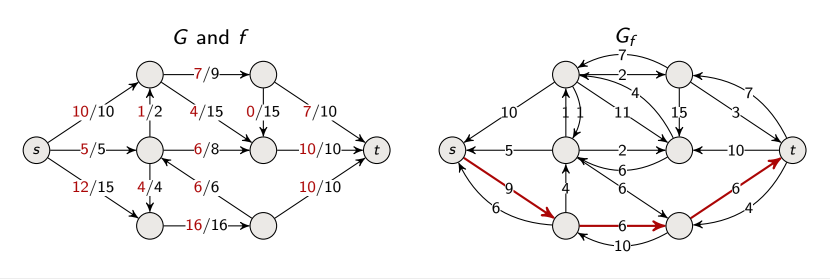

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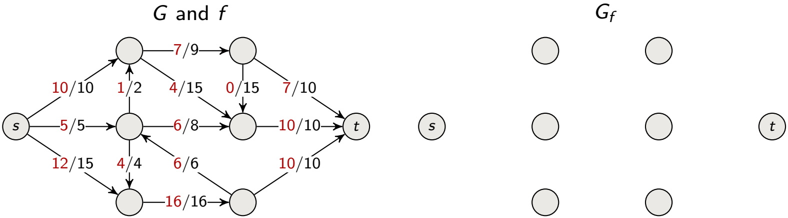

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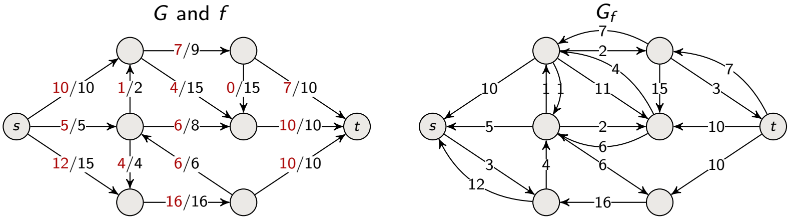

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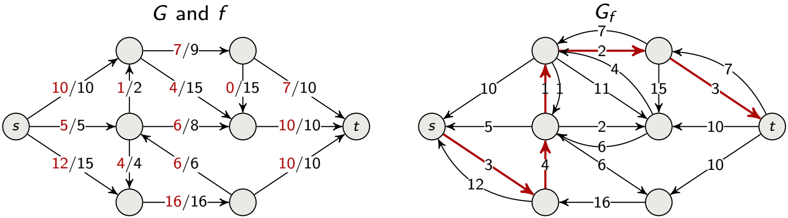

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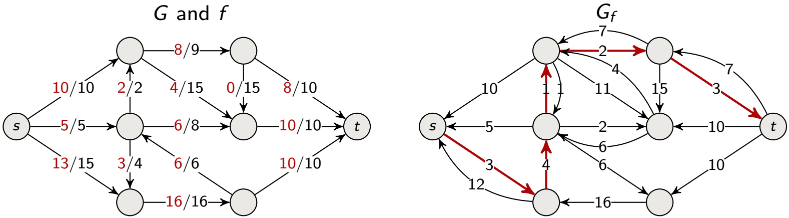
G and f

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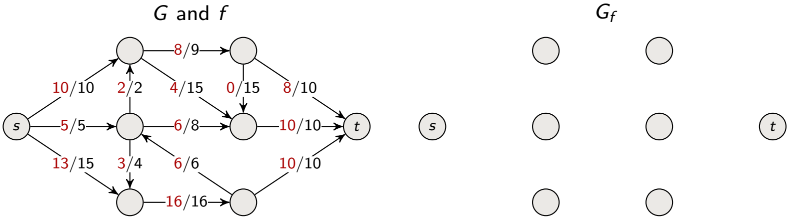

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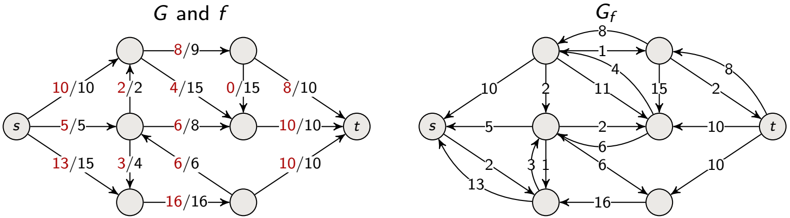

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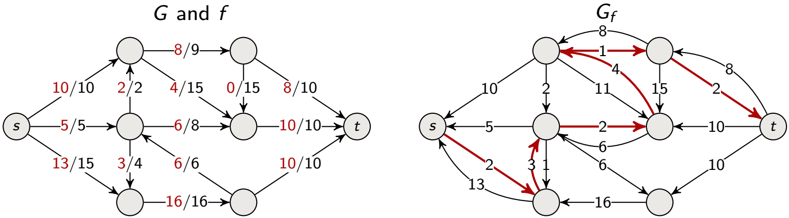

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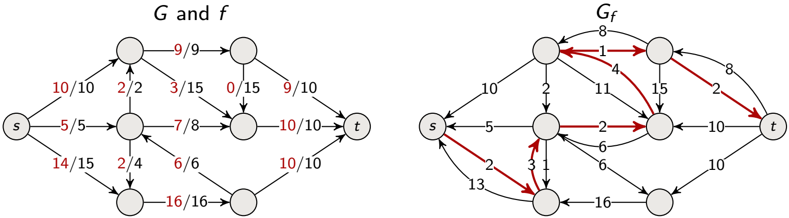

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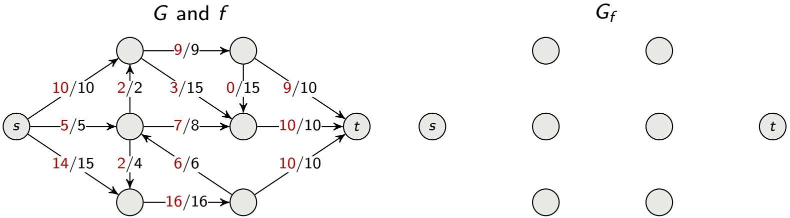

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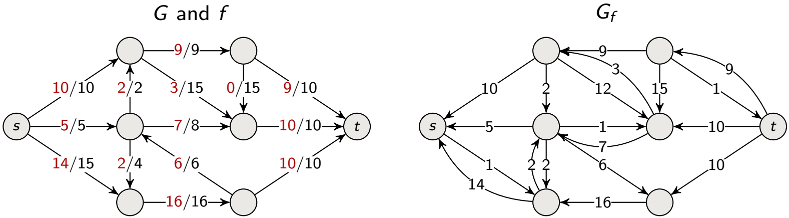

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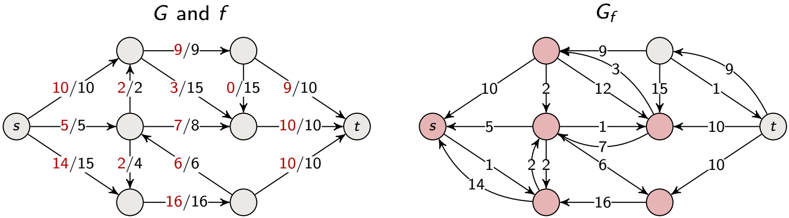

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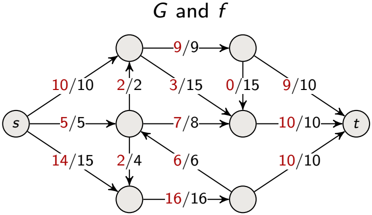
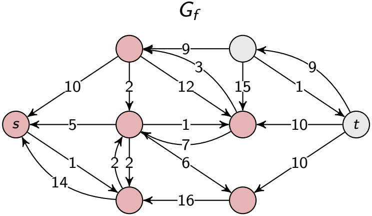
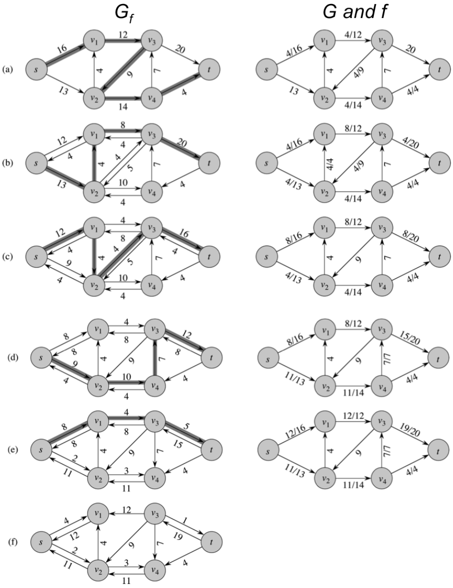
No augmenting path and flow of value 29 is optimal





Study and understand Example!


G_f

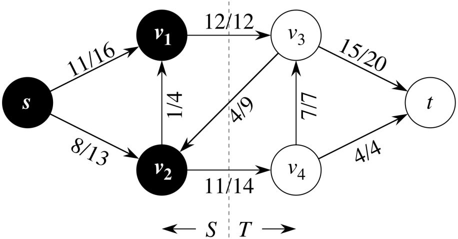
G and f

WHY IS RETURNED FLOW OPTIMAL? (MIN-CUTS)

Cuts in flow networks

A cut of flow network $G(V, E)$ is

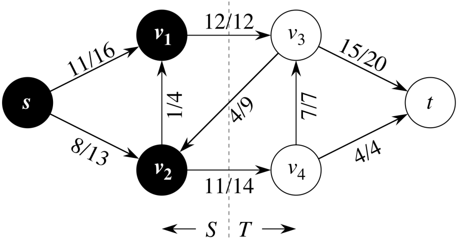
- a partition of V into S and $T = V \setminus S$
- such that $s \in S$ and $t \in T$



Net flow across a cut

The net flow across cut (S, T) is

$$f(S,T) = \underbrace{\sum_{u \in S, v \in T} f(u,v)}_{\text{flow leaving } S} - \underbrace{\sum_{u \in S, v \in T} f(v,u)}_{\text{flow entering } S}$$



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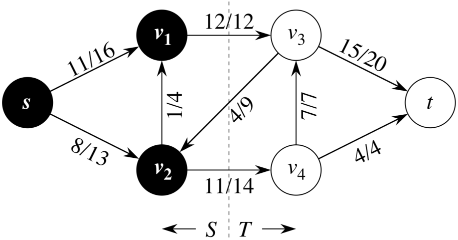
What is the net flow of this cut?



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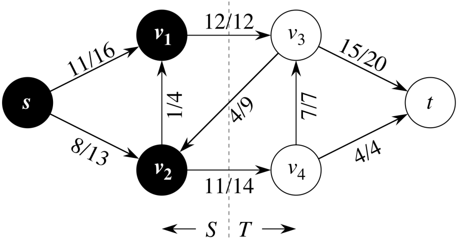
What is the net flow of this cut? $12 + 11 - 4 = 19$



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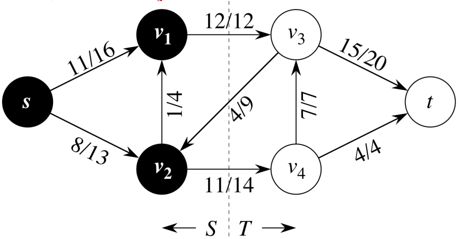
What is the net flow of this cut? $12 + 11 - 4 = 19$ Note that this equals the value of the flow; it's always the case!



Net flow equals flow value for any cut

Theorem

For any cut (S, T), $|f| = f(S, T)$.

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Proof by induction on the size of S.

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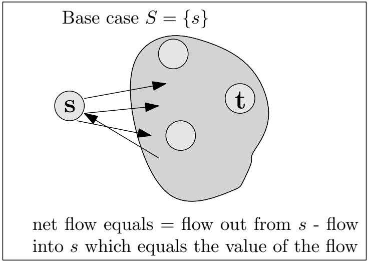
Base case $S = \{s\}$
s
t
net flow equals = flow out from s - flow into s which equals the value of the flow

Net flow equals flow value for any cut

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For any cut (S, T)*,* $|f| = f(S, T)$*.*

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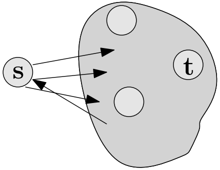

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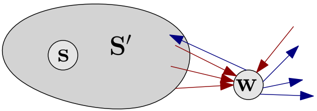
Inductive Step $S = S' \cup \{w\}$

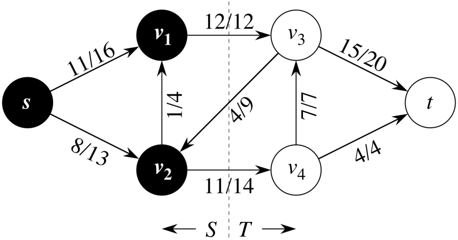


New net flow = Old net flow + $\underbrace{\text{flow on blue edges - flow on red edges}}_{\text{0 by flow conservation}}$

Capacity a cut

The capacity of a cut (S, T) is

$$c(S, T) = \sum_{u \in S, v \in T} c(u, v)$$



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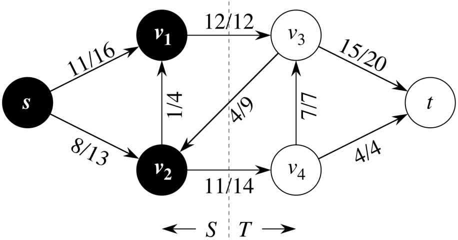
What is the capacity of this cut?



Capacity a cut

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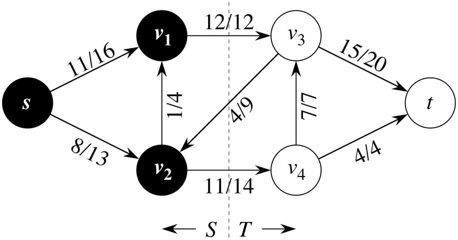
What is the capacity of this cut? $12 + 14 = 26$



Flow is at most capacity of a cut

For any flow f and any cut (S, T):

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Max-flow is at most capacity of a cut

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Therefore: **max-flow** $\leq$ **min-cut**

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We shall prove

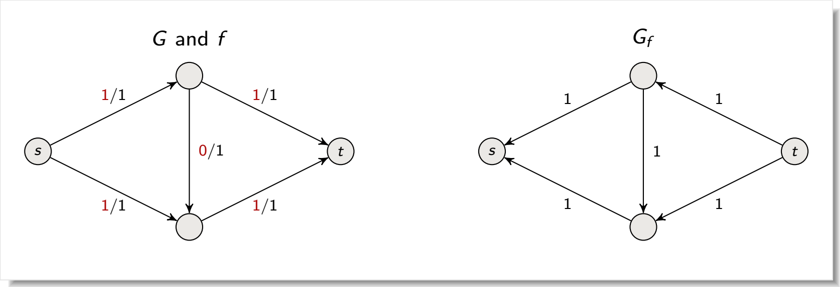
Theorem (max-flow min-cut theorem)

max-flow = min-cut

Examples

Consider f obtained by running Ford-Fulkerson and let

$S = \{v \in V : \text{there is a path from } s \text{ to } v \text{ in } G_f\}$ and $T = V \setminus S$



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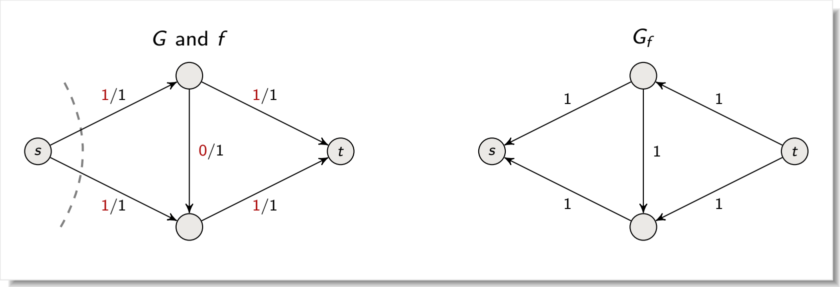

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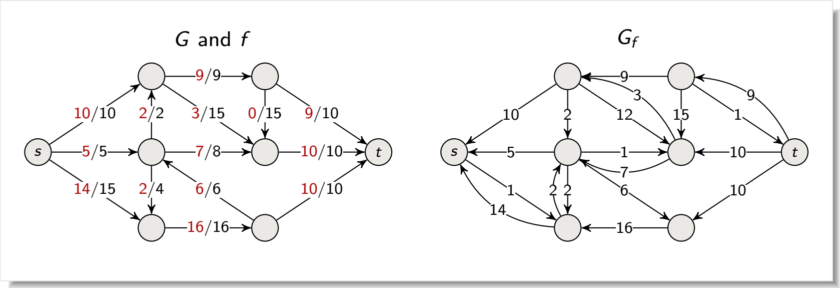

G and f

G_f

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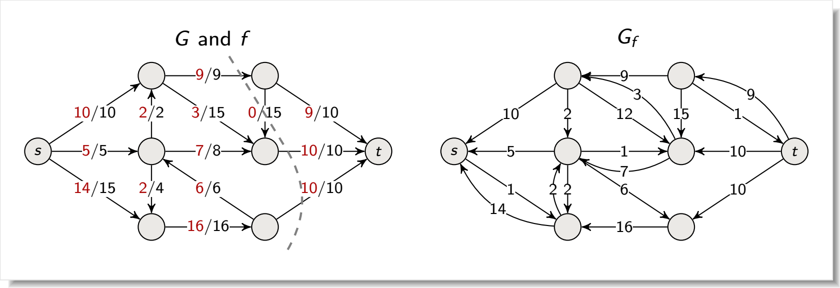

Max-flow min-cut theorem

Let $G = (V, E)$ be a flow network with source s and sink t and capacities c and a flow f.

The following are equivalent:

1. f is a maximum flow
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Proof.

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Proof. (1) $\Rightarrow$ (2): Suppose toward contradiction that G_f has an augmenting path p.

However, then Ford-Fulkerson method would augment f by p to obtain a flow if increased value which contradicts that f is a maximum flow

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Proof. (2) ⇒ (3): $S =$ set of nodes reachable from s in residual network, $T = V \setminus S$

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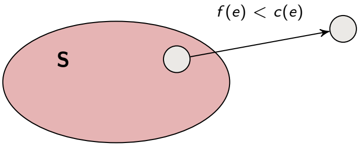
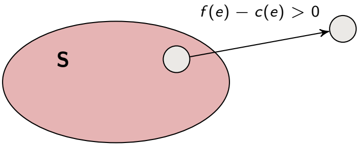
Proof. (2) $\Rightarrow$ (3): $S =$ set of nodes reachable from s in residual network, $T = V \setminus S$

Every edge flowing out of S in G must be at capacity, otherwise we can reach a node outside S in the residual network.



Original graph



Residual network

Max-flow min-cut theorem

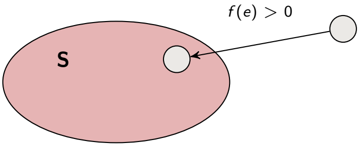
Let $G = (V, E)$ be a flow network with source s and sink t and capacities c and a flow f.

The following are equivalent:

1. f is a maximum flow
2. G_f has no augmenting path
3. $|f| = c(S, T)$ for a minimum cut (S, T)

Proof. (2) $\Rightarrow$ (3): $S =$ set of nodes reachable from s in residual network, $T = V \setminus S$

Every edge flowing into S in G must have flow 0, otherwise we can reach a node outside S in the residual network.



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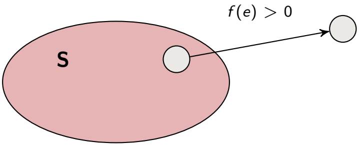

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$$\begin{aligned} |f| &= f(S, T) \\ &= \sum_{u \in S, v \in T} f(u, v) - \sum_{u \in S, v \in T} f(v, u) \end{aligned}$$

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Proof. (3) $\Rightarrow$ (1): Recall that $|f| \leq c(S, T)$ for all cuts (S, T).

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So f is a maximum flow

Summary

- Flow Networks
- Ford-Fulkerson Method
- Cuts
- Max-flow $=$ min cut theorem