Algorithms: FLOWS AND CUTS

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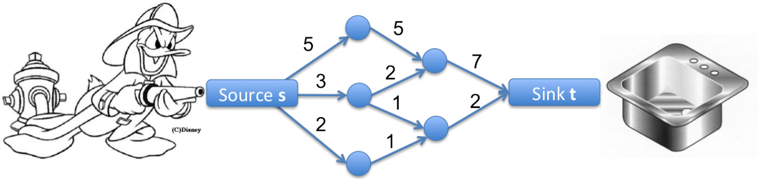
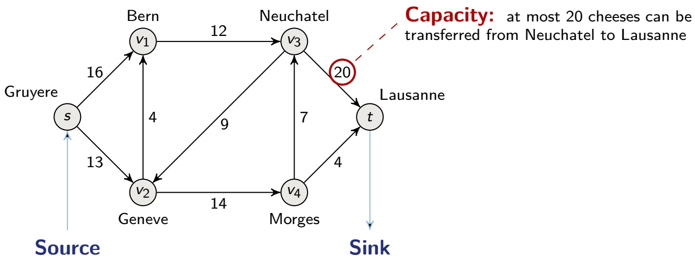


FLOW NETWORKS

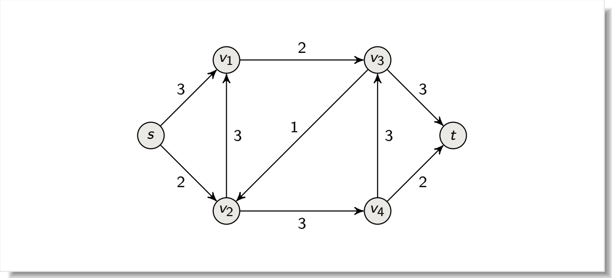
Flow Network

Transfer as much cheese as possible from Gruyere to Lausanne



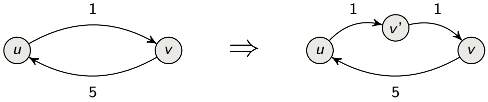
- a graph to model flow through edges (pipes)
- each edge has a capacity an upper bound on the flow rate (pipes have different sizes)
- Want to maximize rate of flow from the source to the sink

Flow Network (formally)



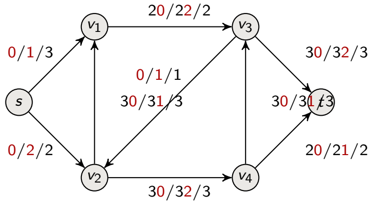
- Directed graph $G = (V, E)$
- Each edge (u, v) has a capacity $c(u, v) \geq 0$ ($c(u, v) = 0$ if $(u, v) \notin E$)
- Source s and sink t (flow goes from s to t)
- No antiparallel edges (assumed w.l.o.g. for simplicity)

Why is “no antiparallel edges” w.l.o.g.?



- If there are two parallel edges (u, v) and (v, u), choose one of them say (u, v)
- Create a new vertex v'
- Replace (u, v) by two new edges (u, v') and (v', v) with $c(u, v') = c(v', u) = c(u, v)$
- Repeat this $O(E)$ times to get an equivalent flow network with no antiparallel edges.

Definition of a flow



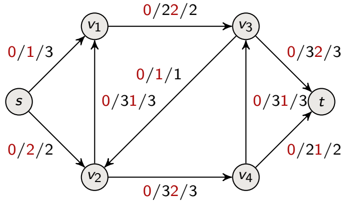
A flow is a function $f : V \times V \to \mathbb{R}$ satisfying:

Capacity constraint: For all $u, v \in V$: $0 \leq f(u, v) \leq c(u, v)$

Flow conservation: For all $u \in V \setminus \{s, t\}$,

$$\underbrace{\sum_{v \in V} f(v, u)}_{\text{flow into } u} = \underbrace{\sum_{v \in V} f(u, v)}_{\text{flow out of } u}$$

Value of a flow

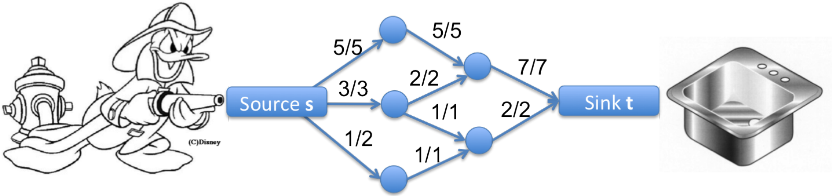


Value of a flow $f = |f|$

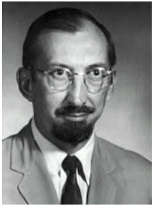
$$= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

$$= \text{flow out of source} - \text{flow into source}$$

What's the value of this flow? 9



L. R. Ford, Jr. (1927-)



D, R, Fulkerson (1924-1976)

MAXIMUM-FLOW PROBLEM

Ford-Fulkerson Method

The Ford-Fulkerson Method'54

FORD-FULKERSON-METHOD(G, s, t):

1. Initialize flow f to 0
2. **while** exists an **augmenting path** p in the **residual network** G_f
3. **augment flow** f along p
4. **return** f

Basic idea:

- As long as there is a path from source to sink, with available capacity on all edges in the path
- send flow along one of these paths and then we find another path and so on

Applying the basic idea to examples

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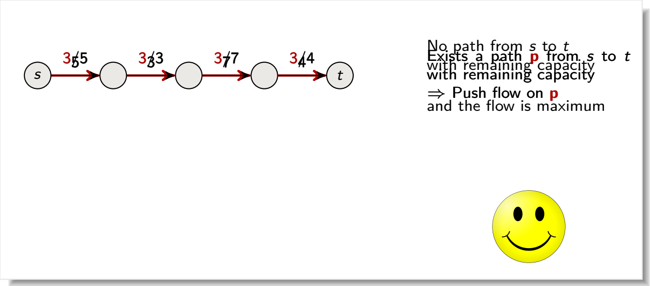

Applying the basic idea to examples

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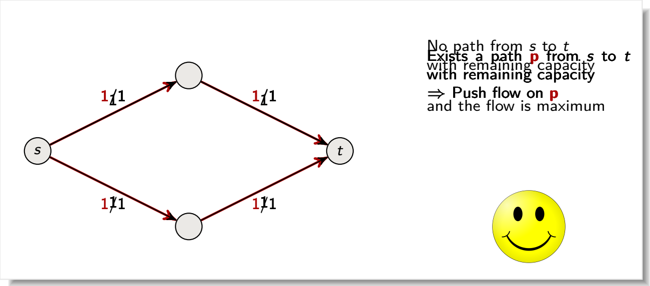

Applying the basic idea to examples

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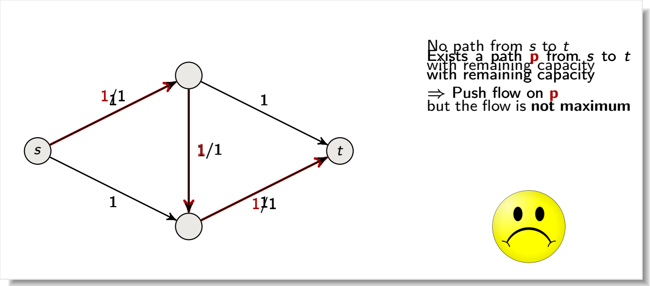

What went wrong? How can we fix it?

The Ford-Fulkerson Method'54

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Residual network

- Given a flow f and a network $G = (V, E)$
- the residual network consists of edges with capacities that represent how we can change the flow on the edges

Residual capacity:

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$

Amount of capacity left (first case)

Amount of flow that can be reversed (second case)

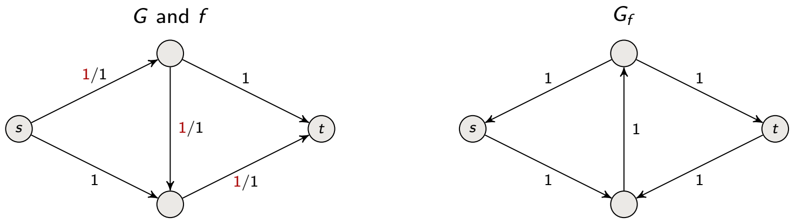
Residual network:

$G_f = (V, E_f)$ where $E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$

Examples

Residual network: $G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$ and

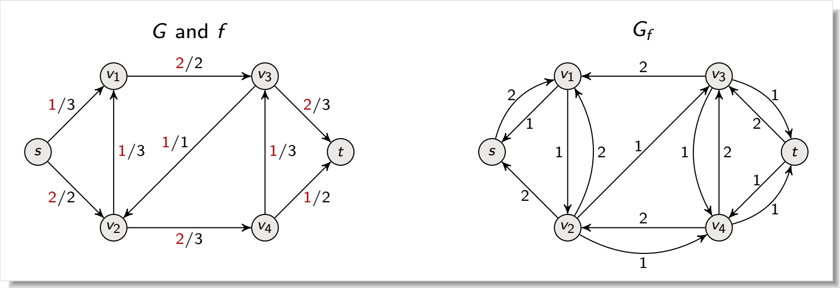
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$



Examples

Residual network: $G_f = (V, E_f)$ where $E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$ and

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$



The Ford-Fulkerson Method'54

FORD-FULKERSON-METHOD(G, s, t):

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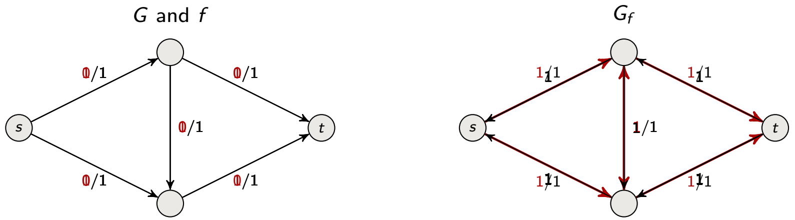
Augm

f is updated flow on an $\mathbf{f_p(u, v) - f_p(v}$

No augmenting path and flow of value 2 is optimal



The Ford-Fulkerson Method'54

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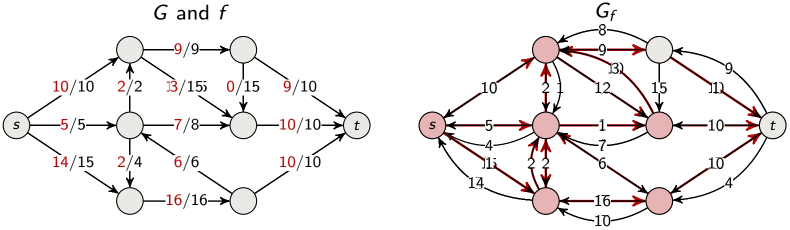
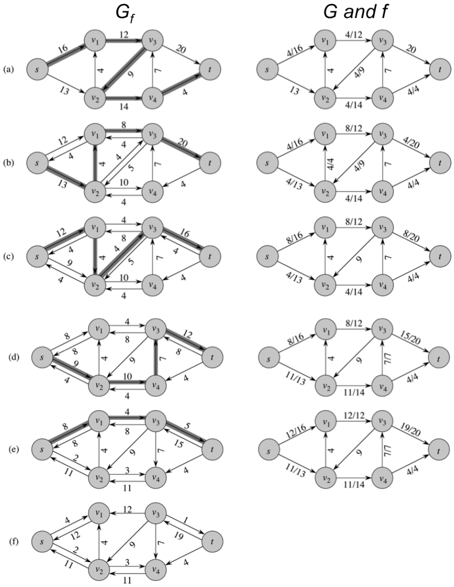
No augmenting path and flow of value 29 is optimal



Study and understand
Example!

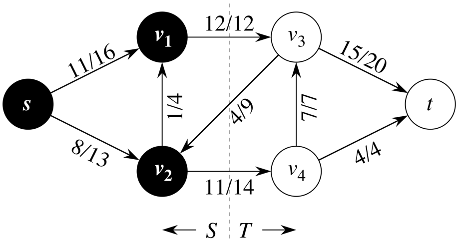


WHY IS RETURNED FLOW OPTIMAL? (MIN-CUTS)

Cuts in flow networks

A cut of flow network $G(V, E)$ is

- a partition of V into S and $T = V \setminus S$
- such that $s \in S$ and $t \in T$

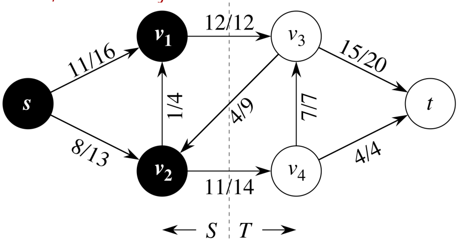


Net flow across a cut

The net flow across cut (S, T) is

$$f(S,T) = \underbrace{\sum_{u \in S, v \in T} f(u,v)}_{\text{flow leaving } S} - \underbrace{\sum_{u \in S, v \in T} f(v,u)}_{\text{flow entering } S}$$

What is the net flow of this cut? $12 + 11 - 4 = 19$ Note that this equals the value of the flow; it's always the case!

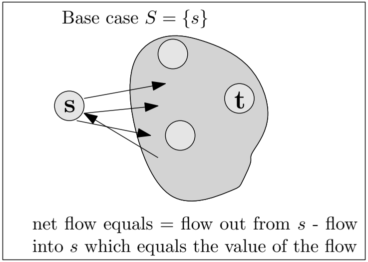


Net flow equals flow value for any cut

Theorem

For any cut (S, T), $|f| = f(S, T)$.

Proof by induction on the size of S.

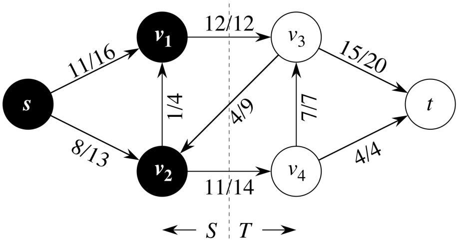


Capacity a cut

The capacity of a cut (S, T) is

$$c(S, T) = \sum_{u \in S, v \in T} c(u, v)$$

What is the capacity of this cut? $12 + 14 = 26$



Flow is at most capacity of a cut

For any flow f and any cut (S, T):

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S, v \in T} f(u, v) - \sum_{u \in S, v \in T} f(v, u) \\
&\leq \sum_{u \in S, v \in T} f(u, v) \\
&\leq \sum_{u \in S, v \in T} c(u, v) \\
&= c(S, T)
\end{aligned}
$$

Max-flow is at most capacity of a cut

Therefore: **max-flow $\leq$ min-cut**

We shall prove

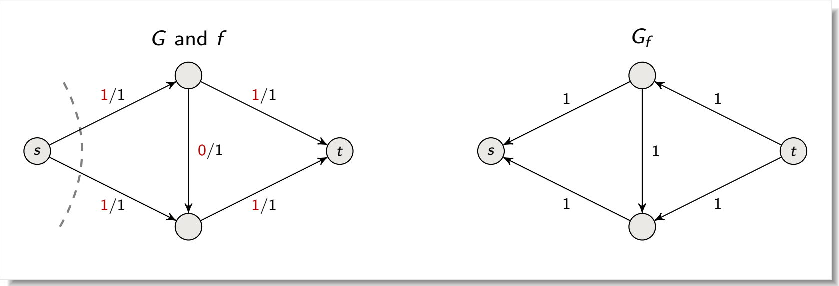
Theorem (max-flow min-cut theorem)

max-flow = min-cut

Examples

Consider f obtained by running Ford-Fulkerson and let

$$S = \{v \in V : \text{there is a path from } s \text{ to } v \text{ in } G_f\} \quad \text{and} \quad T = V \setminus S$$



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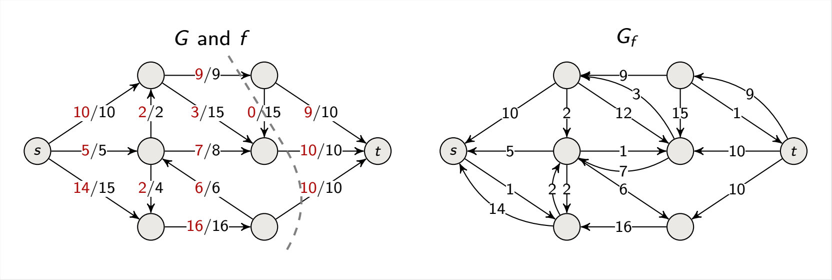

Max-flow min-cut theorem

Let $G = (V, E)$ be a flow network with source s and sink t and capacities c and a flow f.

The following are equivalent:

1. f is a maximum flow
2. G_f has no augmenting path
3. $|f| = c(S, T)$ for a minimum cut (S, T)

Proof. (1) $\Rightarrow$ (2): Suppose toward contradiction that G_f has an augmenting path p.

However, then Ford-Fulkerson method would augment f by p to obtain a flow if increased value which contradicts that f is a maximum flow

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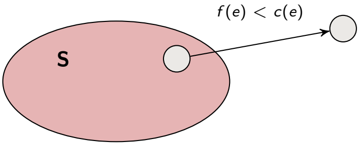
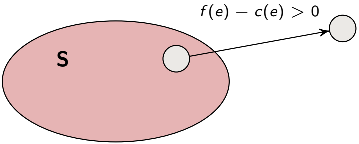
Proof. (2) $\Rightarrow$ (3): S = set of nodes reachable from s in residual network, $T = V \setminus S$

Every edge flowing out of S in G must be at capacity, otherwise we can reach a node outside S in the residual network.

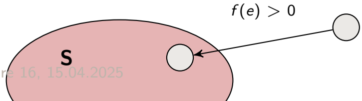


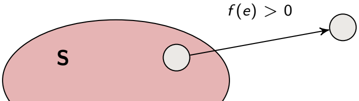
Original graph



Residual network

Every edge flowing into S in G must have flow 0, otherwise we can reach a node outside S in the residual network.





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3. $|f| = c(S, T)$ for a minimum cut (S, T)

Proof. (3) $\Rightarrow$ (1): Recall that $|f| \leq c(S, T)$ for all cuts (S, T).

Therefore, if the value of flow is equal to the capacity of some cut, then it cannot be further improved.

So f is a maximum flow

Summary

- Flow Networks
- Ford-Fulkerson Method
- Cuts
- Max-flow = min cut theorem